

Conformal transformations and equivariance in unbounded KK-theory

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Abstract

We extend unbounded Kasparov theory to encompass conformal group and quantum group equivariance. This new framework allows us to treat conformal actions on both manifolds and noncommutative spaces. As examples, we present unbounded representatives of Kasparov's γ -element for the real and complex Lorentz groups and display the conformal $SL_q(2)$ -equivariance of the standard spectral triple of the Podleś sphere. In pursuing descent for conformally equivariant cycles, we are led to a new framework for representing Kasparov classes. Our new representatives are unbounded, possess a dynamical quality, and also include known twisted spectral triples. We define an equivalence relation on these new representatives whose classes form an abelian group surjecting onto KK. The technical innovation which underpins these results is a novel multiplicative perturbation theory. By these means, we obtain Kasparov classes from the bounded transform with minimal side conditions.

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1 Introduction

In this paper we present a unifying framework for conformal transformations, group equivariance and quantum group equivariance for unbounded Kasparov theory. Our techniques lead to a new class of representatives of Kasparov classes which we call *conformally generated cycles*. These new representatives include known examples of twisted spectral triples.

In particular, the descent maps applied to *conformally equivariant* unbounded Kasparov modules are not unbounded Kasparov modules, but are conformally generated cycles. This applies to both group and quantum group equivariance. We also give unbounded representatives of Kasparov’s γ -element for the real and complex Lorentz groups, a conformally equivariant (higher order) spectral triple for the Heisenberg group with dilation action and display the conformal $SL_q(2)$ -equivariance of the standard spectral triple of the Podleś sphere. We now explain these results and our methods in more detail.

One aspect of the unbounded picture of KK-theory which has not been fully explored is group equivariance. One reason for this is that the definition of equivariance for unbounded Kasparov modules made by Kucerovsky in [Kuc94] fails to capture all the degrees of freedom available in the bounded picture of equivariant KK-theory. In the following we refer to Kucerovsky’s definition as *uniform equivariance*, see Definition 3.10. Perhaps the easiest example is the Dirac spectral triple on a Riemannian manifold, equipped with the action of a group. If the action is isometric, the Dirac operator is invariant. If the action is a conformal one, the Fredholm module defined by the bounded transform yields a bounded equivariant Fredholm module, but it unclear how to think of the spectral triple as being equivariant.

For the case of a compact manifold, conformally equivalent Dirac operators have been addressed in the context of noncommutative geometry by Bär [Bär07]. A conformal change of metric has the effect $\not{D} \rightsquigarrow k^{-1/2} \not{D} k^{-1/2}$ on the Atiyah–Singer Dirac operator. By considering principal symbols, the bounded transform $\not{D}(1 + \not{D}^2)^{-1/2}$ changes only by a compact operator. In §2, we give new tools to identify two self-adjoint regular operators as having “close” bounded transforms in much more general circumstances.

One interpretation of conformal actions and changes of metric is via Connes and Moscovici’s *twisted spectral triples* [CM08]. One of the two main examples [CM08, §2.2] of twisted spectral triples given by Connes and Moscovici is built from a multiplicative perturbation $D \rightsquigarrow kDk$. The other main example [CM08, §2.3] [Mos10, §3.1] is built from a Dirac spectral triple $(C_0(X), L^2(X, S), \not{D})$ on a Riemannian manifold X , equipped with the conformal action of a discrete group G . One extends the algebra $C_0(X)$ to the crossed product $C_0(X) \rtimes G$ and

$$(C_0(X) \rtimes G, L^2(X, S), \not{D})$$

becomes a Lipschitz regular twisted spectral triple. In §6.1, we will interpret this as the dual-Green–Julg map of a *conformally equivariant* unbounded cycle and show that such examples possess well-defined bounded transforms without recourse to the Lipschitz regularity condition of [CM08, Definition 3.1].

In the framework of the spectral action principle, Chamseddine and Connes [CC06] calculate the effect of rescaling the Spectral Standard Model Dirac operator $D \rightsquigarrow e^{-\phi/2} D e^{-\phi/2}$, where the *dilaton* ϕ is interpreted as a scalar field. Apart from the Higgs mass term, the entire Lagrangian of the Standard Model of particle physics is conformally invariant, which was a motivation for this work.

The technical innovation which underpins our results is a multiplicative perturbation theory for self-adjoint regular operators on Hilbert modules. This perturbation theory relates the bounded transforms $D(1 + D^2)^{-1/2}$ and $\mu D \mu^* (1 + (\mu D \mu^*)^2)^{-1/2}$ of D and its multiplicative perturbation $\mu D \mu^*$, for suitable μ . Together with the well-known additive perturbation theory $D \rightsquigarrow D + A$ for (relatively) bounded A , Theorem 2.43 says, roughly, that any perturbation preserving the KK-class of the bounded transform takes the form $\mu D \mu^* + A$.

We introduce several concepts making use of the multiplicative perturbation theory of §2, among which are:

- *Conformal transformations* between unbounded Kasparov modules, Definition 2.7, and a singular version, Definition 2.47;
- *Conformal group equivariance* for unbounded Kasparov modules, Definition 3.19;
- *Conformal quantum group equivariance* for unbounded Kasparov modules, Definition 4.22;
- The equivalence relation of *conformism* for unbounded Kasparov modules, Definition 5.14; and
- *Conformally generated cycles*, Definition 6.1, providing a new picture of KK-theory, generalising unbounded KK-theory.

Conformally generated cycles have a dynamical aspect in addition to a geometrical one. We show that this framework is adapted to all known examples of twisted spectral triples with well-defined bounded transforms. Key features of our approach are the lack of a ‘twist’, in the sense of an algebra automorphism, and the bounded transform which does not depend on any additional smoothness condition such as Lipschitz regularity. We show in §6.1 that Kasparov’s descent map (and the dual-Green–Julg map) applied to group and quantum group conformally equivariant unbounded Kasparov modules give rise to conformally generated cycles whose bounded transforms define the same classes as the descent map (dual-Green–Julg map) applied to the bounded transforms of the original modules.

We begin by considering conformal transformations between (higher order) unbounded Kasparov modules in §2. The motivation for such a framework is conformal changes of metric of Riemannian manifolds and the noncommutative torus, of which we give some details in §2.1. In the simplest instance for unbounded Kasparov modules (A, E, D_1) and (A, E', D_2) , these transformations are a pair (U, μ) with $U : E \rightarrow E'$ unitary and μ a bounded invertible endomorphism (which is even if the module is graded) such that, for all a in a dense subset of A ,

$$U^* D_2 U a - a \mu D_1 \mu^* \tag{1.1}$$

is bounded. The Leibniz rule shows that those a for which (1.1) is bounded naturally form a (not norm-closed) ternary ring of operators, rather than a $*$ -algebra. The implicit presence of ternary rings of operators will be a feature of many of our definitions. For the technical results in §2.3, we require that the ‘conformal factor’ μ be a bounded and invertible operator, although it need not have a globally bounded derivative. We prove the following as Theorem 2.9.

Theorem. *Let (U, μ) be a conformal transformation from the order- $\frac{1}{1-\alpha}$ cycle (A, E_B, D_1) to the order- $\frac{1}{1-\alpha}$ cycle (A, E'_B, D_2) . Then the bounded transforms (A, E_B, F_{D_1}) and (A, E'_B, F_{D_2}) are unitarily equivalent up to locally compact perturbation via the unitary U ; that is*

$$(U^* F_{D_2} U - F_{D_1})a \in \text{End}^0(E)$$

for all $a \in A$. Hence $[(A, E_B, F_{D_1})] = [(A, E'_B, F_{D_2})] \in KK(A, B)$.

On a noncompact manifold, this is not sufficient to describe all conformal changes of metric. One technical issue which arises is that a complete Riemannian manifold, such as the hyperbolic plane, may be conformally equivalent to an incomplete manifold, such as the unit disc, and therefore the self-adjointness of a Dirac operator may not be preserved. With this caveat, we give in §2.5 a framework modelled abstractly on the idea of an open cover extending the idea in (1.1).

We also show in §2.4 that the *logarithmic transform* $D \rightarrow L_D = F_D \log((1 + D^2)^{1/2})$, due to Goffeng, Mesland, and the second named author [GMR19], turns multiplicative perturbations into additive ones. In Theorem 2.46 we prove

Theorem. *Let (U, μ) be a conformal transformation from the order- $\frac{1}{1-\alpha}$ cycle (A, E_B, D_1) to the order- $\frac{1}{1-\alpha}$ cycle (A, E'_B, D_2) . Then the logarithmic transforms (A, E_B, L_{D_1}) and (A, E'_B, L_{D_2}) are related by the unitary U , up to locally bounded perturbation; in particular, A is contained in the closure of the set of $a \in \text{End}^*(E)$ such that*

$$(U^* L_{D_2} U - L_{D_1})a \quad [L_{D_1}, a]$$

is bounded.

We then extend the existing definitions of uniform group equivariance, due to Kucerovsky [Kuc94], to higher order unbounded Kasparov modules to encompass conformal actions in §3, based on the idea of conformal transformation in (1.1). This is necessary to include the full range of equivariance encoded for bounded Kasparov modules, as indicated by the results of Bär [Bär07] and explained using the example of the $ax + b$ group acting on $\mathbb{R}$. In Theorem 3.21 we prove

Theorem. *The bounded transform of a conformally equivariant higher-order unbounded Kasparov module is an equivariant bounded Kasparov module.*

The logarithmic transform again changes multiplicative perturbations coming from conformal actions to additive perturbations. In Theorem 3.27 we prove

Theorem. *The logarithmic transform of a conformally equivariant higher-order unbounded Kasparov module is a uniformly equivariant unbounded Kasparov module.*

These results allow us to represent the γ -elements of Kasparov and Chen for the Lorentz groups and of Julg and Kasparov for the complex Lorentz groups, in §3.3. In §3.4, we give a genuinely noncommutative example, a 2nd order spectral triple for the C^* -algebra of the Heisenberg group which is equivariant for the dilation action.

In §4 we leave the generality of higher order cycles so as not to distract from our main message, and focus on unbounded Kasparov modules. For such modules we study C^* -bialgebra equivariance, following the treatment in the bounded picture by Baaj and Skandalis [BS89]. We give a definition for uniform (non-conformal) equivariance of unbounded Kasparov modules which, to our knowledge, has not previously appeared in the literature (except in the *isometric* case [GB16]). We show how the descent and dual-Green–Julg maps work in the setting of uniform equivariance.

With the definition of uniform equivariance in hand, we define conformal quantum group equivariance for unbounded Kasparov modules in §4.2. The main example to which we apply this framework is the action of $SL_q(2)$ on the Podleś sphere. In Theorems 4.24 and 4.26 we prove

Theorem. *The bounded transform of a conformally quantum group equivariant unbounded Kasparov module is a quantum group equivariant bounded Kasparov module.*

Theorem. *The logarithmic transform of a conformally quantum group equivariant unbounded Kasparov module is a uniformly quantum group equivariant unbounded Kasparov module.*

In §5, we generalise cobordism of bounded Kasparov modules, as defined by Cuntz and Skandalis [CS86], to unbounded Kasparov modules. We show in Theorem 5.13 that cobordism classes of unbounded Kasparov modules are an abelian group which surjects onto the usual KK-groups.

In §5.1, we extend the idea further to define *conformism* of unbounded Kasparov modules, using the framework of cobordism to turn the conformal transformations of §1 into an equivalence relation. We show in Theorem 5.18 that conformism classes of unbounded Kasparov modules are an abelian group which surjects onto the usual KK-groups.

All of the generalisations we have considered so far are brought together in §6 where we introduce conformally generated cycles. These unbounded representatives of Kasparov classes are general enough to include known examples of twisted spectral triples, as we outline at the beginning of §5, as well as the result of applying descent and dual-Green–Julg maps to group and quantum group conformally equivariant Kasparov modules, generalising the constructions for uniform equivariance given in §3.1, in the group case, and §4.2, in the quantum group case.

Finally, in §6.2, we show that conformism extends to an equivalence relation on conformally generated cycles, and the conformism classes of such cycles form an abelian group which surjects onto the usual KK-group.

For the multiplicative perturbation theory of §2.3, we require certain bounds and domain relationships involving fractional powers of positive regular operators on Hilbert modules. Although these are well known in the Hilbert space case, we provide a complete proof in the Hilbert module case in Appendix A.1. For group equivariance, we require certain identifications of Hilbert modules over locally compact Hausdorff spaces and their operators, which we cover in Appendix A.2, based on the approach of Kucerovsky [Kuc94].

For quantum group equivariance and conformally generated cycles, we use the ideas of *matched operators* and *compactly supported states*. These generalise the multipliers of the Pedersen ideal of a C^* -algebra and their positive continuous dual. Given a C^* -algebra C acting on the right of a Hilbert B -module via a nondegenerate homomorphism $C \rightarrow M(B)$, the C -matched operators on E are a subset of the regular operators which form a $*$ -algebra (in fact, a pro- C^* -algebra), as we show in Appendix A.3. In Appendix A.4, we characterise compactly supported states [Har23] on a C^* -algebra in terms of the Pedersen ideal and show that they are weak- $*$ -dense in all states.

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2 Conformal transformations

For us, Kasparov cycles and their generalisations will be over ungraded $\mathbb{C}$ -algebras. We never need the grading of the module, so our results apply to both even and odd cycles with trivially graded algebras. When we consider Kasparov classes, we will write KK generically for classes of even or odd cycles, and unless mentioned all C^* -algebras will be trivially graded and conformal factors are even if the module is graded.

Definition 2.1. [Kas88, Definition 2.2] A bounded Kasparov A - B -module consists of an A - B -correspondence E and a bounded operator F on E such that, for all $a \in A$, the operators

$$(F^* - F)a \quad (1 - F^2)a \quad [F, a]$$

are compact.

We will mostly work with higher order unbounded Kasparov modules due to Wahl [Wah07]. Throughout we use the notation $\langle D \rangle := (1 + D^2)^{1/2}$ for a self-adjoint regular operator D on a Hilbert module.

Definition 2.2. cf. [GM15, Definition A.1] Let D be a self-adjoint regular operator on a right Hilbert B -module E . For $0 \leq \alpha \leq 1$, let

$$\text{Lip}_\alpha^*(D) \subseteq \text{End}_B^*(E)$$

be the subspace consisting of elements $a \in \text{End}^* E$ for which $a \text{ dom } D \subseteq \text{dom } D$ and $[D, a]\langle D \rangle^{-\alpha}$ and $\langle D \rangle^{-\alpha}[D, a]$ extend to bounded adjointable operators. By [GM15, Proposition A.5], $\text{Lip}_\alpha^*(D)$ is a $*$ -algebra.

It can be shown that $\text{Lip}_\alpha^*(D)$ is a Banach $*$ -algebra under an appropriate norm and is closed under the holomorphic functional calculus, but we do not use this here. We will also weaken our definition of unbounded cycles along the lines of [vdDM20, Definition 1.1] since morphisms between cycles may not naturally preserve a given smooth subalgebra.

Definition 2.3. cf. [Wah07, Definition 2.4] [GM15, Definition A.2] [vdDM20, Definition 1.1] Let $0 \leq \alpha < 1$. An order- $\frac{1}{1-\alpha}$ A - B -cycle consists of an A - B -correspondence E and a regular operator D on E such that:

1. D is self-adjoint;
2. $(1 + D^2)^{-1}a$ is compact; and
3. A is contained in the operator norm closure of $\text{Lip}_\alpha^*(D)$.

If we have a dense subalgebra $\mathcal{A}$ of A which is contained in $\text{Lip}_\alpha^*(D)$, we will call the cycle an order- $\frac{1}{1-\alpha}$ $\mathcal{A}$ - B -cycle. If $\alpha = 0$ then we refer to order-1 cycles as unbounded Kasparov modules, and if $B = \mathbb{C}$ we call these cycles spectral triples.

Example 2.4. [GM15, Remark A.0.3] Let X be a complete Riemannian manifold and V a vector bundle over X . If D is a self-adjoint elliptic pseudodifferential operator of order $m > 0$ acting on sections of V then $(C_0(X), L^2(X, V), D)$ is an order- m spectral triple.

The generalisation to “higher order operators” does not interfere with the main topological result for unbounded Kasparov modules.

Theorem 2.5. cf. [Wah07, Definition 2.4] [GM15, Theorem A.6] [vdDM20, Proposition 1.7] Let (A, E_B, D) be an order $\frac{1}{1-\alpha}$ A - B -cycle. Then the bounded transform $D \mapsto F_D := D(1 + D^2)^{-1/2}$ gives a Kasparov module (A, E_B, F_D) .

We recall here a few facts about ternary rings of operators. Ring- or algebra-like objects with ternary product operations are known also as triple systems, and come in Lie, Jordan, and associative varieties, the latter in two kinds. In the context of abstract operator algebras there are C^* - and W^* -ternary rings, due to [Zet83].

Definition 2.6. A *ternary ring of operators* on a Hilbert B -module E is a collection $\mathcal{X} \subseteq \text{End}^*(E)$ which is closed under the operation

$$(x, y, z) \mapsto xy^*z.$$

We will not by default assume that a ternary ring of operators is norm-closed.

In the sense of [RW98, Lemma 2.16], $\mathcal{X}$ is a right pre-Hilbert $\text{span}(\mathcal{X}^*\mathcal{X})$ -module. Its completion $\bar{\mathcal{X}}$ is then a right Hilbert $\overline{\text{span}}(\mathcal{X}^*\mathcal{X})$ -module. By similar considerations on the left, $\bar{\mathcal{X}}$ is a Morita equivalence $\overline{\text{span}}(\mathcal{X}\mathcal{X}^*)$ - $\overline{\text{span}}(\mathcal{X}^*\mathcal{X})$ -bimodule. We remark that, for instance, $\overline{\text{span}}(\mathcal{X}\mathcal{X}^*\mathcal{X}) = \bar{\mathcal{X}}$.

In particular, every norm-closed ternary ring of operators is a Morita equivalence bimodule in a natural way. By [Zet83, Theorem 2.6], any Hilbert C^* -module can be represented as a norm-closed ternary ring of operators on some Hilbert space H .

The implicit presence of ternary rings of operators will be a feature of many of our definitions. This occurs because, just as the Leibniz rule makes the domain of a commutator with a self-adjoint operator D a $*$ -algebra, the domain of a mixed commutator $a \mapsto D_1a - aD_2$ is naturally closed under the ternary product.

We will formulate our definition of conformal transformation for higher order cycles.

Definition 2.7. A *conformal transformation* (U, μ) from one order- $\frac{1}{1-\alpha}$ cycle, (A, E_B, D_1) , to another, (A, E'_B, D_2) , is a unitary map $U : E \rightarrow E'$, intertwining the representations of A , and an (even) invertible operator $\mu \in \text{End}^*(E)$ which is even if the module is graded, satisfying the following. We require that $A \subseteq \overline{\text{span}}(A\mathcal{M}) \cap \overline{\text{span}}(\mathcal{M}A)$, where $\mathcal{M}$ is the set of $a \in \text{End}^*(E)$ such that the operators

$$(U^*D_2Ua - a\mu D_1\mu^*)\mu^{-1*}\langle D_1 \rangle^{-\alpha} \quad \langle D_2 \rangle^{-\alpha}U(U^*D_2Ua - a\mu D_1\mu^*)$$

are bounded, $a, a\mu, a\mu^{-1*} \in \text{Lip}_\alpha^*(D_1)$, and $UaU^* \in \text{Lip}_\alpha^*(D_2)$.

Remarks 2.8.

1. The easiest way for the closure condition to be satisfied is if $1 \in \mathcal{M}$; for nonunital A an approximate unit might be found to lie in $\mathcal{M}$.
2. We have $\mathcal{M}\mathcal{M}^*\mathcal{M} \subseteq \mathcal{M}$ and so $\mathcal{M}$ is a ternary ring of operators, in general not norm-closed.
3. Conformal transformations are generally neither reversible nor composable. This latter occurs very easily for two noncommuting conformal factors μ and ν . We address this issue with the *conformisms* of §5.1. The condition that $\mathcal{M} \subseteq \text{Lip}_\alpha^*(D_2)$ is not strictly necessary for the proof of Theorem 2.9, but it will reappear in §5.1.

In the next section, on page 20, we will prove the following Theorem.

Theorem 2.9. Let (U, μ) be a conformal transformation from the order- $\frac{1}{1-\alpha}$ cycle (A, E_B, D_1) to the order- $\frac{1}{1-\alpha}$ cycle (A, E'_B, D_2) . Then the bounded transforms (A, E_B, F_{D_1}) and (A, E'_B, F_{D_2}) are unitarily equivalent up to locally compact perturbation via the unitary U . That is

$$(U^*F_{D_2}U - F_{D_1})a \in \text{End}^0(E)$$

for all $a \in A$. Hence $[(A, E_B, F_{D_1})] = [(A, E'_B, F_{D_2})] \in KK(A, B)$.

2.1 Motivating examples

Example 2.10. cf. [vdD20, Lemma 2.8] The simplest nontrivial example of a conformal transformation between unbounded cycles can be constructed from an unbounded cycle (A, E_B, D) and a positive number κ . The pair $(\text{id}, \kappa^{1/2})$ is a conformal transformation from (A, E_B, D) to $(A, E_B, \kappa D)$.

On a geodesically complete Riemannian manifold X , there are two standard spectral triples. One relies on a spin structure and takes the form $(C_0(X), L^2(X, S), \not{D})$, where S is a spinor bundle and $\not{D}$ is the Atiyah–Singer Dirac operator. The other depends on only the orientation and Riemannian metric, taking the form $(C_0(X), L^2(\Omega^*X), d + \delta)$ where d is the exterior derivative on differential forms Ω^*X and δ is its adjoint, the codifferential, their sum being the Hodge–de Rham Dirac operator. We consider the effect of a conformal change of metric on both these spectral triples.

Example 2.11. The behaviour of the Atiyah–Singer Dirac operator under conformal transformations was first recorded in [Hit74, Proof of Proposition 1.3]. In the context of noncommutative geometry, see also [Bär07, Proof of Theorem 3.1]. Let $(X, \mathbf{g})$ and $(X, \mathbf{h})$ be Riemannian spin manifolds such that $\mathbf{h} = k^2 \mathbf{g}$. Let $S_{\mathbf{g}}$ and $S_{\mathbf{h}}$ be their associated spinor bundles. There is a fibrewise isometry

$$\psi : S_{\mathbf{g}} \rightarrow S_{\mathbf{h}}.$$

Let $\not{D}_{\mathbf{g}} : \Gamma^\infty(S_{\mathbf{g}}) \rightarrow \Gamma^\infty(S_{\mathbf{g}})$ and $\not{D}_{\mathbf{h}} : \Gamma^\infty(S_{\mathbf{h}}) \rightarrow \Gamma^\infty(S_{\mathbf{h}})$ be the corresponding Dirac operators. Then, by e.g. [Hij86, Proposition 4.3.1],

$$\not{D}_{\mathbf{h}} = k^{(-n-1)/2} \circ \psi \circ \not{D}_{\mathbf{g}} \circ \psi^{-1} \circ k^{(n-1)/2}.$$

Although ψ is a fibrewise isometry, the induced map $V : L^2(X, S_{\mathbf{g}}) \rightarrow L^2(X, S_{\mathbf{h}})$ is not unitary, as the volume form changes. With the relation $\text{vol}_{\mathbf{h}} = k^n \text{vol}_{\mathbf{g}}$, we find that $V^* = k^n V^{-1}$. The polar decomposition is

$$U = V(V^*V)^{-1/2} = k^{-n/2}V$$

and we find that

$$\not{D}_{\mathbf{h}} = k^{-1/2} U \not{D}_{\mathbf{g}} U^* k^{-1/2}$$

or, in other words,

$$U^* \not{D}_{\mathbf{h}} U = k^{-1/2} \not{D}_{\mathbf{g}} k^{-1/2}.$$

In terms of Definition 2.7, if $(X, \mathbf{g})$ is complete and the conformal factor k and its inverse are bounded (which is automatic if X is compact), then $(U, k^{-1/2})$ is a conformal transformation from $(C_0(X), L^2(X, S_{\mathbf{g}}), \not{D}_{\mathbf{g}})$ to $(C_0(X), L^2(X, S_{\mathbf{h}}), \not{D}_{\mathbf{h}})$.

Example 2.12. Next, we consider the Hodge–de Rham Dirac operator. As before, let $(X, \mathbf{g})$ and $(X, \mathbf{h})$ be Riemannian manifolds such that $\mathbf{h} = k^2 \mathbf{g}$. Consider the two inner products on Ω^*X given by $\mathbf{g}$ and $\mathbf{h}$, which we will label $\langle \cdot, \cdot \rangle_{\mathbf{g}}$ and $\langle \cdot, \cdot \rangle_{\mathbf{h}}$. We will call the resulting Hilbert spaces $L^2(\Omega^*X, \mathbf{g})$ and $L^2(\Omega^*X, \mathbf{h})$. There is an obvious map

$$V : L^2(\Omega^*X, \mathbf{g}) \rightarrow L^2(\Omega^*X, \mathbf{h})$$

given by the identity on Ω^*X , in other words, for $\omega \in \Omega^*X \subseteq L^2(\Omega^*X, \mathbf{g})$, $V : \omega \mapsto \omega$. Its adjoint is given on homogenous forms ω by $V^* : \omega \mapsto k^{n-2|\omega|} \omega$. Observe that if n is even the restriction of V to the middle degree forms is unitary. We make the (rather trivial) observation that

$$VV^* : \omega \mapsto k^{n-2|\omega|} \omega \quad V^*V : \omega \mapsto k^{n-2|\omega|} \omega. \quad (2.13)$$

The unitary in the polar decomposition $U = V(V^*V)^{-1/2} = (VV^*)^{-1/2}V$ is given by

$$U : \omega \mapsto k^{(-n+2|\omega|)/2} \omega \quad U^* : \omega \mapsto k^{(n-2|\omega|)/2} \omega.$$

The exterior derivative d does not depend on the metric, but its adjoint the codifferential does, so we use the notation $\delta_{\mathbf{g}}$ and $\delta_{\mathbf{h}}$ to distinguish the two codifferentials acting on Ω^*X . The invariance of the exterior derivative means that $dV = Vd$. With care over which inner product is being used, $(Vd)^* = \delta_{\mathbf{g}}V^*$ and $(dV)^* = V^*\delta_{\mathbf{h}}$. So, $\delta_{\mathbf{g}}V^* = V^*\delta_{\mathbf{h}}$ and we obtain the relations

$$V(d + \delta_{\mathbf{g}})V^* = d(VV^*) + (VV^*)\delta_{\mathbf{h}}$$

and

$$U(d + \delta_{\mathbf{g}})U^* = (VV^*)^{-1/2}V(d + \delta_{\mathbf{g}})V^*(VV^*)^{-1/2} = (VV^*)^{-1/2}d(VV^*)^{1/2} + (VV^*)^{1/2}\delta_{\mathbf{h}}(VV^*)^{-1/2}.$$

On a differential form ω of degree $|\omega|$,

$$\begin{aligned} U(d + \delta_{\mathbf{g}})U^*\omega &= k^{-(n-2(|\omega|+1))/2}d(k^{(n-2|\omega|)/2}\omega) + k^{(n-2(|\omega|-1))/2}\delta_{\mathbf{h}}(k^{-(n-2|\omega|)/2}\omega) \\ &= k\left(k^{-(n-2|\omega|)/2}d(k^{(n-2|\omega|)/2}\omega) + k^{(n-2|\omega|)/2}\delta_{\mathbf{h}}(k^{-(n-2|\omega|)/2}\omega)\right). \end{aligned}$$

For any function $f \in C^\infty(X)$,

$$\begin{aligned} f^{-1}df\omega + f\delta_{\mathbf{h}}f^{-1}\omega &= (d + \delta_{\mathbf{h}})\omega + f^{-1}[d, f]\omega + [f, \delta_{\mathbf{h}}]f^{-1}\omega \\ &= (d + \delta_{\mathbf{h}})\omega + f^{-1}[d, f]\omega - [\delta_{\mathbf{h}}, f]f^{-1}\omega \\ &= (d + \delta_{\mathbf{h}})\omega + f^{-1}[d - \delta_{\mathbf{h}}, f]\omega. \end{aligned}$$

Hence

$$\begin{aligned} &\left(U(d + \delta_{\mathbf{g}})U^* - k^{1/2}(d + \delta_{\mathbf{h}})k^{1/2}\right)\omega \\ &= \left(k(d + \delta_{\mathbf{h}}) + k^{-(n-2|\omega|-2)/2}[d - \delta_{\mathbf{h}}, k^{(n-2|\omega|)/2}] - k^{1/2}(d + \delta_{\mathbf{h}})k^{1/2}\right)\omega \\ &= \left(-k^{1/2}[d + \delta_{\mathbf{h}}, k^{1/2}] + k^{-(n-2|\omega|-2)/2}[d - \delta_{\mathbf{h}}, k^{(n-2|\omega|)/2}]\right)\omega. \end{aligned}$$

In terms of Definition 2.7, if $(X, \mathbf{g})$ is complete and the conformal factor k and its inverse are bounded (which is automatic if X is compact), the data $(U, k^{-1/2})$ define a conformal transformation from $(C_0(X), L^2(\Omega^*X, \mathbf{g}), d + \delta_{\mathbf{g}})$ to $(C_0(X), L^2(\Omega^*X, \mathbf{h}), d + \delta_{\mathbf{h}})$.

Remark 2.14. The extension of the Hodge-de Rham spectral triple to a spectral triple for the $\mathbb{Z}_2$ -graded Clifford algebra bundle is important for Poincaré duality [Kas88]. In the case of a manifold, where the functions and conformal factors are in the centre of the Clifford algebra, it is not difficult to show that our definition of conformal transformation can be modified to handle the graded commutators. We leave a discussion of the general $\mathbb{Z}_2$ -graded case to another place.

Example 2.15. Suppose that we have the data of a continuous family of compact Riemannian spin manifolds $(M_x, \mathbf{g}_x)_{x \in X}$ parameterised by a locally compact Hausdorff space X , as in the families index theorem [LM89, §III.15]. Integration over the fibres of the total space $\mathcal{M} \rightarrow X$ along with the Dirac operators D_x on the fibre spinor bundles S_x yields an unbounded Kasparov module

$$\left(C_0(\mathcal{M}), L^2(\mathcal{M}, S_\bullet, \mathbf{g}_\bullet)_{C_0(X)}, D_\bullet\right). \quad (2.16)$$

Let $k : \mathcal{M} \rightarrow [0, \infty)$ be a family of conformal factors parameterised by X . The commutation of the conformal factors with the algebra means we obtain a new unbounded Kasparov module

$$\left(C_0(\mathcal{M}), L^2(\mathcal{M}, S_\bullet, k^2 \mathbf{g}_\bullet)_{C_0(X)}, k^{-1/2}D_\bullet k^{-1/2}\right).$$

We observe that the integration over the fibres changes, but the compactness of the fibres means we get equivalent measures. That we obtain a new unbounded Kasparov module is straightforward but of more consequence is that the classes defined by F_D and $F_{k^{-1/2}D_\bullet k^{-1/2}}$ in $KK(C_0(\mathcal{M}), C_0(X))$ coincide.

Suppose that we have another family of metrics $\mathbf{h}_\bullet$, for the same family of manifolds, giving an unbounded Kasparov module

$$\left(C_0(\mathcal{M}), L^2(\mathcal{M}, S_\bullet, \mathbf{h}_\bullet)_{C_0(X)}, D_\bullet\right). \quad (2.17)$$

Suppose that $\mathbf{h}_x = k_x^2 \mathbf{g}_x$ for a (pointwise) continuous family $k_\bullet \in C^\infty(M_\bullet)$ of smooth functions and that $\sup_{x \in X} \{\|k_x\|_\infty, \|k_x^{-1}\|_\infty\} < \infty$. Then $(\text{id}, k_\bullet^{-1/2})$ is a conformal transformation from (2.16) to (2.17).

The first appearance of conformal transformations in noncommutative geometry was with the preprint [CC92] on the noncommutative torus, followed up by the same authors in [CT11]; see also [CM14]. This is not to be confused with the twisted spectral triples of [CM08], which will be examined in §6.

Example 2.18. Fix a real number α . Let $C(\mathbb{T}_\alpha^2)$ be the universal C^* -algebra generated by unitaries U and V subject to the relation

$$VU = e^{2\pi i\alpha}UV.$$

There are two self-adjoint (unbounded) derivations δ_1 and δ_2 on $C(\mathbb{T}_\alpha^2)$, given on generators by

$$\delta_1(U) = U \quad \delta_1(V) = 0 \quad \delta_2(U) = 0 \quad \delta_2(V) = V.$$

When $\alpha = 0$, these are the derivatives $-i\partial_{\theta_1}$ and $-i\partial_{\theta_2}$ on the classical torus. There is a trace on $C(\mathbb{T}_\alpha^2)$ given by

$$\varphi(U^m V^n) = \delta_{m,0} \delta_{n,0}.$$

The completion of $C(\mathbb{T}_\alpha^2)$ in the inner product given by φ is $L^2(\mathbb{T}_\alpha^2)$. Fix a complex number τ with $\Im(\tau) > 0$. Then

$$\left(C(\mathbb{T}_\alpha^2), L^2(\mathbb{T}_\alpha^2) \otimes \mathbb{C}^2, D := \begin{pmatrix} & \delta_1 + \tau\delta_2 \\ \delta_1 + \bar{\tau}\delta_2 & \end{pmatrix} \right)$$

is a spectral triple. Now choose a positive invertible element $k \in C(\mathbb{T}_\alpha^2)$ in the domains of δ_1 and δ_2 . Let $k^\circ \in B(L^2(\mathbb{T}_\alpha^2))$ be the operator of right multiplication. Then

$$\left(C(\mathbb{T}_\alpha^2), L^2(\mathbb{T}_\alpha^2) \otimes \mathbb{C}^2, D_{k^2} := \begin{pmatrix} & (k^\circ)^2(\delta_1 + \tau\delta_2) \\ (\delta_1 + \bar{\tau}\delta_2)(k^\circ)^2 & \end{pmatrix} \right)$$

is still a spectral triple. We have that

$$D_k - k^\circ D k^\circ = \begin{pmatrix} & -k^\circ [\delta_1 + \tau\delta_2, k^\circ] \\ [\delta_1 + \bar{\tau}\delta_2, k^\circ] k^\circ & \end{pmatrix}$$

is bounded. Hence $1 \in \mathcal{M}$ and (id, k°) is a conformal transformation from the spectral triple $(C(\mathbb{T}_\alpha^2), L^2(\mathbb{T}_\alpha^2) \otimes \mathbb{C}^2, D)$ to $(C(\mathbb{T}_\alpha^2), L^2(\mathbb{T}_\alpha^2) \otimes \mathbb{C}^2, D_{k^2})$.

Let $\Phi : C(\mathbb{T}_\alpha^2) \rightarrow C(S^1)$ be the expectation coming from averaging over the circle action $U \mapsto zU$, $z \in \mathbb{T}$. Then $(C(\mathbb{T}_\alpha^2), L^2(C(\mathbb{T}_\alpha^2), \Phi)_{C(S^1)}, \delta_2)$ is an unbounded Kasparov module by [BCR15, Proposition 2.9]. Now choose a positive invertible element $k \in C(\mathbb{T}_\alpha^2)$ in the domain of δ_2 . Then (id, k°) is a conformal transformation from $(C(\mathbb{T}_\alpha^2), L^2(C(\mathbb{T}_\alpha^2), \Phi)_{C(S^1)}, \delta_2)$ to the spectral triple

$$\left(C(\mathbb{T}_\alpha^2), L^2(C(\mathbb{T}_\alpha^2), \Phi)_{C(S^1)}, k^\circ \delta_2 k^\circ \right).$$

Example 2.18 can be generalised along the lines of [Sit15], using a real spectral triple satisfying the order zero condition. Theorem 2.9 gives a refinement of [Sit15, Lemma 14] which shows that the class in KK-theory of the conformally perturbed spectral triple is unchanged.

2.2 Technical preliminaries and additive perturbation theory

Throughout this section we fix a countably generated right Hilbert B -module E for some C^* -algebra B . The main tool in our proofs is the integral formula

$$(1 + D^2)^{-\alpha} = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \lambda^{-\alpha} (\lambda + 1 + D^2)^{-1} d\lambda \quad (2.19)$$

whose use in noncommutative geometry is due to Baaß and Julg [BJ83]; for more details we refer to [CP98, Lemma A.4]. We quote the following refinement of Baaß and Julg's bounded transform result which follows easily from the results of [Wah07, §2.1], [Gre12, §7], [GM15, Appendix A].

Theorem 2.20. *Let D be a self-adjoint regular operator on a right Hilbert B -module E . Let S be an adjointable operator such that $S \operatorname{dom} D \subseteq \operatorname{dom} D$ and $[D, S]\langle D \rangle^{-\alpha}$ extends to a bounded operator for some $0 \leq \alpha < 1$. Then*

$$[F_D, S]\langle D \rangle^\beta$$

is bounded for $\beta < 1 - \alpha$.

Theorem 2.20 allows us to study additive perturbations in a more-or-less optimal way, and the following two results can be compared to [CP98, Lemmas B.6–7].

Proposition 2.21. *Let D_0 and D_1 be self-adjoint regular operators on right Hilbert B -modules E_0 and E_1 . Suppose that there is an operator $a \in \operatorname{Hom}_B^*(E_0, E_1)$ such that $a \operatorname{dom} D_0 \subseteq \operatorname{dom} D_1$ and*

$$(D_1 a - a D_0)\langle D_0 \rangle^{-\alpha}$$

extends to an adjointable operator for some $0 \leq \alpha < 1$. Then, fixing $\beta < 1 - \alpha$,

$$(F_{D_1} a - a F_{D_0})\langle D_0 \rangle^\beta$$

is bounded.

Proof. Consider the operators

$$D = \begin{pmatrix} D_0 & \\ & D_1 \end{pmatrix} \quad S = \begin{pmatrix} & 0 \\ a & \end{pmatrix}$$

on $E_0 \oplus E_1$. Then

$$S \operatorname{dom} D = \begin{pmatrix} 0 \\ a \operatorname{dom} D_0 \end{pmatrix} \subseteq \begin{pmatrix} \operatorname{dom} D_0 \\ \operatorname{dom} D_1 \end{pmatrix} = \operatorname{dom} D$$

and

$$[D, S]\langle D \rangle^{-\alpha} = \begin{pmatrix} & 0 \\ (D_1 a - a D_0)\langle D_0 \rangle^{-\alpha} & \end{pmatrix}.$$

By Theorem 2.20,

$$[F_D, S]\langle D \rangle^\beta = \begin{pmatrix} & 0 \\ (F_{D_1} a - a F_{D_0})\langle D_0 \rangle^\beta & \end{pmatrix}$$

is bounded for $\beta < 1 - \alpha$, as required. $\square$

Corollary 2.22. *Let D_0 and D_1 be self-adjoint regular operators on a right Hilbert B -module E with densely intersecting domains. Suppose that there is a bounded operator a such that $a \operatorname{dom} D_0 \subseteq \operatorname{dom} D_0 \cap \operatorname{dom} D_1$ and*

$$(D_1 - D_0)a\langle D_0 \rangle^{-\alpha} \quad [D_0, a]\langle D_0 \rangle^{-\alpha}$$

extend to bounded operators for some $0 \leq \alpha < 1$. Then, fixing $\beta < 1 - \alpha$,

$$(F_{D_1} - F_{D_0})a\langle D_0 \rangle^\beta$$

is bounded.

Proof. We have

$$(D_1 a - a D_0)\langle D_0 \rangle^{-\alpha} = (D_1 - D_0)a\langle D_0 \rangle^{-\alpha} + [D_0, a]\langle D_0 \rangle^{-\alpha}$$

and

$$(F_{D_1} a - a F_{D_0})\langle D_0 \rangle^\beta = (F_{D_1} - F_{D_0})a\langle D_0 \rangle^\beta + [F_{D_0}, a]\langle D_0 \rangle^\beta.$$

By Theorem 2.20, $[F_{D_0}, a]\langle D_0 \rangle^\beta$ is bounded, so $(F_{D_1} - F_{D_0})a\langle D_0 \rangle^\beta$ is also, as required. $\square$

The chief subtlety in using the integral formula (2.19) to study the bounded transform for an unbounded Kasparov module (A, E_B, D) is the commutator $(\lambda + 1 + D^2)^{-1}a - a(\lambda + 1 + D^2)^{-1}$ for $a \in A$, [CP98, Lemma 2.3]. For us, the analogous computation is still the heart of the matter, see Lemma 2.31, but our techniques are different and described next.

Lemma 2.23. *Let A and B be regular operators on E . If B is a symmetric operator, then so is ABA^* , provided that the domain*

$$\text{dom}(ABA^*) = \{x \in \text{dom } A^* \mid A^*x \in \text{dom } B, BA^*x \in \text{dom } A\}$$

is dense. If A is bounded and invertible then ABA^ is regular. If moreover B is self-adjoint then ABA^* is self-adjoint.*

Proof. Given $x, y \in \text{dom}(ABA^*)$, $x, y \in \text{dom}(A^*)$ and $A^*y \in \text{dom}(B)$, the symmetry of B gives

$$\langle ABA^*x | y \rangle = \langle BA^*x | A^*y \rangle = \langle A^*x | BA^*y \rangle = \langle x | ABA^*y \rangle$$

so ABA^* is symmetric. If A is bounded and invertible, [Wor91, §2, Example 2] shows that AB is regular and, by [Wor91, §2, Example 3], ABA^* is regular. Applying the definition of the domain of the adjoint, one readily sees that $\text{dom}((ABA^*)^*) = \text{dom}(ABA^*) = A^{-1*} \text{dom}(B)$. $\square$

In the second statement of Lemma 2.23, the invertibility of A can be relaxed given additional assumptions [Kaa17, §6]. We will consider perturbations of the form $D \rightsquigarrow \mu D \mu^*$ for a self-adjoint regular operator D and an invertible, adjointable operator μ . The following bound is the result of a relation between the domains of fractional powers of $\langle D \rangle$ and $\langle \mu D \mu^* \rangle$, using Theorem A.3 of Appendix A.1.

Lemma 2.24. *Let D be a self-adjoint regular operator and μ an invertible adjointable operator. For all $0 < \alpha \leq 1$ we have*

$$\text{dom}(\mu \langle D \rangle^\alpha \mu^*) = \text{dom}((\mu \langle D \rangle \mu^*)^\alpha) = \text{dom} \langle \mu D \mu^* \rangle^\alpha$$

and the inequalities

$$\|\langle D \rangle^\alpha \mu^* (\mu \langle D \rangle \mu^*)^{-\alpha}\| \leq \|\mu^{-1}\|^\alpha \|\mu\|^{1-\alpha} \quad \left\| (\mu \langle D \rangle \mu^*)^\alpha \mu^{-1*} \langle D \rangle^{-\alpha} \right\| \leq \|\mu\|^\alpha \|\mu^{-1}\|^{1-\alpha}.$$

Proof. The domain statement follows from Theorem A.3. For the first inequality, in the context of Theorem A.3, let $A = \langle D \rangle$ and $B = \mu \langle D \rangle \mu^*$ so that $\mu^* \text{dom } B = \text{dom } A$. We have

$$\begin{aligned} \|\langle D \rangle^\alpha \mu^* (\mu \langle D \rangle \mu^*)^{-\alpha}\| &= \|A^\alpha \mu^* B^{-\alpha}\| \leq \|A \mu^* B^{-1}\|^\alpha \|\mu^*\|^{1-\alpha} \\ &= \left\| \langle D \rangle \mu^* (\mu \langle D \rangle \mu^*)^{-1} \right\|^\alpha \|\mu^*\|^{1-\alpha} \\ &= \|\mu^{-1}\|^\alpha \|\mu\|^{1-\alpha}. \end{aligned}$$

For the second, in the context of Theorem A.3, let $A = \mu \langle D \rangle \mu^*$ and $B = \langle D \rangle$, so that $\mu^{-1*} \text{dom } B = \text{dom } A$. We obtain that

$$\begin{aligned} \left\| (\mu \langle D \rangle \mu^*)^\alpha \mu^{-1*} \langle D \rangle^{-\alpha} \right\| &= \|A^\alpha \mu^{-1*} B^{-\alpha}\| \leq \|A \mu^{-1*} B^{-1}\|^\alpha \|\mu^{-1*}\|^{1-\alpha} \\ &= \left\| (\mu \langle D \rangle \mu^*) \mu^{-1*} \langle D \rangle^{-1} \right\|^\alpha \|\mu^{-1}\|^{1-\alpha} \\ &= \|\mu\|^\alpha \|\mu^{-1}\|^{1-\alpha} \end{aligned}$$

as required. $\square$

We recall tools ensuring convergence of regular self-adjoint operators on a Hilbert module E_B .

Theorem 2.25. [WN92, §1] Let T be a normal regular operator on E and $f \in C_b(\sigma(T))$. Let $(f_n)_{n \in \mathbb{N}} \subseteq C_b(\sigma(T))$ be a sequence of functions with common bound which converge to f uniformly on compact subsets. Then $f_n(T)$ converges to $f(T)$ as $n \rightarrow \infty$ in the strict topology on $M(\text{End}^0(E))$, and hence in the $*$ -strong topology on $\text{End}^*(E)$.

For the final statement, recall that the strict topology on $M(\text{End}^0(E)) = \text{End}^*(E)$ agrees with the $*$ -strong topology on norm-bounded subsets [RW98, Proposition C.7].

The proofs of the following two Theorems are essentially unchanged from the Hilbert space case.

Theorem 2.26. cf. [RS80, Theorem VIII.25(a)], [dO09, Proposition 10.1.18] Let $\mathcal{C} \subseteq E$ be a core for a self-adjoint regular operator T on E . Let $(T_n)_{n \in \mathbb{N}}$ be a sequence of self-adjoint regular operators such that, for all $n \in \mathbb{N}$, $\mathcal{C} \subseteq \text{dom } T_n$ and, for all $\xi \in \mathcal{C}$, $T_n \xi$ converges to $T \xi$ as $n \rightarrow \infty$. Then T_n converges to T in the strong resolvent sense as $n \rightarrow \infty$.

Theorem 2.27. cf. [RS80, Theorem VIII.20(b)], [dO09, Proposition 10.1.9] A sequence $(T_n)_{n \in \mathbb{N}}$ of self-adjoint regular operators on E converges to a self-adjoint regular operator T in the strong resolvent sense if and only if, for all $f \in C_b(\mathbb{R})$, $f(T_n)$ converges strongly to $f(T)$ as $n \rightarrow \infty$.

Let $(\varphi_n)_{n \in \mathbb{N}} \subset C_c(\mathbb{R})$ be a sequence of positive functions, bounded by 1 and converging uniformly on compact subsets to the constant function 1. Let D be a self-adjoint regular operator. By Theorem 2.25, the bounded operators $(\varphi_n(D))_{n \in \mathbb{N}}$ converge $*$ -strongly to 1. We will consider the bounded operators $d_n = D\varphi_n(D)$. On an element $\xi \in \text{dom } D$,

$$d_n \xi = D\varphi_n(D)\xi = \varphi_n(D)(D\xi) \rightarrow D\xi.$$

In particular, by Theorem 2.26, $d_n \rightarrow D$ in the strong resolvent sense. By Theorem 2.27, F_{d_n} converges strongly to F_D as $n \rightarrow \infty$.

Proposition 2.28. Let D be a self-adjoint regular operator and μ an invertible adjointable operator. Then $\mu d_n \mu^*$ converges to $\mu D \mu^*$ in the strong resolvent sense as $n \rightarrow \infty$. Furthermore, $\mu \langle d_n \rangle \mu^*$ converges to $\mu \langle D \rangle \mu^*$ in the strong resolvent sense.

Let a be a bounded operator such that $a \text{ dom } D \subseteq \text{dom } D$. With $a_n = \varphi_n(D)a\varphi_n(D)$, we find that $d_n a_n \langle d_n \rangle^{-1}$ converges strongly to $Da \langle D \rangle^{-1}$ as $n \rightarrow \infty$. In consequence, $[d_n, a_n] \langle d_n \rangle^{-1}$ converges strongly to $[D, a] \langle D \rangle^{-1}$.

Proof. First, apply Theorem 2.26 to the self-adjoint regular operator $\mu D \mu^*$ and the sequence $(\mu d_n \mu^*)_{n \in \mathbb{N}}$ of bounded operators. Noting that $\text{dom}(\mu D \mu^*) = \mu^{-1*} \text{dom } D$, on an element $\mu^{-1*} \xi \in \text{dom}(\mu D \mu^*)$,

$$(\mu d_n \mu^*) \mu^{-1*} \xi = \mu d_n \xi \rightarrow \mu D \xi$$

as $n \rightarrow \infty$. Hence, $\mu d_n \mu^*$ converges to $\mu D \mu^*$ in the strong resolvent sense. On an element $\xi \in \text{dom } D$,

$$\langle d_n \rangle \xi = (1 + (D\varphi_n(D))^2)^{1/2} \xi = (1 + (D\varphi_n(D))^2)^{1/2} \langle D \rangle^{-1} (\langle D \rangle \xi).$$

The function

$$x \mapsto \frac{(1 + (x\varphi_n(x))^2)^{1/2}}{(1 + x^2)^{1/2}} = \left(1 - \frac{1 - \varphi_n(x)^2}{1 + x^{-2}}\right)^{1/2}$$

is bounded above by 1 and below by φ_n and so converges to 1 on compact subsets. Applying Theorem 2.25,

$$\langle d_n \rangle \xi = (1 + (D\varphi_n(D))^2)^{1/2} \langle D \rangle^{-1} (\langle D \rangle \xi) \rightarrow \langle D \rangle \xi$$

and we proceed as before. For the second statement we have

$$d_n a_n \langle d_n \rangle^{-1} = D\varphi_n(D)^2 a \varphi_n(D) \langle D\varphi_n(D) \rangle^{-1} = \varphi_n(D)^2 \left(Da \langle D \rangle^{-1} \right) \langle D \rangle \varphi_n(D) \langle D\varphi_n(D) \rangle^{-1}.$$

The function

$$x \mapsto \frac{(1+x^2)^{1/2}\varphi_n(x)}{(1+(x\varphi_n(x))^2)^{1/2}} = \left(1 - \frac{1-\varphi_n(x)^2}{1+x^2\varphi_n(x)^2}\right)^{1/2}$$

is bounded above by 1 and below by φ_n and so converges to 1 on compact subsets. Applying Theorem 2.25,

$$d_n a_n \langle d_n \rangle^{-1} = \varphi_n(D)^2 \left(Da \langle D \rangle^{-1} \right) \langle D \rangle \varphi_n(D) \langle D \varphi_n(D) \rangle^{-1} \rightarrow Da \langle D \rangle^{-1}$$

strongly, as $n \rightarrow \infty$. For the second part,

$$[d_n, a_n] \langle d_n \rangle^{-1} = d_n a_n \langle d_n \rangle^{-1} - a_n F_{d_n} \rightarrow Da \langle D \rangle^{-1} - a F_D$$

strongly, as required. $\square$

As an application, we prove an operator inequality needed for applications involving summability.

Proposition 2.29. *Let D be a self-adjoint regular operator on a Hilbert B -module E and μ an invertible adjointable operator on E . Then*

$$C^{-1} \mu^{-1*} (1 + D^2)^{-1} \mu^{-1} \leq (1 + (\mu D \mu^*)^2)^{-1} \leq C \mu^{-1*} (1 + D^2)^{-1} \mu^{-1}$$

where $C = \max\{\|\mu\|^2, \|\mu^{-1}\|^2\}$.

Hence if J is a hereditary ideal of $\text{End}^*(B)$, not necessarily closed, then $(1 + (\mu D \mu^*)^2)^{-1} \in J$ if and only if $(1 + D^2)^{-1} \in J$. In particular, this applies if $B = \mathbb{C}$, so that E is a Hilbert space and J is any two-sided ideal of $B(E)$, not necessarily closed [Bla06, §II.5.2], such as Schatten ideals.

Proof. If $\mu^* \mu \text{ dom } D \subseteq \text{dom } D$, we could proceed more straightforwardly. As we do not assume this, we will use the (bounded) operators $d_n = D \varphi_n(D)$ and Proposition 2.28 to write

$$\begin{aligned} 1 + (\mu d_n \mu^*)^2 &= 1 + \mu d_n \mu^* \mu d_n \mu^* \leq 1 + \|\mu\|^2 \mu d_n^2 \mu^* = \mu (\mu^{-1} \mu^{-1*} + \|\mu\|^2 d_n^2) \mu^* \\ &\leq \mu (\|\mu^{-1}\|^2 + \|\mu\|^2 d_n^2) \mu^* = \|\mu\|^2 \mu (\|\mu\|^{-2} \|\mu^{-1}\|^2 + d_n^2) \mu^* \\ &\leq \|\mu\|^2 \max\{1, \|\mu\|^{-2} \|\mu^{-1}\|^2\} \mu (1 + d_n^2) \mu^* = \max\{\|\mu\|^2, \|\mu^{-1}\|^2\} \mu (1 + d_n^2) \mu^*. \end{aligned}$$

Hence, $(1 + (\mu d_n \mu^*)^2)^{-1} \geq C^{-1} \mu^{-1*} (1 + d_n^2)^{-1} \mu^{-1}$, and by Theorem 2.27 and Proposition 2.28, $(1 + (\mu d_n \mu^*)^2)^{-1}$ converges strongly to $(1 + (\mu D \mu^*)^2)^{-1}$ and $(1 + d_n^2)^{-1}$ converges strongly to $(1 + D^2)^{-1}$ as $n \rightarrow \infty$. Thus,

$$C^{-1} \mu^{-1*} (1 + D^2)^{-1} \mu^{-1} \leq (1 + (\mu D \mu^*)^2)^{-1},$$

and similarly,

$$1 + (\mu d_n \mu^*)^2 \geq \min\{\|\mu\|^{-2}, \|\mu^{-1}\|^{-2}\} \mu (1 + d_n^2) \mu^*$$

and

$$(1 + (\mu D \mu^*)^2)^{-1} \leq C \mu^{-1*} (1 + D^2)^{-1} \mu^{-1}$$

as required. $\square$

We use the notation $\mathfrak{T}_{a,b}(x) = ax - xb$ for $a, b, x \in \text{End}^*(E)$. The following inequality controlling $\mathfrak{T}_{a,b}(x)$ is based on Stampfli [Sta70, Theorem 8]; see also Archbold [Arc78].

Lemma 2.30. *Let a and b be elements of a C^* -algebra A . Define the bounded linear operator*

$$\mathfrak{T}_{a,b} : A \rightarrow A \quad x \mapsto ax - xb.$$

If a and b are positive, then

$$\|\mathfrak{T}_{a,b}\| \leq \max\{\|a\| - \|b^{-1}\|^{-1}, \|b\| - \|a^{-1}\|^{-1}\}$$

where $\|a^{-1}\|^{-1}$ is considered to be zero if a is not invertible, and likewise for b .

Proof. Firstly, $\|\mathfrak{T}_{a,b}\| \leq \|a\| + \|b\|$. For any $\lambda \in \mathbb{C}$, $\mathfrak{T}_{a-\lambda,b-\lambda} = \mathfrak{T}_{a,b}$, so $\|\mathfrak{T}_{a,b}\| \leq \|a - \lambda\| + \|b - \lambda\|$. For any $\lambda_1, \lambda_2 \in \mathbb{C}$,

$$\|\mathfrak{T}_{a,b}\| \leq \|a - \lambda_1\| + \|b - \lambda_2\| + |\lambda_1 - \lambda_2|.$$

To obtain the required bound, let

$$\lambda_1 = \frac{1}{2}(\|a\| + \|a^{-1}\|^{-1}) \quad \lambda_2 = \frac{1}{2}(\|b\| + \|b^{-1}\|^{-1})$$

so that, because a and b are positive,

$$\|a - \lambda_1\| = \frac{1}{2}(\|a\| - \|a^{-1}\|^{-1}) \quad \|b - \lambda_2\| = \frac{1}{2}(\|b\| - \|b^{-1}\|^{-1}).$$

Then

$$\begin{aligned} \|\mathfrak{T}_{a,b}\| &\leq \frac{1}{2}(\|a\| - \|a^{-1}\|^{-1}) + \frac{1}{2}(\|b\| - \|b^{-1}\|^{-1}) + \left| \frac{1}{2}(\|a\| + \|a^{-1}\|^{-1}) - \frac{1}{2}(\|b\| + \|b^{-1}\|^{-1}) \right| \\ &= \frac{1}{2} \left((\|a\| - \|b^{-1}\|^{-1}) + (\|b\| - \|a^{-1}\|^{-1}) + \left| (\|a\| - \|b^{-1}\|^{-1}) - (\|b\| - \|a^{-1}\|^{-1}) \right| \right) \\ &= \max\{\|a\| - \|b^{-1}\|^{-1}, \|b\| - \|a^{-1}\|^{-1}\} \end{aligned}$$

as required. $\square$

It is proved in [Sta70, Theorem 8, Corollary 2] that, if A has a faithful irreducible representation, then there is an equality

$$\|\mathfrak{T}_{a,b}\| = \inf_{\lambda \in \mathbb{C}} (\|a - \lambda\| + \|b - \lambda\|)$$

for any $a, b \in A$.

2.3 A multiplicative perturbation theory

The technical tool which allows us to extend the definitions of conformality and equivariance to unbounded Kasparov cycles is a multiplicative perturbation theory. This perturbation theory allows us to relate properties of an unbounded self-adjoint regular operator D and its bounded transform $F_D := D(1 + D^2)^{-1/2} = D\langle D \rangle^{-1}$ to conformally rescaled versions $D_1 = \mu D \mu^*$ and F_{D_1} .

Lemma 2.31. *Let D be a self-adjoint regular operator and μ an invertible adjointable operator on E . Let a be an adjointable operator such that $a\mu^{-1*} \operatorname{dom} D \subseteq \mu^{-1*} \operatorname{dom} D$. Then, with $D_1 = \mu D \mu^*$ and $D_2 = \mu \langle D \rangle \mu^*$, and for all $\lambda \geq 0$*

$$\begin{aligned} -(\lambda + \langle D_1 \rangle^2)^{-1} a + a(\lambda + D_2^2)^{-1} &= (\lambda + \langle D_1 \rangle^2)^{-1} a \mu \mathfrak{T}_{(\mu^* \mu)^{-1}, \mu^* \mu} (\langle D \rangle^{-1}) \mu^{-1} D_2 (\lambda + D_2^2)^{-1} \\ &\quad + D_1 (\lambda + \langle D_1 \rangle^2)^{-1} \left([D_1, a] D_2^{-1} - \mu^{-1*} [F_D, \mu^* a \mu] \mu^{-1} \right) D_2 (\lambda + D_2^2)^{-1} \\ &\quad + (\lambda + \langle D_1 \rangle^2)^{-1} \left(\mu F_D \mu^{-1} [D_1, a] D_2^{-1} + \mu [F_D, \mu^{-1} a \mu] F_D \mu^{-1} \right) D_2^2 (\lambda + D_2^2)^{-1} \end{aligned}$$

as everywhere-defined operators.

Proof. If $\mu^* \mu \operatorname{dom} D \subseteq \operatorname{dom} D$, we could proceed more straightforwardly. As we do not make this assumption, we use the approximation arguments of §2.2. Let $(\varphi_n)_{n \in \mathbb{N}} \subset C_c(\mathbb{R})$ be a sequence of positive functions, bounded by 1 and converging uniformly on compact subsets to the constant function 1. Let $d_n = D \varphi_n(D)$ and set

$$a_n = \mu^{-1*} \varphi_n(D) \mu^* a \mu^{-1*} \varphi_n(D) \mu^*.$$

Note for future reference that we may use the bounded transform $F_{d_n} = d_n \langle d_n \rangle^{-1}$ to write

$$\begin{aligned}
& [\mu d_n \mu^*, a_n] (\mu \langle d_n \rangle \mu^*)^{-1} \\
&= \mu d_n \varphi_n(D) \mu^* a \mu^{-1*} \varphi_n(D) \langle d_n \rangle^{-1} \mu^{-1} - \mu^{-1*} \varphi_n(D) \mu^* a \mu^{-1*} \varphi_n(D) \mu^* \mu F_{d_n} \mu^{-1} \\
&= \mu [d_n, \varphi_n(D) \mu^* a \mu^{-1*} \varphi_n(D)] \langle d_n \rangle^{-1} \mu^{-1} + \mu \varphi_n(D) \mu^* a \mu^{-1*} \varphi_n(D) F_{d_n} \mu^{-1} \\
&\quad - \mu^{-1*} \varphi_n(D) \mu^* a \mu^{-1*} \varphi_n(D) \mu^* \mu F_{d_n} \mu^{-1}
\end{aligned}$$

so that we will be in a position to apply Proposition 2.28 to the first term, while the other two are uniformly bounded in n . Because d_n is bounded, we may write

$$\begin{aligned}
& -(\lambda + \langle \mu d_n \mu^* \rangle^2)^{-1} a_n + a_n (\lambda + (\mu \langle d_n \rangle \mu^*)^2)^{-1} \\
&= (\lambda + \langle \mu d_n \mu^* \rangle^2)^{-1} \left(-a_n (\lambda + (\mu \langle d_n \rangle \mu^*)^2) + (\lambda + \langle \mu d_n \mu^* \rangle^2) a_n \right) (\lambda + (\mu \langle d_n \rangle \mu^*)^2)^{-1} \\
&= (\lambda + \langle \mu d_n \mu^* \rangle^2)^{-1} \left(-a_n (\mu \langle d_n \rangle \mu^*)^2 + \langle \mu d_n \mu^* \rangle^2 a_n \right) (\lambda + (\mu \langle d_n \rangle \mu^*)^2)^{-1}. \tag{2.32}
\end{aligned}$$

Expanding the middle factor and using the identity $F_{d_n} d_n - \langle d_n \rangle = -\langle d_n \rangle^{-1}$ yields

$$\begin{aligned}
& \langle \mu d_n \mu^* \rangle^2 a_n - a_n (\mu \langle d_n \rangle \mu^*)^2 \\
&= a_n + \mu d_n \mu^* \mu d_n \mu^* a_n - a_n \mu \langle d_n \rangle \mu^* \mu \langle d_n \rangle \mu^* \\
&= a_n + \mu d_n \mu^* [\mu d_n \mu^*, a_n] + \mu d_n \mu^* a_n \mu d_n \mu^* - a_n \mu \langle d_n \rangle \mu^* \mu \langle d_n \rangle \mu^* \\
&= a_n + \mu d_n \mu^* [\mu d_n \mu^*, a_n] - \mu d_n [F_{d_n}, \mu^* a_n \mu] \langle d_n \rangle \mu^* + \mu d_n F_{d_n} \mu^* a_n \mu \langle d_n \rangle \mu^* \\
&\quad - a_n \mu \langle d_n \rangle \mu^* \mu \langle d_n \rangle \mu^* \\
&= a_n + \mu d_n \mu^* [\mu d_n \mu^*, a_n] - \mu d_n [F_{d_n}, \mu^* a_n \mu] \langle d_n \rangle \mu^* + \mu F_{d_n} \mu^{-1} \mu d_n \mu^* a_n \mu \langle d_n \rangle \mu^* \\
&\quad - a_n \mu \langle d_n \rangle \mu^* \mu \langle d_n \rangle \mu^* \\
&= a_n + \mu d_n \mu^* [\mu d_n \mu^*, a_n] - \mu d_n [F_{d_n}, \mu^* a_n \mu] \langle d_n \rangle \mu^* + \mu F_{d_n} \mu^{-1} [\mu d_n \mu^*, a_n] \mu \langle d_n \rangle \mu^* \\
&\quad + \mu F_{d_n} \mu^{-1} a_n \mu d_n \mu^* \mu \langle d_n \rangle \mu^* - a_n \mu \langle d_n \rangle \mu^* \mu \langle d_n \rangle \mu^* \\
&= a_n + \mu d_n \mu^* [\mu d_n \mu^*, a_n] - \mu d_n [F_{d_n}, \mu^* a_n \mu] \langle d_n \rangle \mu^* + \mu F_{d_n} \mu^{-1} [\mu d_n \mu^*, a_n] \mu \langle d_n \rangle \mu^* \\
&\quad + \mu [F_{d_n}, \mu^{-1} a_n \mu] d_n \mu^* \mu \langle d_n \rangle \mu^* + a_n \mu (F_{d_n} d_n - \langle d_n \rangle) \mu^* \mu \langle d_n \rangle \mu^* \\
&= a_n + \mu d_n \mu^* [\mu d_n \mu^*, a_n] - \mu d_n [F_{d_n}, \mu^* a_n \mu] \langle d_n \rangle \mu^* + \mu F_{d_n} \mu^{-1} [\mu d_n \mu^*, a_n] \mu \langle d_n \rangle \mu^* \\
&\quad + \mu [F_{d_n}, \mu^{-1} a_n \mu] d_n \mu^* \mu \langle d_n \rangle \mu^* - a_n \mu \langle d_n \rangle^{-1} \mu^* \mu \langle d_n \rangle \mu^* \\
&= a_n \mu \mathfrak{T}_{(\mu^* \mu)^{-1}, \mu^* \mu} (\langle d_n \rangle^{-1}) \mu^{-1} (\mu \langle d_n \rangle \mu^*) \\
&\quad + (\mu d_n \mu^*) \left([\mu d_n \mu^*, a_n] (\mu \langle d_n \rangle \mu^*)^{-1} - \mu^{-1*} [F_{d_n}, \mu^* a_n \mu] \mu^{-1} \right) (\mu \langle d_n \rangle \mu^*) \\
&\quad + \left(\mu F_{d_n} \mu^{-1} [\mu d_n \mu^*, a_n] (\mu \langle d_n \rangle \mu^*)^{-1} + \mu [F_{d_n}, \mu^{-1} a_n \mu] F_{d_n} \mu^{-1} \right) (\mu \langle d_n \rangle \mu^*)^2
\end{aligned}$$

since $\mathfrak{T}_{\mu^{-1} \mu^{-1*}, \mu^* \mu} (\langle d_n \rangle^{-1}) = \mu^{-1} \mu^{-1*} \langle d_n \rangle^{-1} - \langle d_n \rangle^{-1} \mu^* \mu$. Substituting into (2.32) yields

$$\begin{aligned}
& -(\lambda + \langle \mu d_n \mu^* \rangle^2)^{-1} a_n + a_n (\lambda + (\mu \langle d_n \rangle \mu^*)^2)^{-1} \\
&= (\lambda + \langle \mu d_n \mu^* \rangle^2)^{-1} a_n \mu \mathfrak{T}_{(\mu^* \mu)^{-1}, \mu^* \mu} (\langle d_n \rangle^{-1}) \mu^{-1} (\mu \langle d_n \rangle \mu^*) (\lambda + (\mu \langle d_n \rangle \mu^*)^2)^{-1} \\
&\quad + (\mu d_n \mu^*) (\lambda + \langle \mu d_n \mu^* \rangle^2)^{-1} \left([\mu d_n \mu^*, a_n] (\mu \langle d_n \rangle \mu^*)^{-1} - \mu^{-1*} [F_{d_n}, \mu^* a_n \mu] \mu^{-1} \right) \\
&\quad \times (\mu \langle d_n \rangle \mu^*) (\lambda + (\mu \langle d_n \rangle \mu^*)^2)^{-1} \\
&\quad + (\lambda + \langle \mu d_n \mu^* \rangle^2)^{-1} \left(\mu F_{d_n} \mu^{-1} [\mu d_n \mu^*, a_n] (\mu \langle d_n \rangle \mu^*)^{-1} + \mu [F_{d_n}, \mu^{-1} a_n \mu] F_{d_n} \mu^{-1} \right) \\
&\quad \times (\mu \langle d_n \rangle \mu^*)^2 (\lambda + (\mu \langle d_n \rangle \mu^*)^2)^{-1}. \tag{2.33}
\end{aligned}$$

By Proposition 2.28, the right-hand side of (2.33) converges strongly to

$$\begin{aligned}
& (\lambda + \langle \mu D \mu^* \rangle^2)^{-1} a \mu \mathfrak{T}_{(\mu^* \mu)^{-1}, \mu^* \mu} (\langle D \rangle^{-1}) \mu^{-1} (\mu \langle D \rangle \mu^*) (\lambda + (\mu \langle D \rangle \mu^*)^2)^{-1} \\
& + (\mu D \mu^*) (\lambda + \langle \mu D \mu^* \rangle^2)^{-1} \left([\mu D \mu^*, a] (\mu \langle D \rangle \mu^*)^{-1} - \mu^{-1*} [F_D, \mu^* a \mu] \mu^{-1} \right) \\
& \quad \times (\mu \langle D \rangle \mu^*) (\lambda + (\mu \langle D \rangle \mu^*)^2)^{-1} \\
& + (\lambda + \langle \mu D \mu^* \rangle^2)^{-1} \left(\mu F_D \mu^{-1} [\mu D \mu^*, a] (\mu \langle D \rangle \mu^*)^{-1} + \mu [F_D, \mu^{-1} a \mu] F_D \mu^{-1} \right) \\
& \quad \times (\mu \langle D \rangle \mu^*)^2 (\lambda + (\mu \langle D \rangle \mu^*)^2)^{-1}
\end{aligned}$$

and we obtain the required equality of everywhere-defined operators. $\square$

Lemma 2.34. *Let D be a self-adjoint regular operator and μ an invertible, adjointable operator on E . Let a be an adjointable operator such that $a \mu^{-1*} \text{dom } D \subseteq \mu^{-1*} \text{dom } D$. Suppose further that, for some $0 \leq \alpha < 1$,*

$$[F_D, \mu^* a \mu] \langle D \rangle^{1-\alpha} \quad [F_D, \mu^{-1} a \mu] \langle D \rangle^{1-\alpha} \quad [\mu D \mu^*, a] \mu^{-1*} \langle D \rangle^{-\alpha}$$

are bounded. Then, with $D_1 = \mu D \mu^*$ and $D_2 = \mu \langle D \rangle \mu^*$, for $\lambda \geq 0$ and $\beta \leq 1 - \alpha$

$$\left\| D_1 \left((\lambda + \langle D_1 \rangle^2)^{-1} a - a (\lambda + D_2^2)^{-1} \right) \mu \langle D \rangle^\beta \right\| \leq c_1 (\lambda + c_0)^{-1+(\alpha+\beta)/2}$$

where $c_0 = \min\{1, \|\mu^{-1}\|^{-4}\}$ and $c_1 \geq 0$ is independent of λ .

Proof. First, by Lemma 2.24, $\|D_2^{-\beta} \mu \langle D \rangle^\beta\| = \|\langle D \rangle^\beta \mu^* D_2^{-\beta}\| \leq \|\mu^{-1}\|^\beta \|\mu\|^{1-\beta}$ so

$$\begin{aligned}
& \left\| D_1 \left((\lambda + \langle D_1 \rangle^2)^{-1} a - a (\lambda + D_2^2)^{-1} \right) \mu \langle D \rangle^\beta \right\| \\
& \leq \left\| D_1 \left((\lambda + \langle D_1 \rangle^2)^{-1} a - a (\lambda + D_2^2)^{-1} \right) D_2^\beta \right\| \|\mu^{-1}\|^\beta \|\mu\|^{1-\beta},
\end{aligned}$$

By Lemma 2.30, $\|\mathfrak{T}_{(\mu^* \mu)^{-1}, (\mu^* \mu)}\| \leq \max\{\|\mu^{-1}\|^2 - \|\mu^{-1}\|^{-2}, \|\mu\|^2 - \|\mu\|^{-2}\}$. We compute that

$$\begin{aligned}
& \left\| D_1 \left((\lambda + \langle D_1 \rangle^2)^{-1} a - a (\lambda + D_2^2)^{-1} \right) D_2^\beta \right\| \\
& \leq \left\| D_1 (\lambda + \langle D_1 \rangle^2)^{-1} a \mu \mathfrak{T}_{(\mu^* \mu)^{-1}, \mu^* \mu} (\langle D \rangle^{-1}) \mu^{-1} D_2^{1+\beta} (\lambda + D_2^2)^{-1} \right\| \\
& \quad + \left\| D_1^2 (\lambda + \langle D_1 \rangle^2)^{-1} \left([D_1, a] \mu^{-1*} \langle D \rangle^{-\alpha} - \mu^{-1*} [F_D, \mu^* a \mu] \langle D \rangle^{1-\alpha} \right) \times \right. \\
& \quad \quad \times \langle D \rangle^\alpha \mu^* D_2^{-\alpha} D_2^{\alpha+\beta} (\lambda + D_2^2)^{-1} \left. \right\| \\
& \quad + \left\| D_1 (\lambda + \langle D_1 \rangle^2)^{-1} \left(\mu F_D \mu^{-1} [D_1, a] \mu^{-1*} \langle D \rangle^{-\alpha} + \mu [F_D, \mu^{-1} a \mu] \langle D \rangle^{1-\alpha} F_D^2 \right) \right. \\
& \quad \quad \times \langle D \rangle^\alpha \mu^* D_2^{-\alpha} D_2^{1+\alpha+\beta} (\lambda + D_2^2)^{-1} \left. \right\| \\
& \leq \left\| D_1 (\lambda + \langle D_1 \rangle^2)^{-1} \right\| \|a\| \|\mu\| \left\| \mathfrak{T}_{(\mu^* \mu)^{-1}, \mu^* \mu} (\langle D \rangle^{-1}) \right\| \|\mu^{-1}\| \left\| D_2^{1+\beta} (\lambda + D_2^2)^{-1} \right\| \\
& \quad + \left\| D_1^2 (\lambda + \langle D_1 \rangle^2)^{-1} \right\| \left(\left\| [D_1, a] \mu^{-1*} \langle D \rangle^{-\alpha} \right\| + \left\| \mu^{-1*} [F_D, \mu^* a \mu] \langle D \rangle^{1-\alpha} \right\| \right) \\
& \quad \quad \times \left\| \langle D \rangle^\alpha \mu^* D_2^{-\alpha} \right\| \left\| D_2^{\alpha+\beta} (\lambda + D_2^2)^{-1} \right\| \\
& \quad + \left\| D_1 (\lambda + \langle D_1 \rangle^2)^{-1} \right\| \left(\|\mu\| \|\mu^{-1}\| \left\| [D_1, a] \mu^{-1*} \langle D \rangle^{-\alpha} \right\| + \|\mu\| \left\| [F_D, \mu^{-1} a \mu] \langle D \rangle^{1-\alpha} \right\| \right) \\
& \quad \quad \times \left\| \langle D \rangle^\alpha \mu^* D_2^{-\alpha} \right\| \left\| D_2^{1+\alpha+\beta} (\lambda + D_2^2)^{-1} \right\|
\end{aligned}$$

$$\begin{aligned}
&\leq (\lambda + 1)^{-1/2} \|a\| \|\mu\| \max\{\|\mu^{-1}\|^2 - \|\mu^{-1}\|^{-2}, \|\mu\|^2 - \|\mu\|^{-2}\} \|\mu^{-1}\| (\lambda + \|\mu^{-1}\|^{-4})^{(-1+\beta)/2} \\
&\quad + \left(\left\| [D_1, a] \mu^{-1*} \langle D \rangle^{-\alpha} \right\| - \|\mu^{-1}\| \left\| [F_D, \mu^* a \mu] \langle D \rangle^{1-\alpha} \right\| \right) \\
&\quad \times \|\mu^{-1}\|^\alpha \|\mu\|^{1-\alpha} (\lambda + \|\mu^{-1}\|^{-4})^{(-2+\alpha+\beta)/2} \\
&\quad + (\lambda + 1)^{-1/2} \left(\|\mu\| \|\mu^{-1}\| \left\| [D_1, a] \mu^{-1*} \langle D \rangle^{-\alpha} \right\| + \|\mu\| \left\| [F_D, \mu^{-1} a \mu] \langle D \rangle^{1-\alpha} \right\| \right) \\
&\quad \times \|\mu^{-1}\|^\alpha \|\mu\|^{1-\alpha} (\lambda + \|\mu^{-1}\|^{-4})^{(-1+\alpha+\beta)/2} \\
&\leq c'_1 (\lambda + c_0)^{-1+(\alpha+\beta)/2}
\end{aligned}$$

where $c_0 = \min\{1, \|\mu^{-1}\|^{-4}\}$ and $c'_1 \geq 0$ is a constant independent of λ . Hence,

$$\left\| D_1 \left((\lambda + \langle D_1 \rangle^2)^{-1} a - a(\lambda + D_2^2)^{-1} \right) \mu \langle D \rangle^\beta \right\| \leq c_1 (\lambda + c_0)^{-1+(\alpha+\beta)/2}$$

for $c_1 = c'_1 \|\mu^{-1}\|^\beta \|\mu\|^{1-\beta}$. □

Lemma 2.35. *Let D be a self-adjoint regular operator and μ an invertible adjointable operator on E . Let a be an adjointable operator such that $a\mu^{-1*} \text{dom } D \subseteq \mu^{-1*} \text{dom } D$. Suppose further that, for some $0 \leq \alpha < 1$,*

$$[F_D, \mu^* a \mu] \langle D \rangle^{1-\alpha} \quad [F_D, \mu^{-1} a \mu] \langle D \rangle^{1-\alpha} \quad [\mu D \mu^*, a] \mu^{-1*} \langle D \rangle^{-\alpha}$$

are bounded. Then, with $D_1 = \mu D \mu^$ and $D_2 = \mu \langle D \rangle \mu^*$,*

$$D_1 \left(\langle D_1 \rangle^{-1} a - a D_2^{-1} \right) \mu \langle D \rangle^\beta$$

is bounded for $\beta < 1 - \alpha$.

Proof. Using the integral formula (2.19),

$$D_1 \left(\langle D_1 \rangle^{-1} a - a D_2^{-1} \right) \mu \langle D \rangle^\beta = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} D_1 \left((\lambda + \langle D_1 \rangle^2)^{-1} a - a(\lambda + D_2^2)^{-1} \right) \mu \langle D \rangle^\beta d\lambda.$$

By Proposition 2.34, the integrand is bounded and the integral is norm convergent when

$$\int_0^\infty \lambda^{-1/2} (\lambda + c_0)^{-1+(\alpha+\beta)/2} d\lambda$$

is convergent, that is, when $\beta < 1 - \alpha$. □

Theorem 2.36. *Let D_0 be a self-adjoint regular operator and μ an invertible adjointable operator on E . Let a be an adjointable operator such that $a\mu^{-1*} \text{dom } D_0 \subseteq \mu^{-1*} \text{dom } D_0$. Suppose further that, for some $0 \leq \alpha < 1$,*

$$[F_{D_0}, \mu^* a \mu] \langle D_0 \rangle^{1-\alpha} \quad [F_{D_0}, \mu^{-1} a \mu] \langle D_0 \rangle^{1-\alpha} \quad [F_{D_0}, a \mu] \langle D_0 \rangle^{1-\alpha} \quad [\mu D_0 \mu^*, a] \mu^{-1*} \langle D_0 \rangle^{-\alpha}$$

are bounded. Then, with $D_1 = \mu D_0 \mu^$, the operator*

$$(F_{D_1} - F_{D_0}) a \mu \langle D_0 \rangle^\beta$$

is bounded for $\beta < 1 - \alpha$.

Proof. We have

$$\begin{aligned}
(F_{D_1} - F_{D_0})a\mu &= F_{D_1}a\mu - a\mu F_{D_0} - [F_{D_0}, a\mu] \\
&= F_{D_1}a\mu - aD_1\mu^{-1*}\langle D_0 \rangle^{-1} - [F_{D_0}, a\mu] \\
&= F_{D_1}a\mu - D_1a\mu^{-1*}\langle D_0 \rangle^{-1} + [D_1, a]\mu^{-1*}\langle D_0 \rangle^{-1} - [F_{D_0}, a\mu] \\
&= D_1 \left(\langle D_1 \rangle^{-1}a - a(\mu\langle D_0 \rangle\mu^*)^{-1} \right) \mu + [D_1, a]\mu^{-1*}\langle D_0 \rangle^{-1} - [F_{D_0}, a\mu].
\end{aligned}$$

Multiplying on the right by $\langle D \rangle^\beta$, the first term remains bounded by Lemma 2.35. The remaining two terms are bounded owing to the last two of our displayed assumptions. $\square$

Theorem 2.37. *Let D be a self-adjoint regular operator and μ an invertible adjointable operator on E . Let a be an adjointable operator such that $\{\mu^*a\mu, \mu^{-1}a\mu, a\mu, \mu^*a\mu^{-1*}\} \text{dom } D \subseteq \mu^{-1*} \text{dom } D$. Suppose further that, for some $0 \leq \alpha < 1$,*

$$[D, \mu^*a\mu]\langle D \rangle^{-\alpha} \quad [D, \mu^{-1}a\mu]\langle D \rangle^{-\alpha} \quad [D, a\mu]\langle D \rangle^{-\alpha} \quad [\mu D\mu^*, a]\mu^{-1*}\langle D \rangle^{-\alpha}$$

are bounded. Then, with $D_1 = \mu D\mu^$,*

$$(F_{D_1} - F_D)a\mu\langle D \rangle^\beta$$

is bounded for $\beta < 1 - \alpha$. If b is an adjointable operator such that $b^\mu^{-1*} \text{dom } D \subseteq \text{dom } D$, then $(F_{D_1} - F_D)ab\langle D \rangle^\beta$ is bounded. If c is a bounded operator such that $(1 + D^2)^{-1}c$ is compact, then $(F_{D_1} - F_D)abc$ is compact.*

Proof. Applying Theorem 2.20, we find that

$$[F_D, \mu^*a\mu]\langle D \rangle^{1-\gamma} \quad [F_D, \mu^{-1}a\mu]\langle D \rangle^{1-\gamma} \quad [F_{D_0}, a\mu]\langle D \rangle^{1-\gamma}$$

are bounded for $\gamma > \alpha$. Then, by Theorem 2.36, $(F_{D_1} - F_D)a\mu\langle D \rangle^\beta$ is bounded for all $\beta < 1 - \gamma$, and so for all $\beta < 1 - \alpha$. The remaining statements follow immediately. $\square$

Remark 2.38. In Theorem 2.37, that $[\mu D\mu^*, a]\mu^{-1*}\langle D \rangle^{-\alpha}$ is bounded is equivalent to

$$\begin{aligned}
D\mu^{-1}[\mu\mu^*, a]\mu^{-1*}\langle D \rangle^{-\alpha} &= D(\mu^*a\mu^{-1*} - \mu^{-1}a\mu)\langle D \rangle^{-\alpha} \\
&= \mu^{-1}[\mu D\mu^*, a]\mu^{-1*}\langle D \rangle^{-\alpha} - [D, \mu^{-1}a\mu]\langle D \rangle^{-\alpha}
\end{aligned}$$

being bounded, using the assumption that $[D, \mu^{-1}a\mu]\langle D \rangle^{-\alpha}$ is bounded. In other words, that $\mu\mu^*$ and a almost commute.

Corollary 2.39. *Let D be a self-adjoint regular operator and μ an invertible adjointable operator on E . Suppose that, for some $0 \leq \alpha < 1$,*

$$[F_D, \mu]\langle D \rangle^{1-\alpha} \quad [F_D, \mu^*\mu]\langle D \rangle^{1-\alpha}$$

are bounded. Then, with $D_1 = \mu D\mu^$,*

$$(F_{D_1} - F_D)\mu\langle D \rangle^\beta$$

is bounded for $\beta < 1 - \alpha$. If $\mu^ \text{dom } D \subseteq \text{dom } D$, then $(F_{D_1} - F_D)\langle D \rangle^\beta$ is bounded.*

Corollary 2.40. *Let D be a self-adjoint regular operator and μ an invertible adjointable operator on E . Suppose that $\mu \text{dom } D \subseteq \text{dom } D$ and, for some $0 \leq \alpha < 1$,*

$$[D, \mu]\langle D \rangle^{-\alpha} \quad \langle D \rangle^{-\alpha}[D, \mu]$$

are bounded. Then, with $D_1 = \mu D\mu^$, the operator*

$$(F_{D_1} - F_D)\langle D \rangle^\beta$$

is bounded for $\beta < 1 - \alpha$.

Corollary 2.41. Let D_0 and D_1 be self-adjoint regular operators and μ an invertible adjointable operator on E . Suppose that $\mu \operatorname{dom} D_0 \subseteq \operatorname{dom} D_0$ and, for some $0 \leq \alpha < 1$,

$$(\mu^{-1}D_1\mu^{-1*} - D_0)\langle D_0 \rangle^{-\alpha} \quad [D_0, \mu]\langle D_0 \rangle^{-\alpha} \quad \langle D_0 \rangle^{-\alpha}[D_0, \mu]$$

are bounded. Then the operator

$$(F_{D_1} - F_{D_0})\langle D_0 \rangle^\beta$$

is bounded for $\beta < 1 - \alpha$.

Theorem 2.42. Let D be a self-adjoint regular operator and μ an invertible adjointable operator on E . Let a and b be adjointable operators such that $\{\mu^*a, \mu^{-1}a, a, b\mu, b\mu^{-1*}\} \operatorname{dom} D \subseteq \operatorname{dom} D$. Suppose further that, for some $0 \leq \alpha < 1$,

$$\langle D \rangle^{-\alpha}[D, a\mu] \quad \langle D \rangle^{-\alpha}[D, a\mu^{-1*}] \quad \langle D \rangle^{-\alpha}[D, a] \quad [D, b\mu]\langle D \rangle^{-\alpha} \quad [\mu D\mu^*, a^*b]\mu^{-1*}\langle D \rangle^{-\alpha}$$

are bounded. Then, with $D_1 = \mu D\mu^*$, the operator

$$(F_{D_1} - F_D)a^*b\mu\langle D \rangle^\beta$$

is bounded for $\beta < 1 - \alpha$. If c is an adjointable operator such that $c\mu^{-1*} \operatorname{dom} D \subseteq \operatorname{dom} D$, then $(F_{D_1} - F_D)a^*bc^*\langle D \rangle^\beta$ is bounded. If d is an adjointable operator such that $(1 + D^2)^{-1}d$ is compact, then $(F_{D_1} - F_D)a^*bc^*d$ is compact.

Proof. This follows from Theorem 2.37, using [GM15, Proposition A.5] for the appropriate Leibniz rule to relate the differing commutator conditions. $\square$

Now, returning to the concept of conformal transformation, we have:

Proof of Theorem 2.9. Let (U, μ) be a conformal transformation from (A, E_B, D_1) to (A, E'_B, D_2) . By Proposition 2.21 and Lemma 2.24,

$$(U^*F_{D_2}Ua - aF_{\mu D_1\mu^*})\mu\langle D_0 \rangle^\beta$$

is bounded for $a \in \mathcal{M}$. Let $b, c \in \mathcal{M}$ and consider the operators

$$D = \begin{pmatrix} U^*D_2U & \\ & \mu D_1\mu^* \end{pmatrix} \quad B = \begin{pmatrix} & b \\ 0 & \end{pmatrix} \quad C = \begin{pmatrix} & c \\ 0 & \end{pmatrix}$$

on $E \oplus E'$. By assumption and using Lemma 2.24,

$$[D, B]\langle D \rangle^{-\alpha} = \begin{pmatrix} (U^*D_2Ub - b\mu D_1\mu^*)\langle \mu D_1\mu^* \rangle^{-\alpha} \\ 0 \end{pmatrix} \quad \text{and} \quad [D, C]\langle D \rangle^{-\alpha}$$

are bounded. By [GM15, Proposition A.5],

$$[D, B^*C]\langle D \rangle^{-\alpha} = \begin{pmatrix} 0 \\ [\mu D_1\mu^*, b^*c]\langle \mu D_1\mu^* \rangle^{-\alpha} \end{pmatrix}$$

extends to an adjointable operator. Again using Lemma 2.24, $[\mu D_1\mu^*, b^*c]\mu^{-1*}\langle D_1 \rangle^{-\alpha}$ is bounded and we may apply Theorem 2.42 to obtain that

$$(F_{\mu D_1\mu^*} - F_{D_1})b^*c\mu\langle D_1 \rangle^\beta$$

is bounded for $\beta < 1 - \alpha$. Then

$$(U^*F_{D_2}U - F_{D_1})ab^*c = (U^*F_{D_2}Ua - aF_{\mu D_1\mu^*})b^*c - [F_{D_1}, a]b^*c + a(F_{\mu D_1\mu^*} - F_{D_1})b^*c$$

so that $(U^*F_{D_2}U - F_{D_1})ab^*c\mu\langle D_0 \rangle^\beta$ is bounded. For $d \in \mathcal{M}$ and $e \in A$ we find

$$(U^*F_{D_2}U - F_{D_1})ab^*cd^*e = (U^*F_{D_2}U - F_{D_1})a^*bc\mu\langle D_1 \rangle^\beta(\langle D_1 \rangle^{-\beta}\mu^{-1}d^*\langle D_1 \rangle^\beta)\langle D_1 \rangle^{-\beta}e$$

is compact. By the inclusion $A \subseteq \overline{\operatorname{span}}((\mathcal{M}^*\mathcal{M})^2A)$, we are done. $\square$

2.3.1 A partial converse

A partial converse result is possible, in the sense that these kinds of estimates on bounded transforms always arise from an additive and a multiplicative perturbation of the unbounded operator. This is not quite precise due to differences in the differentiability assumptions. The following is nearly a converse to Corollary 2.39.

Theorem 2.43. *Let D_1 and D_2 be self-adjoint regular operators with equal domains such that, for some $0 < \alpha \leq 1$,*

$$(F_{D_1} - F_{D_2})\langle D_1 \rangle^\alpha$$

is bounded on $\text{dom}\langle D_1 \rangle^\alpha$. Then there exist a bounded invertible operator μ and a self-adjoint regular operator T such that

$$D_2 = \mu D_1 \mu^* + T$$

and both

$$\langle D_1 \rangle^{-1/2} T \langle D_1 \rangle^{-1/2+\alpha} \quad \left([F_{D_1}, \mu] - T \langle D_2 \rangle^{-1} \right) \langle D_1 \rangle^\alpha$$

are bounded. Furthermore, if $1/2 \leq \alpha$,

$$T \langle D_1 \rangle^{-1+\alpha} \quad [F_{D_1}, \mu] \langle D_1 \rangle^\alpha$$

are bounded.

Proof. Let $\mu = \langle D_2 \rangle^{1/2} \langle D_1 \rangle^{-1/2}$ and $T = \langle D_2 \rangle^{1/2} (F_{D_2} - F_{D_1}) \langle D_2 \rangle^{1/2}$, defined on $\text{dom } D_1$, so that

$$\begin{aligned} \mu D_1 \mu^* + T &= \langle D_2 \rangle^{1/2} \langle D_1 \rangle^{-1/2} D_1 \langle D_1 \rangle^{-1/2} \langle D_2 \rangle^{1/2} + \langle D_2 \rangle^{1/2} (F_{D_2} - F_{D_1}) \langle D_2 \rangle^{1/2} \\ &= \langle D_2 \rangle^{1/2} (F_{D_1} + (F_{D_2} - F_{D_1})) \langle D_2 \rangle^{1/2} \\ &= D_2. \end{aligned}$$

We have

$$\begin{aligned} [F_{D_1}, \mu] &= \left(F_{D_1} \langle D_2 \rangle^{1/2} - \langle D_2 \rangle^{1/2} F_{D_1} \right) \langle D_1 \rangle^{-1/2} \\ &= \left(\langle D_2 \rangle^{1/2} (F_{D_2} - F_{D_1}) + (F_{D_2} - F_{D_1}) \langle D_2 \rangle^{1/2} \right) \langle D_1 \rangle^{-1/2} \\ &= \left(T \langle D_2 \rangle^{-1/2} + \langle D_2 \rangle^{-1/2} T \right) \langle D_1 \rangle^{-1/2} \\ &= T \langle D_2 \rangle^{-1} + (F_{D_2} - F_{D_1}). \end{aligned}$$

Because the domains of D_1 and D_2 are equal, $(F_{D_2} - F_{D_1}) \langle D_2 \rangle^\alpha$ is bounded and the statement follows from the boundedness of

$$\langle D_2 \rangle^{-1/2} T \langle D_2 \rangle^{-1/2+\alpha} = (F_{D_2} - F_{D_1}) \langle D_2 \rangle^\alpha \quad \left([F_{D_1}, \mu] - T \langle D_2 \rangle^{-1} \right) \langle D_2 \rangle^\alpha = (F_{D_2} - F_{D_1}) \langle D_2 \rangle^\alpha.$$

Suppose that $1/2 \leq \alpha$. It is sufficient to prove that

$$T \langle D_2 \rangle^{-1+\alpha} = \langle D_2 \rangle^{1/2} (F_{D_2} - F_{D_1}) \langle D_2 \rangle^{-1/2+\alpha}$$

is bounded. If $\alpha = 1/2$,

$$T \langle D_2 \rangle^{-1/2} = \langle D_2 \rangle^{1/2} (F_{D_2} - F_{D_1})$$

and we are done. If $1/2 < \alpha \leq 1$, both $1/2$ and $-1/2 + \alpha$ are positive, and we can interpolate between

$$(F_{D_1} - F_{D_2}) \langle D_2 \rangle^\alpha \quad \text{and} \quad \langle D_2 \rangle^\alpha (F_{D_1} - F_{D_2})$$

as in [Les05, Proposition A.1], adjusted for Hilbert modules in [LM19, Lemma 7.7] (see also Appendix A.1). $\square$

2.4 The logarithmic transform: multiplicative to additive

Conformal transformations of unbounded Kasparov modules are not preserved by the exterior product. This is exemplified by the fact that the Cartesian product of two conformally perturbed Riemannian manifolds $(X_1, k_1^2 \mathbf{g}_1)$ and $(X_2, k_2^2 \mathbf{g}_2)$ is not a conformal perturbation of the Cartesian product $(X_1 \times X_2, \mathbf{g}_1 \oplus \mathbf{g}_2)$, unless $k_1(x) = k_2(y)$ for all $x \in X_1$ and $y \in X_2$, i.e. $k_1 = k_2$ is a constant. The *logarithmic dampening* of [GMR19] provides a way of turning conformal transformations into locally bounded perturbations, at the expense of much of the geometrical information encoded by the Dirac operator.

Proposition 2.44. *Let D be a self-adjoint regular operator on a right Hilbert B -module E and let $a \in \text{End}^* E$ preserve $\text{dom } D$. Suppose also that $[F_D, a] \log \langle D \rangle$ is bounded. Then, with*

$$L_D = F_D \log \langle D \rangle = D \log((1 + D^2)^{1/2})(1 + D^2)^{-1/2},$$

the commutator $[L_D, a]$ is bounded.

Proof. By [GMR19, Lemma 1.15], the condition $a \text{ dom } D \subseteq \text{dom } D$ implies that $a \text{ dom } \log \langle D \rangle \subseteq \text{dom } \log \langle D \rangle$ and that $[\log \langle D \rangle, a]$ is bounded. Using also the condition on $[F_D, a]$,

$$[L_D, a] = F_D [\log \langle D \rangle, a] + [F_D, a] \log \langle D \rangle$$

is bounded. □

Corollary 2.45. *Let D_0 and D_1 be self-adjoint regular operators on right Hilbert B -modules E_0 and E_1 . Suppose that there is an operator $a \in \text{Hom}_B^*(E_0, E_1)$ such that $a \text{ dom } D_0 \subseteq \text{dom } D_1$ and*

$$(F_{D_1} a - a F_{D_0}) \log \langle D_0 \rangle$$

extends to an adjointable operator. Then $L_{D_1} a - a L_{D_0}$ is bounded.

Theorem 2.46. *Let (U, μ) be a conformal transformation from the order- $\frac{1}{1-\alpha}$ cycle (A, E_B, D_1) to the order- $\frac{1}{1-\alpha}$ cycle (A, E'_B, D_2) . Then the logarithmic transforms (A, E_B, L_{D_1}) and (A, E'_B, L_{D_2}) are related by the unitary U , up to locally bounded perturbation; in particular, A is contained in the closure of the set of $a \in \text{End}^*(E)$ such that*

$$(U^* L_{D_2} U - L_{D_1}) a \quad [L_{D_1}, a] \quad [L_{D_2}, U a U^*]$$

are bounded.

Proof. Let $a, b, c \in \mathcal{M}$ so that $(U^* F_{D_2} U - F_{D_1}) a b^* c \mu \langle D_0 \rangle^\beta$

$$\begin{aligned} (U^* L_{D_2} U - L_{D_1}) a b^* c \mu &= U^* L_{D_2} U a b^* c \mu - a b^* c \mu L_{D_1} - [L_{D_1}, a b^* c \mu] \\ &= U^* F_{D_2} U (U^* \log \langle D_2 \rangle U a b^* c \mu - a b^* c \mu \log \langle D_1 \rangle) \\ &\quad + (U^* F_{D_2} U - F_{D_1}) a b^* c \mu \log \langle D_1 \rangle - F_{D_1} [\log \langle D_1 \rangle, a b^* c \mu] \end{aligned}$$

is bounded, by the proof of Theorem 2.9. Let $d \in \text{Lip}_\alpha^*(D)$ and multiply on the right by $\mu^{-1} d$. Then $(U^* L_{D_2} U - L_{D_1}) a^* b c d$ is bounded and, by the inclusions $A \subseteq \overline{\text{span}}(\mathcal{M} A) \subseteq \overline{\text{span}}(\mathcal{M} \mathcal{M}^* \mathcal{M} \text{Lip}_\alpha^*(D))$, we are done. □

2.5 The singular case

Conformal factors on noncompact manifolds need not be bounded nor have bounded inverse. In that setting, we can take a suitable open cover and assemble local estimates. This idea motivates the next definition. In the following we stress that $\overline{\text{span}}$ means the norm completion of finite linear combinations.

Definition 2.47. A *singular conformal transformation* $(U, (\mu_i)_{i \in I})$ from one order- $\frac{1}{1-\alpha}$ cycle, (A, E_B, D_1) , to another, (A, E'_B, D_2) , is a unitary map $U : E \rightarrow E'$, intertwining the representations of A , and a family $(\mu_i)_{i \in I} \subseteq \text{End}^*(E)$ of (even) invertible operators such that

$$A \subseteq \overline{\text{span}}_{i \in I} A \mathcal{M}_i \cap \overline{\text{span}}_{i \in I} \mathcal{M}_i A$$

where $\mathcal{M}_i$ is the set of $a \in \text{End}^*(E)$ such that

$$(U^* D_2 U a - a \mu_i D_1 \mu_i^*) \mu_i^{-1*} \langle D_1 \rangle^{-\alpha} \quad \langle D_2 \rangle^{-\alpha} U (U^* D_2 U a - a \mu_i D_1 \mu_i^*)$$

are bounded, $a, a \mu_i, a \mu_i^{-1*} \in \text{Lip}_\alpha^*(D_1)$, and $U a U^* \in \text{Lip}_\alpha^*(D_2)$.

Remark 2.48. As in the non-singular case, $\mathcal{M}_i$ is a ternary ring of operators, generally not closed. In particular, $\overline{\text{span}}(\mathcal{M}_i \mathcal{M}_i^* \mathcal{M}_i) = \overline{\mathcal{M}_i}$.

Theorem 2.49. Let $(U, (\mu_n)_{n \in \mathbb{N}})$ be a singular conformal transformation from (A, E_B, D_1) to (A, E'_B, D_2) . Then the bounded transforms (A, E_B, F_{D_1}) and (A, E'_B, F_{D_2}) are related by the unitary U , up to locally compact perturbation, i.e.

$$(U^* F_{D_2} U - F_{D_1}) a \in \text{End}^0(E)$$

for all $a \in A$.

Proof. As in the Proof of Theorem 2.9, $(U^* F_{D_2} U - F_{D_1}) a b^* c \mu_i \langle D_0 \rangle^\beta$ is bounded for all $a, b, c \in \mathcal{M}_i$. For $d, e \in \mathcal{M}_i$ and $f \in A$ we find

$$(U^* F_{D_2} U - F_{D_1}) a b^* c d^* e f = (U^* F_{D_2} U - F_{D_1}) a^* b c \mu_i \langle D_1 \rangle^\beta (\langle D_1 \rangle^{-\beta} \mu_i^{-1} d^* e \langle D_1 \rangle^\beta) \langle D_1 \rangle^{-\beta} f$$

is compact. The inclusion of $A \subseteq \overline{\text{span}}_{i \in I} (\mathcal{M}_i A) = \overline{\text{span}}_{i \in I} ((\mathcal{M}_i \mathcal{M}_i^*)^2 \mathcal{M}_i A)$ proves the statement. $\square$

Example 2.50. Let us reprise Example 2.11, in which we considered Riemannian spin manifolds $(X, \mathbf{g})$ and $(X, \mathbf{h})$ such that $\mathbf{h} = k^2 \mathbf{g}$. Suppose that $(X, \mathbf{g})$ is geodesically complete, so that $\not{D}_{\mathbf{g}}$ is self-adjoint. It may or may not be the case that $(X, \mathbf{h})$ is complete and $\not{D}_{\mathbf{h}}$ is self-adjoint, depending on the properties of k , although that is guaranteed if k is bounded with bounded inverse. Let $(O_i)_{i \in I}$ be an open cover of X such that k is bounded and invertible when restricted to any O_i . (This can be ensured by choosing a relatively compact cover.) Choose a family $(k_i)_{i \in I}$ of positive smooth functions which are bounded and invertible and agree with k on the corresponding O_i . Let $f \in C_c^\infty(O_i)$, so that

$$\begin{aligned} U^* \not{D}_{\mathbf{h}} U f - f k_i^{-1/2} \not{D}_{\mathbf{g}} k_i^{-1/2} &= k^{-1/2} \not{D}_{\mathbf{g}} k^{-1/2} f - f k_i^{-1/2} \not{D}_{\mathbf{g}} k_i^{-1/2} \\ &= k^{-1/2} [\not{D}_{\mathbf{g}}, f] k_i^{-1/2} \\ &= k_i^{-1/2} [\not{D}_{\mathbf{g}}, f] k_i^{-1/2} \end{aligned}$$

is bounded. Then $(U, (k_i^{-1/2})_{i \in I})$ is a singular conformal transformation from the spectral triple $(C_0(X), L^2(X, S_{\mathbf{g}}), \not{D}_{\mathbf{g}})$ to $(C_0(X), L^2(X, S_{\mathbf{h}}), \not{D}_{\mathbf{h}})$, provided that $(X, \mathbf{h})$ is complete so that the latter is a spectral triple. In the context of Example 2.12, $(U, (k_i^{-1/2})_{i \in I})$ is a conformal transformation from $(C_0(X), L^2(\Omega^* X, \mathbf{g}), d + \delta_{\mathbf{g}})$ to $(C_0(X), L^2(\Omega^* X, \mathbf{h}), d + \delta_{\mathbf{h}})$.

If either or both of $(X, \mathbf{g})$ and $(X, \mathbf{h})$ fails to be complete, the failure of self-adjointness of the Dirac operator(s) means that one requires the technology of half-closed chains and relative spectral triples. We do not pursue this here; for more details, see [Hil10, DGM18, FGMR19].

An abstract treatment of open covers, for the purposes of unbounded KK-theory, can be found in [vdD22]; see, in particular, [vdD22, Lemma 4.3].

In the following example, inspired by the modular cycles of [Kaa21], one should think of $\Delta_- \Delta_+^{-1}$ as the conformal factor, which can be both unbounded and noninvertible. Proposition 2.51 admits a generalisation to the higher-order setting, but it requires the extension of Theorem 2.52 to that case. Later, in Proposition 6.7, we directly generalise the results of [Kaa21].

Proposition 2.51. *Let (A, E_B, D_1) and (A, E_B, D_2) be unbounded Kasparov modules. Let Δ_+ and Δ_- be commuting positive adjointable operators such that*

- $\{\Delta_+, \Delta_-\} \text{ dom } D_1 \subseteq \text{ dom } D_1 \cap \text{ dom } D_2$ and $[D_1, \Delta_+]$, $[D_1, \Delta_-]$ are bounded;
- $A \subseteq \overline{\text{span}}(A\mathcal{N}) \cap \overline{\text{span}}(\mathcal{N}A)$, where $\mathcal{N} = \{T \in \text{Lip}_0^*(D_1) \cap \text{Lip}_0^*(D_2) \mid D_2 T \Delta_+ - T D_1 \Delta_- \text{ is bounded}\}$; and,
- For all $a \in A$, $(a(\Delta_+ + \Delta_-)(\Delta_+ + \Delta_- + \frac{1}{n})^{-1})_{n=1}^\infty$ converges in operator norm to a .

Let $(h_n)_{n \in \mathbb{N}_{\geq 1}} \subseteq C_b^\infty(\mathbb{R}_+^\times)$ be any sequence of positive functions with bounded reciprocals which agree with the function $x \mapsto x^{-1/2}$ on the interval $[\frac{1}{n}, n]$. Then $(1, (h_n(\Delta_+)h_n(\Delta_-)^{-1})_{n \in \mathbb{N}_{\geq 1}})$ is a conformal transformation from (A, E_B, D_1) to (A, E_B, D_2) .

For the proof, we recall a statement of the relevant aspects of the smooth functional calculus.

Theorem 2.52. *cf. [Pow75, Theorem 3], corrected in [BR76, §2] Let D be a self-adjoint regular operator on a Hilbert B -module E . Let S be an adjointable operator on E such that $S \text{ dom } D \subseteq \text{ dom } D$ and $[D, S]$ extends to an adjointable operator. Then, for any function $f \in C_c^\infty(\mathbb{R})$, $f(S) \text{ dom } D \subseteq \text{ dom } D$ and $[D, f(S)]$ extends to an adjointable operator.*

Theorem 2.52 admits an extension to the higher-order case, along the lines of [BEJ84, Lemma 3.2], but we do not pursue this here in the interests of space.

Lemma 2.53. *Let A be a C^* -algebra represented by π on a Hilbert module E . Let $h \in C \subseteq \text{End}^*(E)$ be a strictly positive element of a C^* -algebra C such that, for a dense subset of $a \in A$, the sequence*

$$(\pi(a)h(h + 1/n)^{-1})_{n=1}^\infty$$

converges to $\pi(a)$. Then $\pi(A)$ is contained in the closure of $\pi(A)C$.

Proof. First, note that $(h(h + 1/n)^{-1})_{n=1}^\infty$ is an approximate unit for C . For every $a \in A$ such that the sequence $(\pi(a)h(h + 1/n)^{-1})_{n=1}^\infty \subseteq \pi(a)C$ converges in norm to $\pi(a)$, $\pi(a) \in \overline{\pi(a)C}$. $\square$

Proof of Proposition 2.51. First, the smooth functional calculus of Theorem 2.52 shows that the commutator $[D, h_n(\Delta_+)h_n(\Delta_-)^{-1}]$ is bounded. Second, $\mathcal{M}_n$ consists of those $b \in \text{End}^*(E)$ such that

$$D_2 b - b h_n(\Delta_+)h_n(\Delta_-)^{-1} D_1 h_n(\Delta_+)h_n(\Delta_-)^{-1}$$

extends to an adjointable operator. Let $f_1, f_2 \in C_c^\infty((\frac{1}{n}, n))$ and $a \in \mathcal{N}$, and define $b \in \text{End}^*(E)$ to be the product

$$a f_1(\Delta_+) f_2(\Delta_-) \in \mathcal{N} C_0((\frac{1}{n}, n))(\Delta_+) C_0((\frac{1}{n}, n))(\Delta_-).$$

Then $b h_n(\Delta_+)h_n(\Delta_-)^{-1} = b \Delta_+^{-1/2} \Delta_-^{1/2}$ and, again using the smooth functional calculus,

$$\begin{aligned} D_2 b - b h_n(\Delta_+)h_n(\Delta_-)^{-1} D_1 h_n(\Delta_+)h_n(\Delta_-)^{-1} \\ = (D_2 a \Delta_+ - a D_1 \Delta_-) \Delta_+^{-1} f_1(\Delta_+) f_2(\Delta_-) \\ + a \left[D_1, \Delta_-^{1/2} \Delta_+^{-1/2} f_1(\Delta_+) f_2(\Delta_-) \right] h_n(\Delta_+)h_n(\Delta_-)^{-1} \end{aligned}$$

is bounded. The closure of $C_0((\frac{1}{n}, n))(\Delta_+) C_0((\frac{1}{n}, n))(\Delta_-)$ is $C^*(\Delta_+, \Delta_-)$. By Lemma 2.53, we have $A \subseteq \overline{A C^*(\Delta_+, \Delta_-)}$ and

$$\overline{\text{span}}_{i \in I} A \mathcal{M}_i \cap \overline{\text{span}}_{i \in I} \mathcal{M}_i A \supseteq \overline{\text{span}}(A \mathcal{N} C^*(\Delta_+, \Delta_-)) \cap \overline{\text{span}}(\mathcal{N} C^*(\Delta_+, \Delta_-) A) \supseteq A,$$

as required. $\square$

3 Group-equivariant KK-theory

In this section we begin by recalling the definitions of equivariant KK-theory and the descent map, due to Kasparov [Kas88]. The first attempt to generalise equivariance to unbounded KK-theory is [JV87, §1], for the case of $KK^G(\mathbb{C}, \mathbb{C})$. The first detailed treatment is by Kucerovsky [Kuc94, §8], which we mildly generalise in §3.1 to apply to the higher-order case and allow for local boundedness in the definition. In §3.2, we provide a generalisation to conformal equivariance for unbounded cycles that provides greater flexibility.

The case of compact groups is much easier to handle in both the bounded and unbounded settings. This is because, given the action of a compact group on a Kasparov module, one can integrate using the Haar measure to produce a module for which the operator is actually invariant under the action of the group. This fact has led to the definition of unbounded equivariant KK-theory in the case of a compact group as unbounded Kasparov modules with group actions for which the operator is invariant under the action. Alas, this does not represent the full range of geometrical situations available under equivariant KK-theory.

The following definition introduces notation for tracking the action of operators implementing equivariance. Throughout this section, G is a locally compact group.

Definition 3.1. Let E be a right Hilbert B -module and $\tau \in \text{Aut } A$. We define $\text{End}_B^{*,\tau}(E)$ to be the set of $\mathbb{C}$ -linear maps $T : E \rightarrow E$ for which there exists a map $T^* : E \rightarrow E$ such that

$$(T(x), y)_B = \tau((x, T^*(y))_B).$$

These maps are not B -linear; however they satisfy $T(xb) = T(x)\tau(b)$ since

$$(T(xb), y)_B = \tau((xb, T^*(y))_B) = \tau(b^*)\tau((x, T^*(y))_B) = \tau(b^*)(T(x), y)_B = (T(x)\tau(b), y)_B.$$

This gives an identification of $\text{End}_B^{*,\tau}(E)$ with $\text{Hom}_B^*(E, E \otimes_\tau B)$, where $E \otimes_\tau B$ is the internal tensor product of E with ${}_\tau B$. The adjoint $T^* \in \text{End}_B^{*,\tau^{-1}}(E)$, since

$$(T^*(x), y)_B = (y, T^*(x))_B^* = \tau^{-1}((T(y), x)_B^*) = \tau^{-1}((x, T(y))_B).$$

The composition of $S \in \text{End}_B^{*,\sigma}(E)$ and $T \in \text{End}_B^{*,\tau}(E)$ is $ST \in \text{End}_B^{*,\sigma\tau}(E)$. In particular, if $\tau = \sigma^{-1}$ then ST is an adjointable operator.

Definition 3.2. e.g. [Kas88, §1.2] Let $\beta : G \rightarrow \text{Aut } B$ be an action of a group G on a C^* -algebra B . A G -equivariant Hilbert B -module E is a Hilbert B -module equipped with a continuous $\mathbb{C}$ -linear map $U : G \times E \rightarrow E$ such that

$$U_{gh} = U_g U_h \quad U_g(xb) = U_g(x)\beta_g(b) \quad \beta_g((x, y)_B) = (U_g(x), U_g(y))_B$$

for $g, h \in G$, $x, y \in E$, and $b \in B$. We may equivalently say that $U_g \in \text{End}_B^{*,\beta_g}(E)$ with the conditions

$$U_{gh} = U_g U_h \quad U_{g^{-1}} = U_g^{-1} = U_g^*$$

for all $g, h \in G$.

Definition 3.3. Let $\alpha : G \rightarrow \text{Aut } A$ be an action of a group G on a C^* -algebra A . A G -equivariant A - B -correspondence E is an A - B -correspondence E which is also a G -equivariant Hilbert B -module, such that

$$U_g(ax) = \alpha_g(a)U_g(x)$$

for $g \in G$, $a \in A$ and $x \in E$.

Definition 3.4. [Kas88, Definition 2.2] cf. [Kuc94, Definition 8.5, Remark] A bounded Kasparov A - B -module (A, E_B, F) is G -equivariant if E is a G -equivariant A - B -correspondence and, for all $a \in A$, the map $g \mapsto (U_g F U_g^* - F)a$ is norm-continuous from G into $\text{End}^0(E)$.

Remark 3.5. cf. [Kuc94, Definition 8.5, Remark] By Lemma A.12, the norm continuity of the map $g \mapsto (U_g F U_g^* - F)a$ into $\text{End}^0(E)$ is equivalent to the condition that, when restricted to any compact subset $K \subseteq G$, the function $g \mapsto (U_g F U_g^* - F)a$ is in $\text{End}^0(C(K, E))$.

An important feature of equivariant KK-theory is Kasparov's descent map

$$j_t^G : KK^G(A, B) \rightarrow KK(A \rtimes_t G, B \rtimes_t G)$$

for either topology $t \in \{u, r\}$, universal or reduced [Kas88, Theorem 3.11]. There can be other, exotic, topologies t for which there is a descent map [BEW15, §6] but we will not pursue this.

Definition 3.6. [Kas88, Remarks before Theorem 3.11], [Bla98, Definition 20.6.1] Let E be a G -equivariant A - B -correspondence. The algebra $C_c(G, B)$ acts on the right of $C_c(G, E)$ by

$$(\xi f)(g) = \int_G \xi(h) \beta_h(f(h^{-1}g)) d\mu(h) \quad (\xi \in C_c(G, E), f \in C_c(G, B))$$

where β is the action of G on B . We define a right $C_c(G, B)$ -valued inner product on $C_c(G, E)$ by

$$\langle \xi | \eta \rangle_{C_c(G, B)}(g) = \int_G \beta_{h^{-1}}(\langle \xi(h) | \eta(hg) \rangle_B) d\mu(h) \quad (\xi, \eta \in C_c(G, E)).$$

The algebra $C_c(G, A)$ acts on the left of $C_c(G, E)$ by

$$(f\xi)(g) = \int_G f(h) U_h \xi(h^{-1}g) d\mu(h) \quad (f \in C_c(G, A), \xi \in C_c(G, E))$$

where U is the representation of G on E . For $t \in \{u, r\}$, we denote by $E \rtimes_t G$ the $A \rtimes_t G$ - $B \rtimes_t G$ -correspondence obtained by completing $C_c(G, E)$ in the $C_c(G, B)$ -valued inner product. We may also realise $E \rtimes_t G$ as the internal tensor product $E \otimes_B (B \rtimes_t G)$, but the left action of $A \rtimes_t G$ is difficult to see in this picture.

Proposition 3.7. [Kas88, Theorem 3.11] Let (A, E_B, F) be a G -equivariant bounded Kasparov module. Then, for $t \in \{u, r\}$, $(A \rtimes_t G, (E \rtimes_t G)_{B \rtimes_t G}, \tilde{F})$ is a bounded Kasparov module, where $\tilde{F}$ is the operator given on $\xi \in C_c(G, E) \subseteq E \rtimes_t G$ by $(\tilde{F}\xi)(g) = F(\xi(g))$.

When G acts trivially on B , there is the *dual-Green-Julg* map

$$\Psi^G : KK^G(A, B) \rightarrow KK(A \rtimes_u G, B)$$

which is an isomorphism when G is discrete [Bla98, 20.2.7(b)]. The existence of Ψ^G is proved in the next proposition, and then we present the isomorphism for discrete groups. The universal crossed product is needed because it is universal for covariant representations.

Proposition 3.8. Let (A, E_B, F) be a G -equivariant bounded Kasparov module, with G acting trivially on B . Then $(A \rtimes_u G, E_B, F)$ is a bounded Kasparov module, with the integrated representation of $A \rtimes_u G$.

Proof. With α the action of G on A , π the representation of A on E , and U the representation of G on E , the pair (π, U) is a covariant representation of the C^* -dynamical system (A, G, α) . We obtain by [EKQR06, §A.2] the integrated representation $\pi \rtimes U$ of $A \rtimes_u G$ on E , and it is here that the

universal crossed product is needed. We will consider the dense subalgebra $C_c(G, A) \subseteq A \rtimes_u G$. For an element $f \in C_c(G, A)$,

$$(F^* - F)(\pi \rtimes U)(f) = \int_G (F^* - F)\pi(f(g))U_g d\mu(g).$$

Because f is compactly supported and the integrand norm continuous, the integral converges. The integrand being valued in compact operators, the result is also compact. In the same way,

$$(F^2 - 1)(\pi \rtimes U)(f) = \int_G (F^2 - 1)\pi(f(g))U_g d\mu(g)$$

and

$$[F, (\pi \rtimes U)(f)] = \int_G [F, \pi(f(g))U_g] d\mu(g) = \int_G \left([F, \pi(f(g))]U_g + \pi(f(g))(F - U_g F U_g^*)U_g \right) d\mu(g)$$

are compact. By the density of $C_c(G, A) \subseteq A \rtimes_u G$ we are done. $\square$

Proposition 3.9. *Let $(A \rtimes_u G, E_B, F)$ be a bounded Kasparov module, with G a discrete group and $A \rtimes_u G$ represented nondegenerately on E . Then (A, E_B, F) is a G -equivariant bounded Kasparov module, with the group action given by $(U_g)_{g \in G} \subseteq C_u^*(G) \subseteq M(A \rtimes_u G)$, acting trivially on B .*

Proof. Because G is discrete, A is included in $A \rtimes_u G$. Hence,

$$(F^* - F)a \quad (F^2 - 1)a \quad [F, a]$$

are compact for all $a \in A$. Inside $M(A \rtimes_u G)$ are unitary elements $(U_g)_{g \in G}$ representing G , such that $aU_g \in A \rtimes_u G$ for all $a \in A$ and $g \in G$. Then

$$(F - U_g F U_g^*)a = [F, U_g]U_g^*a = [F, a] - [F, U_g^*a] = [F, a] + [F, aU_g]^*$$

is compact, as required. $\square$

3.1 Uniformly equivariant unbounded KK-theory

Again, throughout this section, G is a locally compact group. The following definition slightly generalises that of Kucerovsky.

Definition 3.10. cf. [Kuc94, Definition 8.7] An order- $\frac{1}{1-\alpha}$ A - B -cycle (A, E_B, D) is *uniformly G -equivariant* if E is a G -equivariant A - B -correspondence and A is contained in the closure of $\mathfrak{Q}$, the set of $a \in \text{End}^*(E)$ such that $a \text{ dom } D \subseteq U_g \text{ dom } D$ for all $g \in G$ and the maps

$$g \mapsto (U_g D U_g^* a - a D) \langle D \rangle^{-\alpha} \quad g \mapsto \langle D \rangle^{-\alpha} U_g^* (U_g D U_g^* a - a D)$$

are $*$ -strongly continuous as a map from G into bounded operators (on $\text{dom } D$). If $U_g D U_g^* = D$ for all $g \in G$, we say that the cycle is *isometrically equivariant*. If $\mathcal{A}$ is a dense $*$ -subalgebra of A contained in $\mathfrak{Q}$, we say that $(\mathcal{A}, E_B, D)$ is a uniformly G -equivariant order- $\frac{1}{1-\alpha}$ $\mathcal{A}$ - B -cycle.

Remarks 3.11.

1. We remark that $\mathfrak{Q} \subseteq \text{Lip}_\alpha^*(D)$ by considering the conditions at $g = e$, the identity of the group. Indeed, $\mathfrak{Q}$ is a right ideal of $\text{Lip}_\alpha^*(D)$.
2. By Lemma A.16, the conditions on $a \in \mathfrak{Q}$ are equivalent to the condition that $a \text{ dom } D \subseteq U_g \text{ dom } D$ and, when restricted to any compact subset $K \subseteq G$, the functions

$$g \mapsto (U_g D U_g^* a - a D) \langle D \rangle^{-\alpha} \quad g \mapsto \langle D \rangle^{-\alpha} U_g^* (U_g D U_g^* a - a D)$$

be in $\text{End}^*(C(K, E))$.

3. When $\alpha = 0$, the conditions on $a \in \mathcal{Q}$ are equivalent to requiring that $[D, a]$ extend to an adjointable operator and

$$g \mapsto (U_g D U_g^* - D)a$$

be $*$ -strongly continuous as a map from G into bounded operators. The higher order generalisation allows for higher order differential operators on manifolds, for example.

To prove that the bounded transform is well-defined, we use the results of Appendix A.2, based on the approach of Kucerovsky [Kuc94, Chapter 8, Appendix A]; see also [AK23, Appendix A].

Theorem 3.12. [Kuc94, Proposition 8.11] *Let (A, E_B, D) be a uniformly G -equivariant order- $\frac{1}{1-\alpha}$ cycle. Then (A, E_B, F_D) is a G -equivariant bounded Kasparov module.*

Proof. The only difference from the non-equivariant case is the need to show that, for every $a \in A$, $g \mapsto (F_D - U_g F_D U_g^*)a$ is norm-continuous as a map from G into $\text{End}^0(E)$.

Fix $b \in \mathcal{Q}$, where $\mathcal{Q}$ is as in Definition 3.10. By definition, the map $f : g \mapsto (U_g D U_g^* b - b D) \langle D \rangle^{-\alpha}$ is $*$ -strongly continuous as a map from G into $\text{End}^*(E)$. By Lemma A.16, this is equivalent to $f|_K$ residing in $\text{End}^*(C(K, E))$ for every compact subset $K \subseteq G$.

Fix a compact subset $K \subseteq G$ and let $\tilde{E} = C(K, E)$. Define $\tilde{D}$ to be the self-adjoint regular operator on $\tilde{E}$ given by D at each point of K . Similarly, let $\tilde{b} \in \text{End}^*(\tilde{E})$ be given by b at each point of K . Let U denote the $\mathbb{C}$ -linear map from $\tilde{E}$ to itself given by $g \mapsto U_g$. Then

$$(U \tilde{D} U^* \tilde{b} - \tilde{b} \tilde{D}) \langle \tilde{D} \rangle^{-\alpha}$$

is bounded. Applying Proposition 2.21, the operator $(F_{U \tilde{D} U^*} - F_{\tilde{D}}) \tilde{b} \langle \tilde{D} \rangle^\beta$ is bounded for all $\beta < 1 - \alpha$. By the functional calculus, $F_{U \tilde{D} U^*} = U F_{\tilde{D}} U^*$. Fixing an element $c \in A$, let $\tilde{c}$ denote the operator on $\tilde{E}$ given by $c \in \text{End}^*(E)$ at every point of K . Since $\langle D \rangle^{-\beta} c \in \text{End}^0(E)$,

$$\langle \tilde{D} \rangle^{-\beta} \tilde{c} \in C(K, \text{End}^0(E)) = \text{End}^0(\tilde{E}).$$

Hence

$$(U F_{\tilde{D}} U^* - F_{\tilde{D}}) \tilde{b} \tilde{c} = (F_{U \tilde{D} U^*} - F_{\tilde{D}}) \tilde{b} \langle \tilde{D} \rangle^\beta \langle \tilde{D} \rangle^{-\beta} \tilde{c}$$

is in $\text{End}^0(\tilde{E}) = \text{End}^0(C(K, E))$.

Define the map $f' : g \mapsto (F_D - U_g F_D U_g^*)bc$ from G into bounded operators on E . By Lemma A.12, the norm-continuity of f' is equivalent to the condition that $f'|_K$ be in $\text{End}^0(C(K, E))$ for every compact subset $K \subseteq G$. By the inclusion of $A \subseteq \overline{\mathcal{Q}A}$, we are done. $\square$

For uniformly equivariant cycles we have the following descent map but first we introduce some notation.

Definition 3.13. We introduce the notation $C_c(G, \mathcal{Q}_G)$ for the compactly supported functions $f : G \rightarrow \mathcal{Q}$ for which the maps

$$h \mapsto f(h) \quad h \mapsto (Df(h) - f(h)U_h D U_h^*)U_h \langle D \rangle^{-\alpha} U_h^* \quad h \mapsto \langle D \rangle^{-\alpha} (Df(h) - f(h)U_h D U_h^*)$$

are $*$ -strongly continuous. Similarly, if $\mathcal{A}$ is (represented) inside $\mathcal{Q}$, we write $C_c(G, \mathcal{A}_G)$ for functions in $C_c(G, \mathcal{Q}_G)$ which land in $\mathcal{A}$.

Proposition 3.14. *Let (A, E_B, D) be a uniformly G -equivariant order- $\frac{1}{1-\alpha}$ cycle. Then for either topology $t \in \{u, r\}$, $(A \rtimes_t G, (E \rtimes_t G)_{B \rtimes_t G}, \tilde{D})$ is an order- $\frac{1}{1-\alpha}$ cycle, where $\tilde{D}$ is the regular operator given on $\xi \in C_c(G, E) \subseteq E \rtimes_t G$ by $(\tilde{D}\xi)(g) = D(\xi(g))$.*

If, for a dense $$ -subalgebra $\mathcal{A} \subseteq A$, $(\mathcal{A}, E_B, D)$ is a uniformly G -equivariant order- $\frac{1}{1-\alpha}$ cycle, $(C_c(G, \mathcal{A}_G), (E \rtimes_t G)_{B \rtimes_t G}, \tilde{D})$ is an order- $\frac{1}{1-\alpha}$ cycle.*

Proof. We have, for $f \in C_c(G, A)$ and $\xi \in C_c(G, E)$

$$((1 + \tilde{D}^2)^{-1} f \xi)(g) = \int_G (1 + D^2)^{-1} f(h) U_h \xi(h^{-1}g) d\mu(h).$$

As f is compactly supported and the integrand continuous, the integral converges. Observe that $(1 + \tilde{D}^2)^{-1} f$ is an element of $C_c(G, \text{End}^0(E))$, given by $g \mapsto (1 + D^2)^{-1} f(g)$. By [Kas88, Proof of Theorem 3.11], $C_c(G, \text{End}^0(E)) \subseteq \text{End}^0(E \rtimes_t G)$, so $(1 + \tilde{D}^2)^{-1} f$ is compact.

Next, note that $C_c(G, \mathfrak{Q})$ contains $\mathfrak{Q} C_c(G)$, whose closure includes $A \rtimes_t G$. Let $f \in C_c(G, \mathfrak{Q})$ and $\xi \in \text{span}(C_c(G) \text{ dom } D) \subseteq C_c(G, \text{dom } D)$. Then we find that

$$\begin{aligned} ([\tilde{D}, f] \langle \tilde{D} \rangle^{-\alpha} \xi)(g) &= \int_G [D, f(h) U_h] \langle \tilde{D} \rangle^{-\alpha} \xi(h^{-1}g) d\mu(h) \\ &= \int_G (Df(h) - f(h) U_h D U_h^*) U_h \langle \tilde{D} \rangle^{-\alpha} U_h^* U_h \xi(h^{-1}g) d\mu(h). \end{aligned}$$

As f is compactly supported and the integrand is continuous, the integral converges. Observe that the closure of $[\tilde{D}, f] \langle \tilde{D} \rangle^{-\alpha}$ is an element of $C_c(G, \text{End}^*(E))$ given by

$$g \mapsto \overline{(Df(g) - f(g) U_g D U_g^*) U_g \langle D \rangle^{-\alpha} U_g^*}.$$

As $C_c(G, \text{End}^*(E)) \subseteq \text{End}^*(E \rtimes_t G)$ (see [Rae88, Lemma 7(1)]), $[\tilde{D}, f] \langle \tilde{D} \rangle^{-\alpha}$ is bounded. Similarly, $\langle \tilde{D} \rangle^{-\alpha} [\tilde{D}, f]$ is bounded. Hence for $f \in C_c(G, \mathfrak{A})$, $[\tilde{D}, f] \langle \tilde{D} \rangle^{-\alpha}$ and $\langle \tilde{D} \rangle^{-\alpha} [\tilde{D}, f]$ are bounded, proving the second statement. $\square$

For uniformly equivariant cycles, we have a dual-Green-Julg map for the universal crossed product.

Proposition 3.15. *Let (A, E_B, D) be a uniformly G -equivariant order- $\frac{1}{1-\alpha}$ cycle, with G acting trivially on B . Then $(A \rtimes_u G, E_B, D)$ is an order- $\frac{1}{1-\alpha}$ cycle, with the integrated representation of $A \rtimes_u G$.*

If, for a dense $$ -subalgebra $\mathfrak{A} \subseteq A$, $(\mathfrak{A}, E_B, D)$ is a uniformly G -equivariant order- $\frac{1}{1-\alpha}$ cycle, with G acting trivially on B , $(C_c(G, \mathfrak{A}_G), E_B, D)$ is an order- $\frac{1}{1-\alpha}$ cycle.*

Proof. With α the action of G on A , π the representation of A on E , and U the representation of G on E , the pair (π, U) is a covariant representation of the C^* -dynamical system (A, G, α) and we obtain the integrated representation $\pi \rtimes U$ of $A \rtimes_u G$ on E . For an element $f \in C_c(G, A)$,

$$(1 + D^2)^{-1} (\pi \rtimes U)(f) = \int_G (1 + D^2)^{-1} \pi(f(g)) U_g d\mu(g).$$

As f is compactly supported and the integrand norm-continuous, the integral converges, and as the integrand is valued in compact operators, the integral is also compact. As in the proof of Proposition 3.14, the closure of $C_c(G, \mathfrak{Q})$ includes $A \rtimes_u G$. Let $f \in C_c(G, \mathfrak{Q})$ and $\xi \in \text{dom } D$; then

$$\begin{aligned} [D, (\pi \rtimes U)(f)] \langle D \rangle^{-\alpha} \xi &= \int_G [D, \pi(f(g)) U_g] \langle D \rangle^{-\alpha} \xi d\mu(g) \\ &= \int_G (D\pi(f(g)) - \pi(f(g)) U_g D U_g^*) U_g \langle D \rangle^{-\alpha} \xi d\mu(g). \end{aligned}$$

As f is compactly supported and the integrand is continuous, the integral converges. By Corollary A.14, $[D, (\pi \rtimes U)(f)] \langle D \rangle^{-\alpha}$ extends to an adjointable operator, as does $\langle D \rangle^{-\alpha} [D, (\pi \rtimes U)(f)]$. $\square$

To display the inverse of the dual Green-Julg map for discrete groups, we require a dense subalgebra $\mathfrak{A}$ of A .

Proposition 3.16. *Let $(\mathcal{A} \rtimes G, E_B, D)$ be an order- $\frac{1}{1-\alpha}$ cycle, with G a discrete group and the representation of $\mathcal{A} \rtimes G$ on E nondegenerate. Then $(\mathcal{A}, E_B, D)$ is a uniformly G -equivariant order- $\frac{1}{1-\alpha}$ cycle, with group action given by $(U_g)_{g \in G} \subseteq C_u^*(G) \subseteq M(A \rtimes_u G)$, acting trivially on B .*

Proof. Because G is discrete, $\mathcal{A}$ is included in $\mathcal{A} \rtimes G$. Hence, $(1 + D^2)^{-1}a$ is compact and $[D, a]$ is bounded for all $a \in \mathcal{A}$. Inside $M(A \rtimes_u G)$ are unitary elements $(U_g)_{g \in G}$ representing G , such that $aU_g \in \mathcal{A} \rtimes G$ for all $a \in \mathcal{A}$ and $g \in G$. Then

$$U_g D U_g^* a - a D = U_g [D, U_g^* a]$$

so that $(U_g D U_g^* a - a D) \langle D \rangle^{-\alpha}$ and $\langle D \rangle^{-\alpha} U_g^* (U_g D U_g^* a - a D)$ are bounded, as required. $\square$

Remark 3.17. It is clear that the bounded transform $(A \rtimes_t G, (E \rtimes_t G)_{B \rtimes_t G}, F_{\tilde{D}} = \tilde{F}_D)$ of the descent $(A \rtimes_t G, (E \rtimes_t G)_{B \rtimes_t G}, \tilde{D})$ of a uniformly G -equivariant cycle (A, E_B, D) is exactly the descent of the bounded transform (A, E_B, F_D) . The same is true for the dual-Green–Julg map.

3.2 Conformally equivariant unbounded KK-theory

It is not clear that Definition 3.10 is the correct generalisation of equivariance to unbounded KK-theory. Definition 3.10 is natural in the sense that the exterior product and descent map are well-defined and Kucerovsky’s conditions [Kuc97, Theorem 13] for the Kasparov product still suffice [Kuc94, Theorem 8.12]. On the other hand, let us examine ‘patient zero’ of noncommutative geometry: a complete Riemannian spin manifold $(X, \mathbf{g})$ with spinor bundle S and Dirac operator $\not{D}$, forming the spectral triple $(C(X), L^2(X, S), \not{D})$. The largest group for which this is uniformly equivariant, in the sense of Definition 3.10, is the isometry group $\text{Iso}(X, \mathbf{g})$. What is the largest group for which the Fredholm module

$$(C(X), L^2(X, S), F_{\not{D}})$$

given by the bounded transform is equivariant, and can a geometric interpretation be put upon it? The answer to this question is that the Fredholm module above is equivariant under the conformal group $\text{Conf}(X, \mathbf{g})$ of X . That this is maximal is confirmed by [Bär07, Theorem 3.1].

Example 3.18. The simplest example exhibiting this discrepancy is the real line and its Dirac spectral triple $(C_0(\mathbb{R}), L^2(\mathbb{R}), i\partial_x)$. We will compare two group actions on $\mathbb{R}$: translations by $\mathbb{R}$ and dilation by $\mathbb{R}_+^\times$, i.e. addition and multiplication, respectively. The affine group $\mathbb{R} \rtimes \mathbb{R}_+^\times$ acts on $\mathbb{R}$ by $\varphi_{(a,b)} : x \mapsto ax + b$, for $(a,b) \in \mathbb{R} \rtimes \mathbb{R}_+^\times$. Let $V_{(a,b)}$ be the pullback by $\varphi_{(a,b)}^{-1} = \varphi_{(a^{-1}, -a^{-1}b)}$ on $L^2(\mathbb{R})$. For $\xi, \eta \in L^2(\mathbb{R})$, we have

$$\int_0^\infty \overline{(V_{(a,b)}\xi)(x)} \eta(x) dx = \int_0^\infty \overline{\xi(a^{-1}(x-b))} \eta(x) dx = \int_0^\infty \overline{\xi(y)} \eta(ay+b) a dy$$

so $V_{(a,b)}^* = a V_{(a,b)}^{-1} = a V_{(a^{-1}, -a^{-1}b)}$. The unitary part of the polar decomposition of $V_{(a,b)}$ is, therefore, $U_{(a,b)} = a^{-1/2} V_{(a,b)}$. By the chain rule, for $\xi \in C_c^\infty(\mathbb{R})$,

$$(U_{(a,b)} \partial_x U_{(a,b)}^* \xi)(x) = a^{-1/2} (\partial_x U_{(a,b)}^* \xi)(a^{-1}(x-b)) = a^{-3/2} (U_{(a,b)}^* \xi)'(a^{-1}(x-b)) = a^{-1} \xi'(x)$$

so that $U_{(a,b)} i\partial_x U_{(a,b)}^* = a^{-1} i\partial_x$. For the subgroup $\mathbb{R}$ ($a = 1$), the spectral triple $(C_0(\mathbb{R}), L^2(\mathbb{R}), i\partial_x)$ is isometrically equivariant in the sense of Definition 3.10. On the other hand, when $a \neq 1$, for $f \in C_c^\infty(\mathbb{R})$,

$$U_{(a,b)} i\partial_x U_{(a,b)}^* f - f i\partial_x = (a^{-1} - 1) i\partial_x f + [i\partial_x, f]$$

is as unbounded as $i\partial_x$, so condition 4 of Definition 3.10 is not satisfied. On the other hand,

$$(U_{(a,b)} F_{i\partial_x} U_{(a,b)}^* - F_{i\partial_x}) f = (F_{a^{-1}i\partial_x} - F_{i\partial_x}) f = i\partial_x \left((a^2 + (i\partial_x)^2)^{-1/2} - (1 + (i\partial_x)^2)^{-1/2} \right) f$$

is compact, as $y \mapsto y \left((a^2 + y^2)^{-1/2} - (1 + y^2)^{-1/2} \right)$ is in $C_0(\mathbb{R})$. Hence $(C_0(\mathbb{R}), L^2(\mathbb{R}), F_{i\partial_x})$ is equivariant for all of $\mathbb{R} \rtimes \mathbb{R}_+^\times$. In this section, we will make a definition of equivariance in unbounded KK-theory which can cope with this and similar examples. (We remark that multiplication by -1 , although an isometry, is not orientation-preserving and has the effect of multiplying by -1 in $KK_1(C_0(\mathbb{R}), \mathbb{C})$, rather than preserving the class.)

Definition 3.19. An order- $\frac{1}{1-\alpha}$ A - B -cycle (A, E_B, D) is *conformally equivariant* if E is a G -equivariant A - B -correspondence and there exists a $*$ -strongly continuous family $(\mu_g)_{g \in G} \subseteq \text{End}^*(E)$ of (even) invertible operators satisfying the following. We require that $A \subseteq \overline{\text{span}}(A\mathcal{Q}) \cap \overline{\text{span}}(\mathcal{Q}A)$, where $\mathcal{Q}$ is the set of $a \in \text{Lip}_\alpha^*(E)$ such that for all $g \in G$ we have $\{a\mu_g, a\mu_g^{-1*}\} \text{dom } D \subseteq \text{dom } D \cap U_g \text{dom } D$, and the maps

$$\begin{aligned} g \mapsto (U_g D U_g^* a - a \mu_g D \mu_g^*) \mu_g^{-1*} \langle D \rangle^{-\alpha} & \quad g \mapsto [D, a \mu_g] \langle D \rangle^{-\alpha} & \quad g \mapsto [D, a \mu_g^{-1*}] \langle D \rangle^{-\alpha} \\ g \mapsto U_g \langle D \rangle^{-\alpha} U_g^* (U_g D U_g^* a - a \mu_g D \mu_g^*) & \quad g \mapsto \langle D \rangle^{-\alpha} [D, a \mu_g] & \quad g \mapsto \langle D \rangle^{-\alpha} [D, a \mu_g^{-1*}] \end{aligned}$$

are $*$ -strongly continuous from G into bounded operators (but need not be globally bounded). We call $\mu = (\mu_g)_{g \in G}$ the conformal factor.

Remarks 3.20.

1. When $\mu_g = 1$ for all $g \in G$, this Definition reduces to Definition 3.10 of uniformly equivariant G -cycles.
2. Also, if $\mu_e = 1$, for elements $a \in \text{End}^*(E)$ satisfying that

$$[D, a \mu_g] \langle D \rangle^{-\alpha}$$

is bounded, a is automatically in $\text{Lip}_\alpha^*(D)$.

3. Note also that it is sufficient that $1 \in \mathcal{Q}$ for the closure conditions to be satisfied; in the nonunital case, an approximate unit might be used.

Theorem 3.21. *Let (A, E_B, D) be a conformally G -equivariant order- $\frac{1}{1-\alpha}$ cycle. Then (A, E_B, F_D) is a G -equivariant bounded Kasparov module.*

Proof. The only difference from the non-equivariant case is the need to show that, for every $a \in A$, $g \mapsto (F_D - U_g F_D U_g^*) a$ is norm-continuous as a map from G into $\text{End}^0(E)$.

By definition, for every $a \in \mathcal{Q}$, the maps $f_0 : g \mapsto \mu_g^{-1}$ and

$$\begin{aligned} f_{1,a} : g \mapsto (U_g D U_g^* a - a \mu_g D \mu_g^*) \mu_g^{-1*} \langle D \rangle^{-\alpha} & \quad f_{2,a} : g \mapsto \langle D \rangle^{-\alpha} U_g^* (U_g D U_g^* a - a \mu_g D \mu_g^*) \\ f_{3,a} : g \mapsto [D, a \mu_g] \langle D \rangle^{-\alpha} & \quad f_{4,a} : g \mapsto \langle D \rangle^{-\alpha} [D, a \mu_g] \\ f_{5,a} : g \mapsto [D, a \mu_g^{-1*}] \langle D \rangle^{-\alpha} & \quad f_{6,a} : g \mapsto \langle D \rangle^{-\alpha} [D, a \mu_g^{-1*}] \end{aligned}$$

are $*$ -strongly continuous as a map from G into $\text{End}^*(E)$. By Lemma A.16, this is equivalent to $f_{i,a}|_K$ residing in $\text{End}^*(C(K, E))$ for every compact subset $K \subseteq G$.

Fix a compact subset $K \subseteq G$ and let $\tilde{E} = C(K, E)$. Define $\tilde{D}$ to be the self-adjoint regular operator on $\tilde{E}$ given by D at each point of K . Let U denote the $\mathbb{C}$ -linear map from $\tilde{E}$ to itself given by $g \mapsto U_g$. Let $\tilde{\mu} \in \text{End}^*(\tilde{E})$ be given by $g \mapsto \mu_g$. For every $a \in \text{End}^*(E)$, let $\tilde{a}$ be given by a at each point of G . Then, for every $a \in \mathcal{Q}$,

$$\begin{aligned} (U \tilde{D} U^* \tilde{a} - \tilde{a} \tilde{\mu} \tilde{D} \tilde{\mu}^*) \tilde{\mu}^{-1*} \langle \tilde{D} \rangle^{-\alpha} & \quad \langle \tilde{D} \rangle^{-\alpha} U^* (U \tilde{D} U_g^* \tilde{a} - \tilde{a} \tilde{\mu} \tilde{D} \tilde{\mu}^*) \\ [\tilde{D}, \tilde{a} \tilde{\mu}] \langle \tilde{D} \rangle^{-\alpha} & \quad \langle \tilde{D} \rangle^{-\alpha} [\tilde{D}, \tilde{a} \tilde{\mu}] & \quad [\tilde{D}, \tilde{a} \tilde{\mu}^{-1*}] \langle \tilde{D} \rangle^{-\alpha} & \quad \langle \tilde{D} \rangle^{-\alpha} [\tilde{D}, \tilde{a} \tilde{\mu}^{-1*}] & \quad [\tilde{D}, \tilde{a}] \end{aligned}$$

are adjointable endomorphisms of $\tilde{E}$. Let $a, b, c, d \in \mathcal{Q}$. As in the Proof of Theorem 2.9,

$$[\tilde{\mu}\tilde{D}\tilde{\mu}^*, \tilde{b}^*\tilde{c}]\tilde{\mu}^{-1*}\langle\tilde{D}\rangle^{-\alpha}$$

is bounded. We apply Theorem 2.42 to obtain that $(F_{\tilde{\mu}\tilde{D}\tilde{\mu}^*} - F_{\tilde{D}})\tilde{b}^*\tilde{c}d^*\langle\tilde{D}\rangle^\beta$ is bounded for $\beta < 1 - \alpha$. Furthermore, as

$$(U\tilde{D}U^* - \tilde{a}\tilde{\mu}\tilde{D}\tilde{\mu}^*)\tilde{\mu}^{-1*}\langle\tilde{D}\rangle^{-\alpha}$$

is bounded, Proposition 2.21, shows that

$$(UF_{\tilde{D}}U^* - \tilde{a}F_{\tilde{\mu}\tilde{D}\tilde{\mu}^*})\tilde{\mu}\langle\tilde{D}\rangle^\beta$$

is too. Taking care because U is only $\mathbb{C}$ -linear, we have

$$\begin{aligned} (UF_{\tilde{D}}U^* - F_{\tilde{D}})\tilde{a}\tilde{b}^*\tilde{c}d^* &= U[F_{\tilde{D}}, U^*]\tilde{a}\tilde{b}^*\tilde{c}d^* = U[F_{\tilde{D}}, U^*\tilde{a}\tilde{b}^*\tilde{c}]d^* - [F_{\tilde{D}}, \tilde{a}\tilde{b}^*\tilde{c}]d^* \\ &= U(F_{\tilde{D}}U^*\tilde{a} - U^*\tilde{a}F_{\tilde{\mu}\tilde{D}\tilde{\mu}^*})\tilde{b}^*\tilde{c}d^* + \tilde{a}(F_{\tilde{\mu}\tilde{D}\tilde{\mu}^*}\tilde{b}^*\tilde{c}d^* - \tilde{b}^*\tilde{c}F_{\tilde{D}}) - [F_{\tilde{D}}, \tilde{a}\tilde{b}^*\tilde{c}]d^* \\ &= U(F_{\tilde{D}}U^*\tilde{a} - U^*\tilde{a}F_{\tilde{\mu}\tilde{D}\tilde{\mu}^*})\tilde{b}^*\tilde{c}d^* + \tilde{a}(F_{\tilde{\mu}\tilde{D}\tilde{\mu}^*} - F_{\tilde{D}})\tilde{b}^*\tilde{c}d^* - [F_{\tilde{D}}, \tilde{a}]\tilde{b}^*\tilde{c}d^* \end{aligned}$$

so that $(UF_{\tilde{D}}U^* - F_{\tilde{D}})\tilde{a}\tilde{b}^*\tilde{c}d^*\langle\tilde{D}\rangle^\beta$ is bounded. Letting $e \in A$ we have

$$(UF_{\tilde{D}}U^* - F_{\tilde{D}})\tilde{a}\tilde{b}^*\tilde{c}d^*\tilde{e} \tag{3.22}$$

is in $\text{End}^0(\tilde{E}) = \text{End}^0(C(K, E))$.

Define the map $f' : g \mapsto (F_D - U_g F_D U_g^*)ab^*cd^*e$ from G into bounded operators on E . By Lemma A.12, the norm-continuity of f' is equivalent to the condition that $f'|_K$ be in $\text{End}^0(C(K, E))$ for every compact subset $K \subseteq G$. By the inclusion of $A \subseteq \overline{\mathcal{Q}\mathcal{Q}^*\mathcal{Q}\mathcal{Q}^*A}$, we are done. $\square$

Example 3.23. Let $(X, \mathbf{g})$ be a complete Riemannian spin manifold with spinor bundle S and Dirac operator $\not{D}$. Let G be a locally compact group with a conformal action φ on X , so that $\varphi_g^*(\mathbf{g}) = k_g^2 \mathbf{g}$ for $g \in G$. If the conformal factors $(k_g)_{g \in G}$ are each bounded and invertible (for instance, if X is compact), then $(C_0(X), L^2(X, S_{\mathbf{g}}), \not{D})$ is a conformally G -equivariant spectral triple with conformal factors $(k_{g^{-1}}^{-1/2})_{g \in G}$.

Example 3.24. Let $(X, \mathbf{g})$ be a complete oriented Riemannian manifold with Hodge-de Rham operator $d + \delta$. Let G be a locally compact group with a conformal action φ on X , so that $\varphi_g^*(\mathbf{g}) = k_g^2 \mathbf{g}$ for $g \in G$. If the conformal factors $(k_g)_{g \in G}$ are each bounded and invertible (for instance, if X is compact), then $(C_0(X), L^2(\Omega^*X), d + \delta)$ is a conformally G -equivariant spectral triple with conformal factors $(k_{g^{-1}}^{-1/2})_{g \in G}$.

Example 3.25. Let P be a principal circle bundle over a compact Hausdorff space X . Let $\Phi : C(P) \rightarrow C(X)$ be the conditional expectation given by averaging over the circle action. By [CNNR11, Proposition 2.9],

$$(C(P), L^2(P, \Phi)_{C(X)}, N = -i\partial_\theta) \tag{3.26}$$

is an unbounded Kasparov module, where N is the number operator on the spectral subspaces, equivalent to the vertical Dirac operator $-i\partial_\theta$ acting on each fibre. Let G be a group acting on P and X , compatibly with the surjection $P \rightarrow X$. Suppose that φ acts differentiably between the fibres. Since the circle is one-dimensional, $\varphi_g^*(d\theta^2) = k_g^2 d\theta^2$ for a family of functions $(k_g)_{g \in G} \in C(P)$. We obtain that (3.26) is conformally G -equivariant with conformal factors $(k_{g^{-1}}^{-1/2})_{g \in G}$.

One limitation of conformal equivariance is that the exterior product becomes ill-defined. This is exemplified by the fact that the conformal group of the Cartesian product of Riemannian manifolds is generically smaller than the product of the conformal groups. Example 3.25 also demonstrates that the internal Kasparov product is generally not constructive for conformally equivariant cycles.

However, at the bounded level of KK-theory, the exterior product is known to exist by Kasparov's technical theorem. Recall the logarithmic transform of §2.4, which will provide a way of turning conformal equivariance into uniform equivariance, making the exterior product constructive, at the expense of much of the geometric information encoded by the Dirac operator.

Theorem 3.27. *Let (A, E_B, D) be a conformally G -equivariant order- $\frac{1}{1-\alpha}$ cycle with conformal factor μ . Then (A, E_B, L_D) is a uniformly G -equivariant unbounded Kasparov module.*

Proof. The only difference from the non-equivariant case is the need to show that A is contained in the closure of the set of $a \in \text{End}^*(E)$ such that $[L_D, a]$ extends to an adjointable operator and $g \mapsto (L_D - U_g L_D U_g^*)a$ is $*$ -strongly continuous as a map from G into $\text{End}^*(E)$.

Fix a compact subset $K \subseteq G$ and let $\tilde{E} = C(K, E)$. As in the Proof of Theorem 3.21, define $\tilde{D}$ to be the self-adjoint regular operator on $\tilde{E}$ given by D at each point of K . Let U denote the $\mathbb{C}$ -linear map from $\tilde{E}$ to itself given by $g \mapsto U_g$. Let $\tilde{\mu} \in \text{End}^*(\tilde{E})$ be given by $g \mapsto \mu_g$. For every $a \in \text{End}^*(E)$, let $\tilde{a}$ be given by a at each point of G . Let $a, b, c \in \mathcal{Q}$; then as in (3.22)

$$(UF_{\tilde{D}}U^* - F_{\tilde{D}})\tilde{a}\tilde{b}^*\tilde{c}\tilde{\mu}\langle\tilde{D}\rangle^\beta$$

is bounded for $\beta < 1 - \alpha$. Hence,

$$\begin{aligned} (UL_{\tilde{D}}U^* - L_{\tilde{D}})\tilde{a}\tilde{b}^*\tilde{c}\tilde{\mu} &= UL_{\tilde{D}}U^*\tilde{a}\tilde{b}^*\tilde{c}\tilde{\mu} - \tilde{a}^*b\tilde{\mu}L_{\tilde{D}} - [L_{\tilde{D}}, \tilde{a}\tilde{b}^*\tilde{c}\tilde{\mu}] \\ &= UF_{\tilde{D}}U^*(U\log\langle\tilde{D}\rangle U^*\tilde{a}\tilde{b}^*\tilde{c}\tilde{\mu} - \tilde{a}\tilde{b}^*\tilde{c}\tilde{\mu}\log\langle\tilde{D}\rangle) \\ &\quad + (UF_{\tilde{D}}U^* - F_{\tilde{D}})\tilde{a}\tilde{b}^*\tilde{c}\tilde{\mu}\log\langle\tilde{D}\rangle - F_{\tilde{D}}[\log\langle\tilde{D}\rangle, \tilde{a}\tilde{b}^*\tilde{c}\tilde{\mu}] \end{aligned}$$

is bounded. By the invertibility of $\tilde{\mu}$, $(UL_{\tilde{D}}U^* - L_{\tilde{D}})\tilde{a}\tilde{b}^*\tilde{c} \in \text{End}^*(C(K, E))$.

Let $d \in A$ and define the map $f' : g \mapsto (L_D - U_g L_D U_g^*)ab^*cd^*$ from G into bounded operators on E . By Lemma A.16, the $*$ -strong-continuity of f' is equivalent to the condition that $f'|_K$ be in $\text{End}^*(C(K, E))$ for every compact subset $K \subseteq G$, which it is. By the inclusion of $A \in \overline{\text{span}}(\mathcal{Q}\mathcal{Q}^*\mathcal{Q}A)$, we are done. $\square$

3.3 The γ -element for the real and complex Lorentz groups

In this section, we lift to unbounded KK-theory the γ -elements constructed for $SO(2n+1, 1)$, $SO(2n, 1)$, and $SU(n, 1)$ by Kasparov [Kas84], Chen [Che96], and Julg and Kasparov [JK95], respectively. We have opted to present them with notation close to the original sources, in the interests of space. For a unified treatment, see [AJV19, §5.3].

In each case, the Bernstein–Gelfand–Gelfand (BGG) complex [ČS09] for a sphere, considered as a symmetric space, is cleft in twain. For the real Lorentz groups, the BGG complex is the de Rham complex and, for the complex Lorentz groups, it is the Rumin complex [Rum94]. In the case of $SO(2n+1, 1)$, the symmetric space is S^{2n} . The sphere being even dimensional, the middle-degree forms are split into the two eigenspaces of the Hodge star operator, which division is conformally invariant and, indeed, appears in the BGG complex. In the cases of $SO(2n, 1)$ and $SU(n, 1)$, the symmetric space is S^{2n-1} . The sphere being odd-dimensional necessitates the addition of the L^2 harmonic forms on a real or complex hyperbolic space to be added to the half-complex, along with an operator related to the Poisson transform. The sphere S^{2n-1} is considered as the boundary of $\mathbb{R}H^{2n} = SO(2n, 1)/S(O(n) \times O(1))$ or $\mathbb{C}H^n = SU(n, 1)/S(U(n) \times U(1))$.

It is possible that the framework of conformally equivariant unbounded KK-theory could be used to treat the other rank-one groups, $Sp(n, 1)$ and the real form $F_{4(-20)}$, lifting the construction in [Jul19]; however, there, the resulting complex contains differential operators of different orders. In rank two, there is a construction by Yuncken [Yun11] of the γ -element in bounded KK-theory of $SL(3, \mathbb{C})$, using the BGG complex of the flag manifold. A similar construction is proposed for the other rank-two complex semisimple groups [Yun18]. The BGG complex, in full generality, has

been put on a sound analytical footing in [DH22] and subsequently fitted into bounded KK-theory in [Gof24], although with limitations on equivariance. The lifting of these constructions to the unbounded picture remains a difficult task, likely to require a substantial renovation of the axioms of an unbounded Kasparov module, beyond what is done here. A step in this direction is the treatment of ‘mixed-order’ situations in noncommutative geometry which will appear in [FGM].

3.3.1 The case of $SO(2n+1, 1)$

Following [Kas84, §4], we begin with the sphere S^{2n} on which $SO(2n+1, 1)$ acts conformally and its Hodge–de Rham Dirac operator. As we have seen, we can build a conformally $SO(2n+1, 1)$ -equivariant spectral triple

$$(C(S^{2n}), L^2(\Omega^* S^n), d + \delta).$$

In order to obtain the KK-class of the γ -element, we split the complexified exterior algebra into two subspaces, each preserved by the Dirac operator. On a $2n$ -dimensional manifold, the codifferential is equal to $\delta = d^* = -\star d \star$ and the Hodge star satisfies that

$$\star^2 : \alpha \mapsto (-1)^{|\alpha|} \alpha \quad \star^* : \alpha \mapsto (-1)^{|\alpha|} \star \alpha$$

for homogeneous $\alpha \in \Omega^* S^n$. The Hodge star and Hodge-de Rham operator are related by

$$(d + \delta) \star \alpha = (d \star - (-1)^{|\alpha|} \star d) \alpha = \star((-1)^{|\alpha|+1} \star d \star - (-1)^{|\alpha|} d) \alpha = (-1)^{|\alpha|+1} \star (d - \delta) \alpha.$$

Define the map $\epsilon : \alpha \mapsto i^{|\alpha|(|\alpha|+1)-n} \alpha = (-1)^{|\alpha|(|\alpha|+1)/2} i^{-n} \alpha$, so that

$$(\star \epsilon)^2 \alpha = i^{|\alpha|(|\alpha|+1)-n} \star \epsilon \star \alpha = i^{|\alpha|(|\alpha|+1)-n} i^{(2n-|\alpha|)((2n-|\alpha|)+1)-n} (-1)^{|\alpha|} \alpha = \alpha$$

and

$$(\star \epsilon)^* \alpha = (-1)^{(2n-|\alpha|)(2n-|\alpha|+1)/2} i^n (-1)^{|\alpha|} \star \alpha = (-1)^{|\alpha|(|\alpha|+1)/2} i^{-n} \star \alpha = \star \epsilon \alpha,$$

meaning that $\star \epsilon$ is a self-adjoint unitary. We have

$$(d + \delta) \star \epsilon \alpha = i^{|\alpha|(|\alpha|+1)-n} (d + \delta) \star \alpha = i^{2|\alpha|+2+|\alpha|(|\alpha|+1)-n} \star (d - \delta) \alpha$$

and

$$\epsilon d \alpha = i^{2|\alpha|+2+|\alpha|(|\alpha|+1)-n} d \alpha \quad \epsilon \delta \alpha = -i^{2|\alpha|+2+|\alpha|(|\alpha|+1)-n} \delta \alpha.$$

Hence $\star \epsilon$ commutes with $d + \delta$ and we can decompose the exterior algebra into

$$\Omega^* S^{2n} = \Omega_1^* \oplus \Omega_2^* := \text{im} \left(\frac{1}{2} (1 + \star \epsilon) \right) \oplus \text{im} \left(\frac{1}{2} (1 - \star \epsilon) \right).$$

We thus have a spectral triple

$$(C(S^{2n}), L^2(\Omega_1^*), d + \delta)$$

which is still conformally $SO(2n+1, 1)$ -equivariant and isometrically $SO(2n+1)$ -equivariant. By forgetting the action of the algebra, we obtain a representative $(\mathbb{C}, L^2(\Omega_1^*), d + \delta)$ of a class $\gamma \in KK^{SO(2n+1, 1)}(\mathbb{C}, \mathbb{C})$. The only harmonic forms on S^{2n} are scalar multiples of $1 \in \Omega^0 S^{2n}$ and the volume form $\text{vol} \in \Omega^{2n} S^{2n}$. One can check that

$$\star \epsilon 1 = i^{-n} \text{vol} \quad \star \epsilon \text{vol} = i^n 1 \quad \frac{1}{2} (1 + \star \epsilon) (1 + i^{-n} \text{vol}) = 1 + i^{-n} \text{vol}.$$

Hence the only harmonic forms in Ω_1^* are scalar multiples of $(1 + i^{-n} \text{vol})$. The form $(1 + i^{-n} \text{vol})$ being $SO(2n+1)$ -invariant, the restriction $r^{SO(2n+1, 1), SO(2n+1)}(\gamma)$ represents $1 \in KK^{SO(2n+1)}(\mathbb{C}, \mathbb{C})$. By [AJV19, Proposition 5.9], because γ is the image of an element of $KK^{SO(2n+1, 1)}(C(S^{2n}), \mathbb{C})$ and restricts to $1 \in KK^{SO(2n+1)}(\mathbb{C}, \mathbb{C})$, γ is really the γ -element of $SO(2n+1, 1)$.

3.3.2 The case of $SO(2n, 1)$

Following [Che96, §3.1], we begin with the sphere S^{2n-1} , on which $SO(2n, 1)$ acts conformally, and its Hodge–de Rham operator. As in the even-dimensional case, we can build a conformally $SO(2n, 1)$ -equivariant spectral triple

$$(C(S^{2n-1}), L^2(\Omega^* S^{2n-1}), d + \delta).$$

To obtain the correct class in $KK_0^{SO(2n, 1)}(\mathbb{C}, \mathbb{C})$ for the γ -element, we will cut the differential forms in two, as we did for $SO(2n + 1, 1)$, and add an additional operator.

Let D^{2n} be the open unit ball with Euclidean metric. The Poincaré disc model is a conformal identification of the hyperbolic space $\mathbb{R}H^{2n}$ with D^{2n} . As we saw in Example 2.12 (in particular (2.13)) the pullback map $L^2(\Omega^n \mathbb{R}H^{2n}) \rightarrow L^2(\Omega^n D^{2n})$ is automatically unitary because the forms are of middle degree. Let $I : \text{dom}(I) \subset L^2(\Omega^n \mathbb{R}H^{2n}) \rightarrow L^2(\Omega^n S^{2n-1})$ be the restriction to the boundary S^{2n-1} of the ball. Let $\mathcal{H} \subseteq L^2(\Omega^n \mathbb{R}H^{2n})$ be the L^2 harmonic forms on the real hyperbolic $2n$ -space and let $\mathcal{H}_\infty \subset \mathcal{H}$ be those forms in the domain of I . We have a complex

$$0 \longrightarrow \mathcal{H}_\infty \xrightarrow{I} \Omega^n S^{2n-1} \xrightarrow{d} \Omega^{n+1} S^{2n-1} \xrightarrow{d} \dots \xrightarrow{d} \Omega^{2n-1} S^{2n-1} \longrightarrow 0$$

which is invariant under the pullback by the action φ of $SO(2n, 1)$. When we complete the spaces of the complex to Hilbert spaces, pullback by the action of $SO(2n, 1)$ is not unitary. On $L^2(\Omega^n S^{2n-1})$ the unitaries $(U_g)_{g \in G}$ implementing the group action φ act by

$$U_g : \xi \mapsto k_{g^{-1}}^{-(-(2n-1)+2n)/2} \varphi_{g^{-1}}^*(\xi) = k_{g^{-1}}^{-1/2} \varphi_{g^{-1}}^*(\xi).$$

As in Example 2.12,

$$U_g dU_g^* - k_{g^{-1}}^{-1/2} dk_{g^{-1}}^{-1/2}$$

is bounded. However, on the hyperbolic space $\mathbb{R}H^{2n}$, the group $SO(2n, 1)$ acts by isometries. Because the map I commutes with pullback by the group action, $U_g I U_g^* = k_{g^{-1}}^{-1/2} I$, which is not the same behaviour as the rest of the complex displays, the overall exponent of the conformal factor being $-1/2$ rather than -1 . On all of $L^2(\Omega^* S^{2n-1})$ the Laplacian $\Delta = d\delta + \delta d$ transforms so that

$$U_g \Delta^{1/4} U_g^* - k_{g^{-1}}^{-1/2} \Delta^{1/4}$$

is of order $-1/2$. We will replace the operator I in the complex with $\Delta^{1/4} I$, in the hope of obtaining the right conformal scaling.

We need also an operator on $\mathcal{H}$ to act as the conformal factor, because neither functions on S^{2n-1} nor on $\bar{D}^{2n}$ are represented naturally on $\mathcal{H}$. By [Che96, Proposition 3.2], there is a polar decomposition $I = \Delta^{1/4} B$, where $B : \mathcal{H} \rightarrow L^2(\Omega^n S^{2n-1})$ is an isometry with range $\Omega^n S^{2n-1} \cap \ker d$. The operator $B^* k_{g^{-1}}^{-1/2} B$ is positive and invertible on $\mathcal{H}$ because

$$B^* k_{g^{-1}}^{-1/2} B \geq B^* \|k_{g^{-1}}^{1/2}\|^{-1} B = \|k_{g^{-1}}^{1/2}\|^{-1} 1_{\mathcal{H}}.$$

We compute that both

$$\Delta^{1/4} I (B^* k_{g^{-1}}^{-1/2} B) - k_{g^{-1}}^{-1/2} \Delta^{1/4} I = [\Delta^{1/2} P_{\ker d}, k_{g^{-1}}^{-1/2}] B$$

and

$$\begin{aligned} U_g \Delta^{1/4} I U_g^* - k_{g^{-1}}^{-1/2} \Delta^{1/4} I (B^* k_{g^{-1}}^{-1/2} B) \\ = \left(U_g \Delta^{1/4} U_g^* - k_{g^{-1}}^{-1/2} \Delta^{1/4} \right) k_{g^{-1}}^{-1/2} \Delta^{1/4} B - k_{g^{-1}}^{-1/2} \Delta^{1/4} \left[\Delta^{1/4} P_{\ker d}, k_{g^{-1}}^{-1/2} \right] B \end{aligned}$$

are bounded. With $D = \Delta^{1/4}I + I^*\Delta^{1/4} + d + \delta$, the Hodge decomposition theorem $\Omega^n S^{2n-1} = \ker(\Delta) \oplus \text{Im}(d) \oplus \text{Im}(\delta)$ shows that $D^2|_{\mathcal{H}_\infty} = B^*d\delta B$ has at most a finite dimensional kernel, while $D^2|_{\Omega^n S^{2n-1}} = \Delta^{1/2}P_{\ker d}\Delta^{1/2} + \delta d = d\delta + \delta d = \Delta$. On the rest of the complex, D^2 agrees with Δ and so D has compact resolvent. Therefore,

$$(\mathbb{C}, \mathcal{H} \oplus L^2(\Omega^{\geq n} S^{2n}), \Delta^{1/4}I + I^*\Delta^{1/4} + d + \delta)$$

is a conformally $SO(2n, 1)$ -equivariant spectral triple with conformal factors $\mu_g = B^*k_{g^{-1}}^{-1/2}B \oplus k_{g^{-1}}^{-1/2}$. Its bounded transform (more exactly its phase) is the γ -element constructed by Chen [Che96, §3.1].

To show that we have obtained the γ -element, independent of the bounded transform, we would need a representation of $C(\bar{D}^{2n})$ so as to apply [AJV19, Proposition 5.10]. For this purpose, Chen shows that the phase of the larger complex

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{H}_\infty & \xrightarrow{I} & \Omega^n S^{2n-1} & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^{2n-1} S^{2n-1} & \longrightarrow & 0 \\ & & \oplus & & \oplus & & & & \oplus & & \oplus \\ 0 & \longrightarrow & \Omega^0 \mathbb{R}H^{2n} & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^n \mathbb{R}H^{2n} / \mathcal{H}_\infty & \xrightarrow{d} & \Omega^{n+1} \mathbb{R}H^{2n} & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^{2n} \mathbb{R}H^{2n} & \longrightarrow & 0 \end{array}$$

gives a Fredholm module for $C(\bar{D}^{2n})$. Unfortunately, at the level of unbounded Kasparov modules, the construction cannot be carried through because the Hodge-de Rham operator on $\mathbb{R}H^{2n}$ does not have compact resolvent. Although we do not pursue it here, this defect can be remedied by appealing to the framework of relative spectral triples [FGMR19, Fri25]. The larger complex can be assembled into a relative spectral triple for $C_0(\mathbb{R}H^{2n}) \triangleleft C(\bar{D}^{2n})$ in the sense of [Fri25, Definition 2.8] cf. [Fri25, Example 2.15]. We can show that the K -homology class of the relative spectral triple extends to a class for $C(\bar{D}^{2n})$ by showing that the boundary map applied to the class of the relative spectral triple is zero. To compute the boundary map as in [HR00, §8.5], one uses the phase rather than the bounded transform. Since the phase already gives a Fredholm module for all of $C(\bar{D}^{2n})$ the boundary map is zero and we conclude that we do obtain a K -homology class for $C(\bar{D}^{2n})$.

3.3.3 The case of $SU(n, 1)$

Following [JK95], we consider the sphere S^{2n-1} , on which $SU(n, 1)$ acts by CR-automorphisms. This is not a conformal group action. We replace the de Rham complex with the Rumin complex [Rum94], a refinement depending on a contact structure. A detailed discussion of the Rumin complex would be beyond the scope of this paper, especially as the construction of the γ -element does not use the whole complex. A treatment of the Rumin complex in the context of spectral noncommutative geometry and unbounded KK-theory can be found in the upcoming [FGM]. The analytical underpinnings of the Rumin complex, and the much more general class of Rockland complexes, have recently been examined in [DH22]. We limit ourselves to outlining those points which we require.

Let X be a $(2n - 1)$ -dimensional contact manifold with contact structure $H \subseteq TX$. By this, it is meant that there exists a one-form θ such that $H = \ker \theta$ and $d\theta|_H$ is nondegenerate. The nondegeneracy of $d\theta|_H$ is equivalent to $\theta \wedge (d\theta)^{n-1}$ being a volume form. Such a one-form θ is a *contact form* and is not unique. However, if τ is another contact form, then the equality $\ker \tau = \ker \theta$ implies that $\tau = f\theta$ for a nonvanishing smooth function f on X . Conversely, $f\theta$ will be a contact form for any nonvanishing smooth function f on X .

The *Rumin complex* associated to a contact manifold X is a refinement of the de Rham complex of X , depending only on the contact structure (and not on the choice of contact form). For the construction of the Rumin complex on X , we do require a choice of θ , to define two differential ideals of Ω^*X ,

- $\mathcal{I}$, the ideal generated by θ and $d\theta$, and
- $\mathcal{J}$, the ideal of forms $\omega \in \Omega^*X$ such that $\theta \wedge \omega$ and $d\theta \wedge \omega$ are zero.

The Rumin complex is built by combining the quotient complex $\Omega^*X/\mathcal{I}^*$ and the subcomplex $\mathcal{J}^*$. These complexes are spliced together using a map $D_H : \Omega^{n-1}X/\mathcal{I}^{n-1} \rightarrow \mathcal{J}^n$. The *Rumin differential* D_H is given by $\omega \mapsto d\tilde{\omega}$ where $\tilde{\omega}$ is the unique lift of ω such that $\theta \wedge d\tilde{\omega} = 0$. Surprisingly, D_H is well-defined, is a second-order differential operator, and completes the Rumin complex

$$0 \longrightarrow \Omega^0 X \xrightarrow{d_H} \Omega^1 X/\mathcal{I}^1 \xrightarrow{d_H} \dots \xrightarrow{d_H} \Omega^{n-1} X/\mathcal{I}^{n-1} \xrightarrow{D_H} \mathcal{J}^n \xrightarrow{d_H} \dots \xrightarrow{d_H} \mathcal{J}^{2n-1} \longrightarrow 0,$$

whose cohomology coincides with the de Rham cohomology. Here, we have denoted the exterior differential on the quotient complex and subcomplex by d_H . The mixture of first- and second-order operators means that the construction of a spectral triple from the Rumin complex requires careful thought; see the upcoming [FGM]. For the construction of the γ -element of $SU(n, 1)$, however, this issue will not arise, as we shall see.

Let us fix a contact form θ and choose a Riemannian metric $\mathbf{g}$ on X . We require that these be compatible, in the sense that H is orthogonal to the *Reeb field*, the (unique) vector field Z such that $\theta(Z) = 1$ and $\iota_Z(d\theta) = 0$. Using the metric on $\Omega^k X$ induced by $\mathbf{g}$, we obtain a version

$$\star_H : \Omega^k X/\mathcal{I}^k \rightarrow \mathcal{J}^{2n-1-k} \quad \star_H : \mathcal{J}^k \rightarrow \Omega^{2n-1-k} X/\mathcal{I}^{2n-1-k}$$

of the Hodge star operator by the relation $\bar{\alpha} \wedge \star_H \beta = (\alpha, \beta)\theta \wedge (d\theta)^{n-1}$. We thereby obtain formal adjoints of the operators in the Rumin complex, viz. $d_H^* = (-1)^k \star_H d_H \star_H$ and $D^* = (-1)^n \star_H D_H \star_H$. We also obtain the *Rumin Laplacian*, given by

$$\Delta_H = \begin{cases} (n-1-k)d_H d_H^* + (n-k)d_H^* d_H & \text{on } \Omega^k X/\mathcal{I}^k, 0 \leq k \leq n-2 \\ (d_H d_H^*)^2 + D_H^* D_H & \text{on } \Omega^{n-1} X/\mathcal{I}^{n-1} \\ D_H D_H^* + (d_H^* d_H)^2 & \text{on } \mathcal{J}^n \\ (n-k)d_H d_H^* + (n-1-k)d_H^* d_H & \text{on } \mathcal{J}^k, n+1 \leq k \leq 2n-1 \end{cases}.$$

The Rumin Laplacian is hypoelliptic, fourth-order on $\Omega^{n-1}X/\mathcal{I}^{n-1}$ and $\mathcal{J}^n$ and second-order elsewhere.

The contact form θ determines a symplectic form $d\theta$ on H . A CR-structure on X is the additional datum of a complex structure J on H such that $d\theta(X, JY) = \mathbf{g}(X, Y)$ for all $X, Y \in H$. A CR-automorphism of X is a diffeomorphism φ such that the Jacobian φ' preserves and acts complex-linearly on $H \subseteq TX$. Because the Rumin complex depends only on the contact structure, the operators d_H and D_H are unchanged. Again, because the contact structure is preserved, the pullback $\varphi^*(\theta)$ of the contact form must be $f\theta$ for some nonvanishing smooth function on X . Hence

$$\varphi^*(\mathbf{g})(X, Y) = (fd\theta + df \wedge \theta)(X, JY) = fd\theta(X, JY) = f\mathbf{g}(X, Y)$$

for all $X, Y \in H$. The induced metric on TX/H is multiplied by f^2 . One can check that the induced metric on the Rumin complex is multiplied by f^{-k} on $\Omega^k X/\mathcal{I}^k$ and f^{-k-1} on $\mathcal{J}^k$. In this sense, CR-automorphisms behave in a similar way to conformal diffeomorphisms.

To construct the γ -element for $SU(n, 1)$, following [JK95, §6(b)], we begin with the Rumin complex on the sphere S^{2n-1} , on which the group acts by CR-automorphisms. To obtain the correct class in $KK^{SU(n,1)}(\mathbb{C}, \mathbb{C})$ for the γ -element, we will cut the Rumin complex in two, as we did for $SO(2n+1, 1)$ and $SO(2n, 1)$, and add an additional operator, as we did for the latter. The extra map is the Szegő map S constructed in [JK95, Theorem 2.12] from $\Omega^{n-1}S^{2n-1}/\mathcal{I}^{n-1}$ to the L^2 harmonic n -forms $\mathcal{H}^n \subseteq \Omega^n \mathbb{C}H^{2n}$ on the complex hyperbolic space. The sphere S^{2n-1} can be attached to $\mathbb{C}H^{2n}$ as its boundary, forming the closed disc $\bar{D}^{2n}$. The Szegő map takes $\omega \in \Omega^{n-1}S^{2n-1}/\mathcal{I}^{n-1}$, lifts it uniquely to $\tilde{\omega}$ such that $\theta \wedge d\tilde{\omega} = 0$ (as in the construction of D_H), extends $\tilde{\omega}$ to $\eta \in \Omega^{n-1}\mathbb{C}H^{2n}$ so that $d\eta \in L^2(\Omega^n \mathbb{C}H^{2n})$, and then projects η down to $S\omega \in \mathcal{H}^n$. It turns out that such a process

gives a well-defined map, whose kernel is $\ker D_H$. We dissect the Rumin complex and graft in the Szegő map S , obtaining

$$0 \longrightarrow \Omega^0 S^{2n-1} \xrightarrow{d_H} \Omega^1 S^{2n-1} / \mathcal{I}^1 \xrightarrow{d_H} \dots \xrightarrow{d_H} \Omega^{n-1} S^{2n-1} / \mathcal{I}^{n-1} \xrightarrow{S} \mathcal{H}_\infty^n \longrightarrow 0$$

where $\mathcal{H}_\infty^n$ is the image of S , dense in $\mathcal{H}^n$. This complex is invariant under pullback by the action φ of $SU(n, 1)$. When we complete the spaces of the complex to Hilbert spaces, pullback by the action of $SU(n, 1)$ is not unitary. The unitary action is, for $\omega \in L^2(\Omega^k / \mathcal{I}^k)$ and $\xi \in \mathcal{H}^n$,

$$U_g \omega = f_{g^{-1}}^{\frac{n-k}{2}} \varphi_{g^{-1}}^* \omega \quad U_g \xi = \varphi_{g^{-1}}^* \xi,$$

where $(f_g)_{g \in SU(n, 1)}$ is a family of nonvanishing, positive, smooth functions on S^{2n-1} . By similar computations to those for Example 2.12, for the unitary implementors U_g we have that

$$U_g d_H U_g^* \omega = f_{g^{-1}}^{\frac{n-(k+1)}{2}} d_H f_{g^{-1}}^{-\frac{n-k}{2}} \omega = f_{g^{-1}}^{-\frac{1}{2}} d_H \omega + f_{g^{-1}}^{\frac{n-(k+1)}{2}} [d_H, f_{g^{-1}}^{-\frac{n-k}{2}}] \omega$$

so that $U_g d_H U_g^* - f_{g^{-1}}^{-1/4} d_H f_{g^{-1}}^{-1/4}$ is bounded. On the hyperbolic space $\mathbb{C}H^n$, the group $SU(n, 1)$ acts by isometries. Because the map S commutes with pullback by the group action, $U_g S U_g^* = S f_{g^{-1}}^{-1/2}$. Unlike in the case of $SO(2n, 1)$, there is no discrepancy between the conformal behaviours of d_H and S . It remains to construct a conformal factor on $\mathcal{H}^n$. By [JK95, Proof of Theorem 6.6(ii)], there is a polar decomposition $S = \Phi(S) \Delta_H^{1/4}$, where $\Phi(S) : L^2(\Omega^{n-1} S^{2n-1} / \mathcal{I}^{n-1}) \rightarrow \mathcal{H}^n$ is a coisometry with kernel $\ker D_H$. The operator $\Phi(S) f_{g^{-1}}^{-1/4} \Phi(S)^*$ is positive and invertible on $\mathcal{H}^n$ because

$$\Phi(S) f_{g^{-1}}^{-1/4} \Phi(S)^* \geq \Phi(S) \|f_{g^{-1}}^{1/4}\|^{-1} \Phi(S)^* = \|f_{g^{-1}}^{1/4}\|^{-1}.$$

We compute that both

$$S f_{g^{-1}}^{-1/4} - \left(\Phi(S) f_{g^{-1}}^{-1/4} \Phi(S)^* \right) S = \Phi(S) \left[(1 - \ker D) \Delta_H^{-1/4}, f_{g^{-1}}^{-1/4} \right]$$

and

$$U_g S U_g^* - \left(\Phi(S) f_{g^{-1}}^{-1/4} \Phi(S)^* \right) S f_{g^{-1}}^{-1/4} = \Phi(S) \left[(1 - \ker D) \Delta_H^{-1/4}, f_{g^{-1}}^{-1/4} \right] f_{g^{-1}}^{-1/4}$$

are bounded. The operator $d_H + d_H^* + S + S^*$ has compact resolvent by an argument very similar to the case of $SO(2n, 1)$, using this time the compactness of the resolvent of the Rumin Laplacian [JK95, Corollary 5.20]. For example, on $\Omega^{n-1} S^{2n-1} / \mathcal{I}^{n-1}$ one can check that

$$\begin{aligned} (d_H + d_H^* + S + S^*)^2|_{\Omega^{n-1} S^{2n-1} / \mathcal{I}^{n-1}} &= \Delta_H^{1/4} (1 - \ker D) \Delta_H^{1/4} + d_H d_H^* \\ &= (D_H^* D_H)^{1/2} + d_H d_H^* \\ &= \Delta_H^{1/2}, \end{aligned}$$

and the other cases are similar. In summary, we have constructed a conformally $SU(n, 1)$ -equivariant spectral triple

$$(\mathbb{C}, L^2(\Omega^{\leq n-1} S^{2n-1} / \mathcal{I}^{\leq n-1}) \oplus \mathcal{H}^n, d_H + d_H^* + S + S^*)$$

with conformal factors $\mu_g = f_{g^{-1}}^{-1/4} \oplus \Phi(S) f_{g^{-1}}^{-1/4} \Phi(S)^*$. The phase of this spectral triple is exactly the Fredholm module of [JK95, Corollary 6.10] whose class is $\gamma \in KK^{SU(n, 1)}(\mathbb{C}, \mathbb{C})$.

To show that we have obtained the γ -element without directly using the result of Julg and Kasparov, it would be necessary, as in the case of $SO(2n, 1)$ to expand the complex to accommodate a representation of $\bar{D}^{2n}$. However, as before, the resolvent would not be compact. Furthermore, it is unclear whether sufficient analytical tools are available to obtain bounded commutators.

3.4 C*-algebra of the Heisenberg group

In this section we give a truly noncommutative example of conformal equivariance, building a conformally equivariant higher-order spectral triple for the C*-algebra of the Heisenberg group. An element of the 3-dimensional Heisenberg group H^3 can be written as

$$\begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix}$$

for $a, b, c \in \mathbb{R}$. There is an action of $\mathbb{R}_+^\times$ on H^3 by automorphisms, given for $t \in \mathbb{R}_+^\times$ by

$$\begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & ta & t^2c \\ & 1 & tb \\ & & 1 \end{pmatrix}.$$

We will construct a conformally equivariant higher-order spectral triple for $C^*(H^3)$. Define a Clifford algebra-valued function $\ell : H^3 \rightarrow \mathcal{C}\ell_3$ by

$$\ell : \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \mapsto (a\gamma_1 + b\gamma_2)(a^2 + b^2)^{1/2} + c\gamma_3.$$

With

$$g = \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \quad h = \begin{pmatrix} 1 & a' & c' \\ & 1 & b' \\ & & 1 \end{pmatrix} \quad gh = \begin{pmatrix} 1 & a + a' & c + c' + ab' \\ & 1 & b + b' \\ & & 1 \end{pmatrix}$$

we can check that

$$\begin{aligned} \ell(gh) - \ell(h) &= ((a + a')\gamma_1 + (b + b')\gamma_2)((a + a')^2 + (b + b')^2)^{1/2} + (c + c' + ab')\gamma_3 \\ &\quad - (a'\gamma_1 + b'\gamma_2)(a'^2 + b'^2)^{1/2} + c'\gamma_3 \\ &= (a'\gamma_1 + b'\gamma_2)((a + a')^2 + (b + b')^2)^{1/2} - (a'^2 + b'^2)^{1/2} \\ &\quad + (a\gamma_1 + b\gamma_2)((a + a')^2 + (b + b')^2)^{1/2} + (c + ab')\gamma_3 \end{aligned}$$

and

$$(1 + \ell(h)^2)^{1/2} = \left(1 + (a'^2 + b'^2)^2 + c'^2\right)^{1/2}.$$

Hence $(\ell(gh) - \ell(h))(1 + \ell(h)^2)^{-1/4}$ is uniformly bounded in $h \in G$. A computation then shows that, for $f \in C_c(H^3)$, the operator $[M_\ell, f](1 + M_\ell^2)^{-1/4} = [M_\ell, f]\langle M_\ell \rangle^{-1/2}$ is bounded where M_ℓ is multiplication by ℓ . We arrive at the order-2 spectral triple $(C^*(H^3), L^2(H^3, \mathbb{C}^2), M_\ell)$. The local compactness of the resolvent is a consequence of $(1 + \ell^2)^{-1} \in C_0(H^3, \mathcal{C}\ell_3)$ and the isomorphism $C_0(H^3) \rtimes H^3 \cong K(L^2(H^3))$. Let $V_t \in B(L^2(H^3))$ be given by the pullback

$$V_t \xi(a, b, c) = \xi(t^{-1}a, t^{-1}b, t^{-2}c)$$

on $\xi \in L^2(H^3)$. Then

$$\langle V_t^* \xi | \eta \rangle = \int \xi(t^{-1}a, t^{-1}b, t^{-2}c) \eta(a, b, c) da db dc = \int \xi(x, y, z) \eta(tx, ty, t^2z) t^4 dx dy dz = t^4 \langle \xi | V_{t^{-1}} \eta \rangle$$

so that $V_t^* = t^4 V_{t^{-1}}$. The unitary in the polar decomposition is given by $U_t = t^{-2} V_t$. Noting that

$$\ell(ta, tb, t^2c) = t^2 \ell(a, b, c)$$

we see that the operator M_ℓ transforms as

$$\begin{aligned}
(U_t M_\ell U_t^* \xi)(a, b, c) &= t^{-2} (M_\ell U_t^* \xi)(t^{-1}a, t^{-1}b, t^{-2}c) \\
&= t^{-2} \ell(t^{-1}a, t^{-1}b, t^{-2}c) (U_t^* \xi)(t^{-1}a, t^{-1}b, t^{-2}c) \\
&= t^{-2} \ell(a, b, c) \xi(a, b, c) \\
&= t^{-2} (M_\ell \xi)(a, b, c)
\end{aligned}$$

on a vector $\xi \in L^2(H^3, \mathbb{C}^2)$. In summary, we have:

Proposition 3.28. *The data $(C^*(H^3), L^2(H^3, \mathbb{C}^2), M_\ell)$, together with the action $(U_t)_{t \in \mathbb{R}}$ of the group $\mathbb{R}_+^\times$ and conformal factors given by $\mu_t = t^{-1}$, constitute a conformally $\mathbb{R}_+^\times$ -equivariant 2nd-order spectral triple.*

The C^* -algebra of the Heisenberg group can be identified with a continuous field of Moyal planes (with one classical plane) over $\mathbb{R}$ [ENN93, §4]. In this picture, the group action is dilation on $\mathbb{R}$ and a corresponding scaling of the parameters of the Moyal planes. A generalisation of the construction in this section to all Carnot groups and their dilations will appear in [FGM].

4 Quantum-group-equivariant KK theory

Conformal group actions of a nontrivial kind are already rare in the classical setup of Riemannian manifolds, as the Ferrand–Obata theorem [Fer96, Theorem A] shows. The conformal group of a Riemannian metric must be the isometry group of a conformally equivalent metric, unless the manifold is conformally equivalent to a round sphere S^n or Euclidean space $\mathbb{R}^n$. It seems that the rarity of large conformal groups carries over to the noncommutative setting. A possible example of a noncommutative geometry with interesting conformal group is the Podleś sphere. As we shall see in §4.4, this hope is realised; however the conformal geometry of the Podleś sphere is not governed by a group but rather by a quantum group. Quantum-group-equivariant KK-theory, in the bounded picture, is due to Baaj and Skandalis [BS89]. A detailed account can be found in [Ver02]. We first recall the notions of a C^* -bialgebra and a *locally compact quantum group*.

Definition 4.1. e.g. [Tim08, Definitions 4.1.1,3] A C^* -bialgebra is a C^* -algebra S equipped with a *comultiplication* map, a coassociative, nondegenerate $*$ -homomorphism $\Delta : S \rightarrow M(S \otimes S)$ such that $\Delta(S)(S \otimes 1)$ and $(1 \otimes S)\Delta(S)$ are contained in $S \otimes S$. A C^* -bialgebra S is *simplifiable* if

$$\overline{\text{span}}(\Delta(S)(S \otimes 1)) = S \otimes S = \overline{\text{span}}((1 \otimes S)\Delta(S)).$$

A *von Neumann bialgebra* is a von Neumann algebra M with a *comultiplication* map, a coassociative, unital, normal $*$ -homomorphism $\Delta : M \rightarrow M \otimes M$, the von Neumann tensor product.

Commutative C^* -bialgebras are in duality with certain topological semigroups, the simplifiability property being related to regularity; see [Val85, §3] for precise statements.

Definition 4.2. e.g. [Tim08, Chapter 8] A *locally compact quantum group* $\mathbb{G}$ is given by the equivalent data of either:

- A simplifiable C^* -bialgebra $C_0^r(\mathbb{G})$ with left- and right-invariant, KMS, faithful weights; or
- A von Neumann bialgebra $L^\infty(\mathbb{G})$ with left- and right-invariant, normal, semifinite, faithful weights.

For the precise meaning of the adjectives on the weights, see e.g. [Tim08, §8.1.1-2], but we will not use these details. From such data, one obtains:

- The Hilbert space $L^2(\mathbb{G})$, on which $L^\infty(\mathbb{G})$ and $C_0^r(\mathbb{G})$ are represented, obtained by the GNS construction from the left Haar weight (of either algebra);
- The *universal* function algebra $C_0^u(\mathbb{G})$, which surjects onto $C_0^r(\mathbb{G})$;
- The *dual* locally compact quantum group $\hat{\mathbb{G}}$, for which $L^2(\hat{\mathbb{G}}) \cong L^2(\mathbb{G})$, and the C^* -algebras $C_r^*(\mathbb{G}) := C_0^r(\hat{\mathbb{G}})$ and $C_u^*(\mathbb{G}) := C_0^u(\hat{\mathbb{G}})$;
- The *multiplicative unitary* $W \in M(C_0^r(\mathbb{G}) \otimes C_0^r(\hat{\mathbb{G}})) \subseteq B(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$ satisfying the equation $W_{12}W_{13}W_{23} = W_{23}W_{12}$ and, for $a \in C_0^r(\mathbb{G})$, $\Delta(a) = W^*(1 \otimes a)W$ on $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$; and
- A Banach algebra $L^1(\mathbb{G}) := L^\infty(\mathbb{G})_*$, the predual of $L^\infty(\mathbb{G})$.

We next recall the details of C^* -bialgebra-coactions on C^* -algebras and Hilbert modules.

Definition 4.3. [EKQR06, Definitions 1.39, A.3] Let B and C be C^* -algebras. The *C -multiplier algebra* of $B \otimes C$ is

$$M_C(B \otimes C) = \{m \in M(B \otimes C) \mid m(1 \otimes C) \cup (1 \otimes C)m \in B \otimes C\}.$$

If E is a Hilbert B -module, the *C -multiplier module* of $E \otimes S_{B \otimes S}$ is the Hilbert $M_C(B \otimes C)$ -module

$$M_C(E \otimes C) = \{m \in \text{Hom}_{B \otimes C}^*(B \otimes C, E \otimes C) \mid m(1 \otimes C) \cup (1 \otimes C)m \in E \otimes C\}.$$

Definition 4.4. [BS89, §2], [Ver02, §3.1] A coaction of a C^* -bialgebra S on a C^* -algebra B is a coassociative nondegenerate $*$ -homomorphism $\delta_B : B \rightarrow M_S(B \otimes S)$. A coaction of S on a Hilbert B -module E is a coassociative $\mathbb{C}$ -linear map $\delta_E : E \rightarrow M_S(E \otimes S)$ such that

- $\delta_E(\xi)\delta_B(b) = \delta_E(\xi b)$ and $\langle \delta_E(\xi) \mid \delta_E(\eta) \rangle_{M_S(B \otimes S)} = \delta_B(\langle \xi \mid \eta \rangle_B)$ for all $\xi, \eta \in E$ and $b \in B$; and
- $\delta_E(E)(B \otimes S)$ is dense in $E \otimes S$.

Let $E \otimes_{\delta_B}(B \otimes S)$ be the internal tensor product of Hilbert modules where the left action of B on $B \otimes S$ is given by δ_B . For an element $\xi \in E$, denote by $T_\xi \in \text{Hom}_{B \otimes S}^*(B \otimes S, E \otimes_{\delta_B}(B \otimes S))$ the map $b \otimes s \mapsto \xi \otimes_{\delta_B}(b \otimes s)$. A unitary $V_E \in \text{Hom}_{B \otimes S}^*(E \otimes_{\delta_B}(B \otimes S), E \otimes S)$ is *admissible* if

- $V_E T_\xi \in M_S(E \otimes S)$ for all $\xi \in E$; and
- $(V_E \otimes_{\mathbb{C}} 1)(V_E \otimes_{\delta_B \otimes \text{id}_S} 1) = (V_E \otimes_{\text{id}_B \otimes \Delta_S} 1) \in \text{Hom}_{B \otimes S \otimes S}^*(E \otimes_{\delta_B^2}(B \otimes S \otimes S), E \otimes S \otimes S)$, where $\delta_B^2 = (\delta_B \otimes \text{id}_S)\delta_B = (\text{id}_B \otimes \Delta_S)\delta_B$.

A coaction on E can equivalently be described by an admissible unitary V_E using the identity $V_E T_\xi = \delta_E(\xi)$ for $\xi \in E$.

If A is a C^* -algebra with an S -coaction δ_A , an A - B -correspondence E is *S -equivariant* if it possesses a Hilbert B -module coaction δ_E such that

$$\delta_A(a)\delta_E(\xi) = \delta_E(a\xi)$$

for all $a \in A$ and $\xi \in E$. In terms of the admissible unitary, this is equivalent to $V_E(a \otimes 1)V_E^* = \delta_A(a)$.

Definition 4.5. cf. [Pod95, Definition 1.4(b)], [BSV03, §5.2] Let S be a C^* -bialgebra. An S -coaction δ_B on a C^* -algebra B satisfies the *Podleś condition* (sometimes called simply *continuity*) if $\overline{\text{span}}(\delta_B(B)(1 \otimes S)) = B \otimes S$. An S -coaction δ_E on a Hilbert B -module E then automatically satisfies

$$\overline{\text{span}}(\delta_E(E)(1 \otimes S)) = \overline{\text{span}}(\delta_E(E)\delta_B(B)(1 \otimes S)) = \overline{\text{span}}(\delta_E(E)(B \otimes S)) = E \otimes S$$

and $\overline{V_E(E \otimes_{\delta_B}(1 \otimes S))}$ is dense in $E \otimes S$.

Definition 4.6. An *action* of a locally compact quantum group $\mathbb{G}$ on a C^* -algebra B is a $C_0^r(\mathbb{G})$ -coaction on B satisfying the Podleś condition. A $\mathbb{G}$ -action on a Hilbert B -module E is a $C_0^r(\mathbb{G})$ -coaction on E .

Definition 4.7. [BS89, Définition 3.1] cf. [NV10, §4] Let A and B be C^* -algebras equipped with coactions of a C^* -bialgebra S . A bounded Kasparov A - B -module (A, E_B, F) is *S -equivariant* if E is an S -equivariant A - B -correspondence and for all $a \in A$ and $s \in S$

$$(V_E(F \otimes_{\delta_B} 1)V_E^* - F \otimes 1)a \otimes s$$

is compact. If A and B are C^* -algebras with $\mathbb{G}$ -actions, a bounded Kasparov module (A, E_B, F) is $\mathbb{G}$ -equivariant if it is $C_0^r(\mathbb{G})$ -equivariant.

4.1 Uniform quantum group equivariance

From now on, we leave the generality of higher-order Kasparov modules and focus on unbounded Kasparov modules in the interests of readability. We make the following definition in the unbounded setting. To our knowledge, except in the case of the isometric coaction of a compact quantum group (see e.g. [GB16, Definition 2.3.1]), such a definition has not appeared in the published literature (but see [Gof09, Definition 3.3.1]).

Definition 4.8. Let A and B be C^* -algebras equipped with coactions of a C^* -bialgebra S . For $a \in \text{Lip}_0^*(E)$ let

$$\mathcal{S}_a = \{s \in S \mid a \otimes s \text{ dom}(D \otimes 1) \subseteq V_E \text{ dom}(D \otimes_{\delta_B} 1) \text{ and } (V_E(D \otimes_{\delta_B} 1)V_E^* - D \otimes 1)a \otimes s \in \text{End}^*(E \otimes S)\}.$$

An unbounded Kasparov A - B -module (A, E_B, D) is *uniformly S -equivariant* if E is an S -equivariant A - B -correspondence and A is contained in the closure of

$$\mathcal{Q} = \left\{ a \in \text{Lip}_0^*(D) \mid \overline{\mathcal{S}_a} = S \right\}.$$

If $V_E(D \otimes_{\delta_B} 1)V_E^* = D \otimes 1$, we say that the cycle is *isometrically equivariant*.

If A and B are C^* -algebras with $\mathbb{G}$ -actions, an unbounded Kasparov module (A, E_B, D) is *uniformly $\mathbb{G}$ -equivariant* if it is uniformly $C_0^r(\mathbb{G})$ -equivariant.

If $\mathcal{A}$ is a dense $*$ -subalgebra of A such that $\mathcal{A} \subseteq \mathcal{Q}$, we say that $(\mathcal{A}, E_B, D)$ is *S -equivariant* (or $\mathbb{G}$ -equivariant, as the case may be).

Remark 4.9. The dense subset $\mathcal{S}_a \subseteq S$ need not be the same for different $a \in \mathcal{Q}$. For many locally compact quantum groups, there may be a natural choice, fixed for all a . For a discrete quantum group $\mathbb{G}$, i.e. when $C_0(\mathbb{G})$ is isomorphic as an algebra to the C^* -algebraic direct sum

$$\bigoplus_{\lambda \in \Lambda} M_{n_\lambda}(\mathbb{C})$$

of finite-dimensional matrix algebras, $\mathcal{S}_a$ would contain all elements of the algebraic direct sum. In this case, the admissible unitary would be labelled by the index set $\lambda \in \Lambda$, so that

$$V_E^\lambda \in \text{Hom}_B^*(E \otimes_{\delta_B} (B \otimes \mathbb{C}^{n_\lambda}), E \otimes \mathbb{C}^{n_\lambda})$$

and the equivariance condition becomes that

$$(V_E^\lambda(D \otimes_{\delta_B} 1)V_E^{\lambda*} - D \otimes 1)a \otimes 1_{\mathbb{C}^{n_\lambda}}$$

be bounded for all $\lambda \in \Lambda$. For the dual $\hat{G}$ of a group G , we suspect it always makes sense to assume that $\mathcal{S}_a$ contains the right ideal $C_r^*(G)^\infty$ of smooth elements [WN92, §§2–3], as in Example 4.11.

Theorem 4.10. *A uniformly S -equivariant unbounded Kasparov module (A, E_B, D) gives rise to an S -equivariant bounded Kasparov module (A, E_B, F_D) .*

Proof. The only difference from the non-equivariant case is the need to show that, for every $a \in A$ and $s \in S$, $(F_D \otimes 1 - V_E(F_D \otimes_{\delta_B} 1)V_E^*)a \otimes s$ is compact. Let $b \in \mathcal{Q}$ and $s \in \mathcal{S}_a$ so that

$$(V_E(D \otimes_{\delta_B} 1)V_E^* - D \otimes 1)b \otimes s$$

extends to an adjointable operator. By Corollary 2.22,

$$(V_E(F_D \otimes_{\delta_B} 1)V_E^* - F_D \otimes 1)(b \otimes s)\langle D \rangle^\beta \otimes 1$$

is bounded for all $\beta < 1$. With $c \in A$,

$$(V_E(F_D \otimes_{\delta_B} 1)V_E^* - F_D \otimes 1)bc \otimes s = (V_E(F_D \otimes_{\delta_B} 1)V_E^* - F_D \otimes 1)(b \otimes s)(\langle D \rangle^\beta \otimes 1)\langle D \rangle^{-\beta} c \otimes 1$$

is compact and, by the density of $\mathcal{S}_a \subseteq S$ and the inclusion of $A \subseteq \overline{\mathcal{Q}A}$, we are done. $\square$

Example 4.11. Let G be a connected Lie group with a left-invariant Riemannian metric $\mathbf{g}$, such as the affine group $\mathbb{R} \rtimes \mathbb{R}_+^\times$ of the real line as the real hyperbolic plane. The left-invariant Riemannian metric on G is exactly determined by the inner product $\mathbf{g}_e$ on the tangent space $T_e G = \mathfrak{g}$ at the identity $e \in G$. The left-invariant differential operators and differential forms on G can be identified with $U(\mathfrak{g})$ and $\Lambda^*(\mathfrak{g})$, respectively. The Clifford algebra $\mathcal{C}\ell(\mathfrak{g})$ acts on the left of $\Lambda^*(\mathfrak{g})$. The Hodge–de Rham Dirac operator $d + \delta$ on $(G, \mathbf{g})$ can be written as

$$d + \delta = \sum_{i=1}^{\dim \mathfrak{g}} X_i \otimes \gamma_i,$$

where $X_i \in \mathfrak{g} \subseteq U(\mathfrak{g})$ and $\gamma_i \in \mathfrak{g} \subseteq \mathcal{C}\ell(\mathfrak{g})$. We have an isometrically G -equivariant spectral triple

$$(C_0(G), L^2(G, \Lambda^*(\mathfrak{g})), d + \delta).$$

By Baaj–Skandalis duality [BS89, §6], it is reasonable to expect that

$$(\mathbb{C}, (C_r^*(G) \otimes \Lambda^*(\mathfrak{g}))_{C_r^*(G)}, d + \delta)$$

is a uniformly $\hat{G}$ -equivariant $\mathbb{C}$ - $C_r^*(G)$ -unbounded Kasparov module where, by an abuse of notation, $d + \delta \in U(\mathfrak{g}) \otimes \mathcal{C}\ell(\mathfrak{g})$ is considered to be an unbounded operator on $C_r^*(G) \otimes \Lambda^*(\mathfrak{g})$; see [WN92, §3]. To see this, first consider the coaction on the module $(C_r^*(G) \otimes \Lambda^*(\mathfrak{g}))_{C_r^*(G)}$. The admissible unitary is a map from

$$(C_r^*(G) \otimes \Lambda^*(\mathfrak{g})) \otimes_{\delta_{C_r^*(G)}} (C_r^*(G) \otimes C_r^*(G)) = C_r^*(G) \otimes \Lambda^*(\mathfrak{g}) \otimes C_r^*(G)$$

to

$$(C_r^*(G) \otimes \Lambda^*(\mathfrak{g})) \otimes_{\mathbb{C}} C_r^*(G) = C_r^*(G) \otimes \Lambda^*(\mathfrak{g}) \otimes C_r^*(G).$$

Under these identifications,

$$T_{x \otimes \psi} : C_r^*(G) \otimes C_r^*(G) \rightarrow C_r^*(G) \otimes \Lambda^*(\mathfrak{g}) \otimes C_r^*(G) \quad y \otimes z \mapsto x_{(1)}y \otimes \psi \otimes x_{(2)}z$$

$$x_{(1)} \otimes \psi \otimes x_{(2)} = \delta(x \otimes \psi) = VT_x = V(x_{(1)} \otimes \psi \otimes x_{(2)})$$

so V is just the identity in $\text{End}_{C_r^*(G)}^*(C_r^*(G) \otimes \Lambda^*(\mathfrak{g}) \otimes C_r^*(G))$. Because $X_i \in \mathfrak{g}$, in the universal enveloping algebra $U(\mathfrak{g})$, $\Delta X_i = \bar{X}_i \otimes 1 + 1 \otimes X_i$ and

$$(d + \delta) \otimes_{\delta_{C_r^*(G)}} 1 = \sum_i (X_i \otimes \gamma_i) \otimes_{\Delta_{U(\mathfrak{g})}} 1 = \sum_i (X_i \otimes \gamma_i \otimes 1 + 1 \otimes \gamma_i \otimes X_i).$$

Therefore,

$$V((d + \delta) \otimes_{\delta_{C_r^*(G)}} 1)V^* - (d + \delta) \otimes 1 = 1 \otimes \gamma_i \otimes X_i.$$

For $(\mathbb{C}, (C_r^*(G) \otimes \Lambda^*(\mathfrak{g}))_{C_r^*(G)}, d + \delta)$ to be $C_r^*(G)$ -equivariant, we require a dense subalgebra of $C_r^*(G)$ in the common domain of the derivations $\mathfrak{g}$. There is in fact such a subalgebra, the right ideal $C_r^*(G)^\infty$ of smooth elements for the G -action on $C_r^*(G)$ by unitary multipliers [WN92, §§2–3].

4.2 Descent and the dual-Green–Julg map for uniform equivariance

Crossed products are not defined in the generality of Hopf C^* -algebra–coactions. One needs a well-defined notion of duality and, for that, we restrict to locally compact quantum groups. (It is possible to work in the greater generality of a weak Kac system [Ver02, §2.2], but we forgo this in the interests of readability.)

We use the symbol Σ for the flip map on a tensor product.

Definition 4.12. [Tim08, Definition 7.3.1] cf. [BS93, Proposition 3.2, Définition 3.3] A locally compact quantum group $\mathbb{G}$ is *regular* if

$$\overline{\text{span}}\{(\omega \otimes 1)(W\Sigma) \mid \omega \in B(L^2(\mathbb{G}))_*\} = K(L^2(\mathbb{G})).$$

Equivalently, $\mathbb{G}$ is regular if the reduced crossed product $C_0^r(\mathbb{G}) \rtimes_r \mathbb{G} \cong K(L^2(\mathbb{G}))$; see Definition 4.14 below.

Lemma 4.13. *Let E be a Hilbert B -module with a $\mathbb{G}$ action, $\mathbb{G}$ acting trivially on B . Then $C_u^*(\mathbb{G})$ is represented on E . Conversely, if $\mathbb{G}$ is a regular quantum group, a (nondegenerate) representation of $C_u^*(\mathbb{G})$ on a Hilbert B -module gives rise to a $\mathbb{G}$ action on E which is trivial on B .*

Proof. Let E be a Hilbert B -module with a $\mathbb{G}$ action, $\mathbb{G}$ acting trivially on B . The fundamental unitary V_E is then an element of $\text{End}^*(E \otimes C_0^r(\mathbb{G}))$ and can be thought of as an element of $\text{End}^*(E \otimes L^2(\mathbb{G}))$ by the left regular representation of $C_0^r(\mathbb{G})$. By [Kus01, Proposition 5.2], there is a nondegenerate representation of $C_u^*(\mathbb{G})$ on E .

On the other hand, suppose that $C_u^*(\mathbb{G})$ is represented nondegenerately by π on a Hilbert B -module E . Let $\hat{V} \in M(C_0^r(\mathbb{G}) \otimes C_u^*(\mathbb{G}))$ be the unitary of [Kus01, Proposition 4.2]. By [Kus01, Corollary 4.3], we obtain an element $X = (\pi \otimes \text{id})(\Sigma \hat{V} \Sigma) \in \text{End}^*(E \otimes S)$ such that $(1 \otimes \Delta)(X) = X_{12}X_{13}$. The only thing stopping X from being the admissible unitary of an action of $\mathbb{G}$ on E (with trivial action on B) is the possible failure of $(1 \otimes C_0^r(\mathbb{G}))X(E \otimes 1)$ to be contained in $E \otimes C_0^r(\mathbb{G})$. One might expect

$$(A \otimes 1)U(1 \otimes B) \subseteq A \otimes B$$

to hold automatically for a unitary $U \in M(A \otimes B)$ but this is not the case, as [LPRS87, Remark after Lemma 1.2] shows. If, however, we assume $\mathbb{G}$ to be regular, by [BS93, Proposition A.3(d)],

$$\overline{\text{span}}(1 \otimes C_0^r(\mathbb{G}))X(\pi(C_u^*(\mathbb{G})) \otimes 1) = \pi(C_u^*(\mathbb{G})) \otimes C_0^r(\mathbb{G})$$

and therefore

$$(1 \otimes C_0^r(\mathbb{G}))X(E \otimes 1) = (1 \otimes C_0^r(\mathbb{G}))X(\pi(C_u^*(\mathbb{G}))E \otimes 1) \subseteq E \otimes C_0^r(\mathbb{G}),$$

as required. $\square$

It is unclear if the converse statement of Lemma 4.13 is true without the assumption of regularity.

Definition 4.14. cf. [Ver02, Définitions 4.2, 5.1, Lemmes 4.1, 5.2] Let A be a C^* -algebra with a $\mathbb{G}$ -action. The *reduced* crossed product $A \rtimes_r \mathbb{G}$ is given by

$$\overline{\text{span}}(\delta_A(A)(1 \otimes C_r^*(\mathbb{G}))) \subseteq M(A \otimes K(L^2(\mathbb{G}))).$$

Let ${}_\pi E$ be a $\mathbb{G}$ -equivariant A - B -correspondence, with $\mathbb{G}$ acting trivially on B . There is an *integrated* representation of the *universal* crossed product $A \rtimes_u \mathbb{G}$ on E whose image is

$$\overline{\text{span}}(\pi(A)C_u^*(\mathbb{G})) \subseteq \text{End}^*(E).$$

If $\mathbb{G}$ is regular, the algebra $A \rtimes_u \mathbb{G}$ is universal for such integrated representations; if $\mathbb{G}$ is not regular $A \rtimes_u \mathbb{G}$ is universal for a slightly larger class of representations; see [Ver02, Définition 4.2] and [Vae05, §2.3]. There is a canonical surjection $A \rtimes_u \mathbb{G} \rightarrow A \rtimes_r \mathbb{G}$.

Let E be a right Hilbert B -module with an action of $\mathbb{G}$. For either topology $t \in \{u, r\}$, the crossed product Hilbert module $E \rtimes_t \mathbb{G}$ is given by the internal tensor product $E \otimes_B (B \rtimes_t \mathbb{G})$. By [Ver02, Lemme 5.2], $\text{End}_B^0(E) \rtimes_t \mathbb{G}$ is naturally identified with $\text{End}_{B \rtimes_t \mathbb{G}}^0(E \rtimes_t \mathbb{G})$.

In the locally compact quantum group setting, there is a descent map

$$j_t^{\mathbb{G}} : KK^{\mathbb{G}}(A, B) \rightarrow KK(A \rtimes_t \mathbb{G}, B \rtimes_t \mathbb{G})$$

for either topology $t \in \{u, r\}$, universal or reduced, generalising Kasparov's descent map for classical groups. If $\mathbb{G}$ is the dual of a classical group, descent is due to Baaj and Skandalis [BS89, Théorème 6.19], and in general due to Vergnioux [Ver02, Proposition 5.3]. In the locally compact quantum group setting, a refinement of the reduced descent is possible, to a map

$$J_r^{\mathbb{G}} : KK^{\mathbb{G}}(A, B) \rightarrow KK^{\hat{\mathbb{G}}}(A \rtimes_r \mathbb{G}, B \rtimes_r \mathbb{G})$$

whose composition with the forgetful functor $KK^{\hat{\mathbb{G}}} \rightarrow KK$ is $j_r^{\mathbb{G}}$. If $\mathbb{G}$ is regular, $C_0^r(\mathbb{G}) \rtimes_r \mathbb{G} \cong K(L^2(\mathbb{G})) \cong C_r^*(\mathbb{G}) \rtimes_r \hat{\mathbb{G}}$ and the maps $J_r^{\mathbb{G}}$ and $J_r^{\hat{\mathbb{G}}}$ are mutually inverse isomorphisms [BS93, Remarque 7.7(b)].

Proposition 4.15. [Ver02, Proposition 5.3] *Let (A, E_B, F) be a $\mathbb{G}$ -equivariant bounded Kasparov module. For $t \in \{u, r\}$, let ι be the inclusion $\text{End}^0(E) \rightarrow M(\text{End}^0(E) \rtimes_t \mathbb{G}) \cong \text{End}_{B \rtimes_t \mathbb{G}}^*(E \rtimes_t \mathbb{G})$. Then $(A \rtimes_t \mathbb{G}, (E \rtimes_t \mathbb{G})_{B \rtimes_t \mathbb{G}}, \iota(F))$ is a bounded Kasparov module.*

When G acts trivially on B , there is a dual-Green–Julg map for the universal crossed product

$$\Psi^{\mathbb{G}} : KK^{\mathbb{G}}(A, B) \rightarrow KK(A \rtimes_u \mathbb{G}, B)$$

which is an isomorphism when $\mathbb{G}$ is discrete [Ver02, Proposition 5.11].

Proposition 4.16. [Ver02, Proposition 5.11] *Let (A, E_B, F) be a $\mathbb{G}$ -equivariant bounded Kasparov module, with $\mathbb{G}$ acting trivially on B . Then $(A \rtimes_u \mathbb{G}, E_B, F)$ is a bounded Kasparov module, with the integrated representation of $A \rtimes_u \mathbb{G}$.*

Proposition 4.17. [Ver02, Proposition 5.11] *Let $(A \rtimes_u \mathbb{G}, E_B, F)$ be a bounded Kasparov module, with $\mathbb{G}$ a discrete quantum group and $A \rtimes_u \mathbb{G}$ represented nondegenerately on E . Then (A, E_B, F) is a $\mathbb{G}$ -equivariant bounded Kasparov module, with the coaction of $C_0^r(\mathbb{G})$ on E given by the action of $C_u^*(\mathbb{G}) \subseteq M(A \rtimes_u \mathbb{G})$ on E , acting trivially on B .*

In the unbounded setting, we have the following picture of descent.

Proposition 4.18. *Let (A, E_B, D) be a uniformly $\mathbb{G}$ -equivariant unbounded Kasparov module. For $t \in \{u, r\}$, let ι be the inclusion $\text{End}^0(E) \rightarrow M(\text{End}^0(E) \rtimes_t \mathbb{G}) \cong \text{End}_{B \rtimes_t \mathbb{G}}^*(E \rtimes_t \mathbb{G})$. Then $(A \rtimes_t \mathbb{G}, (E \rtimes_t \mathbb{G})_{B \rtimes_t \mathbb{G}}, \iota(D))$ is an unbounded Kasparov module.*

If, for a dense $$ -subalgebra $\mathcal{A} \subseteq A$, $(\mathcal{A}, E_B, D)$ is a uniformly $\mathbb{G}$ -equivariant unbounded Kasparov module, with G acting trivially on B , the data*

$$\left(\overline{\text{span}}\{(1 \otimes \omega)((\iota(a)^* \otimes s^*)X) \mid a \in \mathcal{A}, s \in \mathcal{I}_a, \omega \in L^1(\mathbb{G})\}, (E \rtimes_t \mathbb{G})_{B \rtimes_t \mathbb{G}}, \iota(D) \right)$$

defines an unbounded Kasparov module, where X is a unitary on $(E \rtimes_t \mathbb{G}) \otimes C_0^r(\mathbb{G})$ described in the proof.

Proof. Note that the image of the representation of $A \rtimes_t \mathbb{G}$ is $\overline{\text{span}}(\iota(A)C_t^*(\mathbb{G})) \subseteq \text{End}^*(E \rtimes_t \mathbb{G})$. Using the identification $\text{End}_B^0(E) \rtimes_t \mathbb{G} \cong \text{End}_{B \rtimes_t \mathbb{G}}^0(E \rtimes_t \mathbb{G})$, we see that, for $a \in A$ and $f \in C_t^*(\mathbb{G})$,

$$(1 + \iota(D)^2)^{-1/2}(\iota(a)f) = \iota((1 + D^2)^{-1/2}a)f$$

is compact, cf. [Ver02, Démonstration du Proposition 5.3]. By the universality of the crossed product [Ver02, §4.1] [Vae05, §2.3], the morphism $\text{End}^0(E) \rtimes_u \mathbb{G} \rightarrow \text{End}^0(E) \rtimes_t \mathbb{G}$ gives rise to the morphism $\iota : \text{End}^0(E) \rightarrow M(\text{End}^0(E) \rtimes_t \mathbb{G}) \cong \text{End}^*(E \rtimes_t \mathbb{G})$ and a unitary $X \in M((\text{End}^0(E) \rtimes_t \mathbb{G}) \otimes C_0^r(\mathbb{G})) \cong \text{End}^*((\text{End}^0(E) \rtimes_t \mathbb{G}) \otimes C_0^r(\mathbb{G}))$ such that

$$X(\iota(T) \otimes 1)X^* = (\iota \otimes \text{id})\delta_{\text{End}^0(E)}(T)$$

for $T \in \text{End}^0(E)$. Let $a \in \mathcal{Q}$ and $s \in \mathcal{S}_a$; then for $X^*(\iota(a) \otimes s_1) \in \text{End}^*((E \rtimes_t \mathbb{G}) \otimes C_0^r(\mathbb{G}))$, the commutator

$$\begin{aligned} & [\iota(D) \otimes 1, X^*(\iota(a) \otimes s)] \\ &= X^*(X(\iota(D) \otimes 1)X^*(\iota(a) \otimes s) - (\iota \otimes \text{id})((a \otimes s)(D \otimes 1))) \\ &= X^*(\iota \otimes \text{id})\left(\delta_{\text{End}^0(E)}(D)(a \otimes s) - (a \otimes s)(D \otimes 1)\right) \\ &= X^*(\iota \otimes \text{id})\left((V_E(D \otimes_{\delta_B} 1)V_E^* - D \otimes 1)(a \otimes s) + [D, a] \otimes s\right) \end{aligned}$$

is adjointable. The representation of $A \rtimes_t \mathbb{G}$ on $E \rtimes_t \mathbb{G}$ consists of

$$\begin{aligned} \overline{\text{span}}(\iota(A)C_t^*(\mathbb{G})) &= \overline{\text{span}}\left\{\iota(a)(1 \otimes \omega)(X) \mid a \in A, \omega \in L^1(\mathbb{G})\right\} \\ &= \overline{\text{span}}\left\{\iota(a)(1 \otimes \eta_1^*)X(1 \otimes \eta_2^*) \mid a \in A, \eta_1, \eta_2 \in L^2(\mathbb{G})\right\} \\ &= \overline{\text{span}}\left\{(1 \otimes \eta_1^*)(\iota(a)^* \otimes s^*)X(1 \otimes \eta_2^*) \mid a \in A, s \in C_0^r(\mathbb{G}), \eta_1, \eta_2 \in L^2(\mathbb{G})\right\} \\ &\subseteq \overline{\text{span}}\left\{(1 \otimes \eta_1^*)(\iota(a)^* \otimes s^*)X(1 \otimes \eta_2^*) \mid a \in \mathcal{Q}, s \in \mathcal{S}_a, \eta_1, \eta_2 \in L^2(\mathbb{G})\right\} \end{aligned}$$

by the density of $\mathcal{S}_a^* \subseteq C_0^r(\mathbb{G})$ and the inclusion $A \subseteq \overline{\mathcal{Q}}$. $\square$

We also have a realisation of the dual-Green–Julg map on uniformly equivariant unbounded Kasparov modules.

Proposition 4.19. *Let (A, E_B, D) be a uniformly $\mathbb{G}$ -equivariant unbounded Kasparov module, with $\mathbb{G}$ acting trivially on B . Then $(A \rtimes_u \mathbb{G}, E_B, D)$ is an unbounded Kasparov module, with the integrated representation of $A \rtimes_u \mathbb{G}$.*

If, for a dense $$ -subalgebra $\mathcal{A} \subseteq A$, $(\mathcal{A}, E_B, D)$ is a uniformly $\mathbb{G}$ -equivariant unbounded Kasparov module, with G acting trivially on B , then*

$$\left(\overline{\text{span}}\{(1 \otimes \omega)((a^* \otimes s^*)V_E) \mid a \in \mathcal{A}, s \in \mathcal{S}_a, \omega \in L^1(\mathbb{G})\}, E_B, D\right)$$

is an unbounded Kasparov module.

Proof. The only point which is not immediate is the boundedness of commutators with D . Let $a \in \mathcal{Q}$ and $s \in \mathcal{S}_a$ and let $\omega \in L^1(\mathbb{G})$, so that

$$(1 \otimes \omega)((a^* \otimes s^*)V_E)$$

is in the integrated representation of $A \rtimes_u \mathbb{G}$ on E . By the uniform equivariance condition,

$$[D, (1 \otimes \omega)((a^* \otimes s^*)V_E)] = (1 \otimes \omega)\left((V_E(D \otimes 1)V_E^*(a \otimes s) - (a \otimes s)(D \otimes 1))^* V_E\right)$$

is bounded. The representation of $A \rtimes_t \mathbb{G}$ on $E \rtimes_t \mathbb{G}$ consists of

$$\begin{aligned} \overline{\text{span}}(AC_u^*(\mathbb{G})) &= \overline{\text{span}} \left\{ a(1 \otimes \omega)(V_E) \mid a \in A, \omega \in L^1(\mathbb{G}) \right\} \\ &= \overline{\text{span}} \left\{ a(1 \otimes \eta_1^*)V_E(1 \otimes \eta_2^*) \mid a \in A, \eta_1, \eta_2 \in L^2(\mathbb{G}) \right\} \\ &= \overline{\text{span}} \left\{ (1 \otimes \eta_1^*)(a^* \otimes s^*)V_E(1 \otimes \eta_2^*) \mid a \in A, s \in C_0^r(\mathbb{G}), \eta_1, \eta_2 \in L^2(\mathbb{G}) \right\} \\ &\subseteq \overline{\text{span}} \left\{ (1 \otimes \eta_1^*)(a^* \otimes s^*)V_E(1 \otimes \eta_2^*) \mid a \in \mathcal{Q}, s \in \mathcal{S}_a, \eta_1, \eta_2 \in L^2(\mathbb{G}) \right\} \end{aligned}$$

by the density of $\mathcal{S}_a L^2(\mathbb{G}) \subseteq C_0^r(\mathbb{G}) L^2(\mathbb{G}) \subseteq L^2(\mathbb{G})$ and the inclusion $A \subseteq \overline{\mathcal{Q}}$. $\square$

For the inverse map, more structure is required, including the presence of a dense subalgebra $\mathcal{A}$ of A . A discrete quantum group $\mathbb{G}$ has a compact dual, whose polynomial algebra we denote by $\mathcal{O}(\hat{\mathbb{G}})$. We write $\mathcal{A} \rtimes \mathbb{G}$ for the subalgebra of $A \rtimes_u \mathbb{G}$ generated by $\mathcal{A}$ and $\mathcal{O}(\hat{\mathbb{G}})$.

Proposition 4.20. *Let $(\mathcal{A} \rtimes \mathbb{G}, E_B, D)$ be an unbounded Kasparov module, with $\mathbb{G}$ a discrete quantum group and the representation of $\mathcal{A} \rtimes \mathbb{G}$ on E nondegenerate. Then $(\mathcal{A}, E_B, D)$ is a uniformly $\mathbb{G}$ -equivariant unbounded Kasparov module, with the $\mathbb{G}$ -action on E given by Lemma 4.13 and trivial on B .*

Proof. Because $\mathbb{G}$ is discrete, $\mathcal{A}$ is included in $\mathcal{A} \rtimes \mathbb{G}$. Hence $(1 + D^2)^{-1}a$ is compact and $[D, a]$ is bounded for all $a \in \mathcal{A}$. The inclusion $C_u^*(\mathbb{G}) \subseteq M(A \rtimes_u \mathbb{G})$ gives a (nondegenerate) representation π of $C_u^*(\mathbb{G})$ on E . Because $\mathbb{G}$ is discrete, it is regular. Applying Lemma 4.13, we obtain an action of $\mathbb{G}$ on E , acting trivially on B . Let V_E be the admissible unitary. Discreteness means that $C_0(\mathbb{G})$ is isomorphic as an algebra to the C^* -algebraic direct sum

$$\bigoplus_{\lambda \in \Lambda} M_{n_\lambda}(\mathbb{C})$$

of finite-dimensional matrix algebras. The admissible unitary is the direct sum over the index set $\lambda \in \Lambda$ of

$$V_E^\lambda \in \pi(\mathcal{O}(\hat{\mathbb{G}})) \otimes M_{n_\lambda}(\mathbb{C}) \subseteq \pi(C_u^*(\mathbb{G})) \otimes M_{n_\lambda}(\mathbb{C}) \subseteq \text{End}_B^*(E \otimes \mathbb{C}^{n_\lambda}),$$

cf. [VY20, §4.2.3] for the inclusion in the polynomial subalgebra. Then, for $a \in \mathcal{A}$,

$$(V_E^\lambda(D \otimes 1)V_E^{\lambda*} - D \otimes 1)a \otimes 1_{\mathbb{C}^{n_\lambda}} = V_E^\lambda[D \otimes 1, V_E^{\lambda*}(a \otimes 1)] - [D, a] \otimes 1$$

is bounded for all $\lambda \in \Lambda$, because $V_E^{\lambda*}(a \otimes 1) \in \pi(\mathcal{O}(\hat{\mathbb{G}}))\mathcal{A} \otimes M_{n_\lambda}(\mathbb{C}) \subseteq \mathcal{A} \rtimes \mathbb{G} \otimes M_{n_\lambda}(\mathbb{C})$. $\square$

Remark 4.21. It is clear that the bounded transform $(A \rtimes_t \mathbb{G}, (E \rtimes_t \mathbb{G})_{B \rtimes_t \mathbb{G}}, F_{\iota(D)} = \iota(F_D))$ of the descent $(A \rtimes_t \mathbb{G}, (E \rtimes_t \mathbb{G})_{B \rtimes_t \mathbb{G}}, \iota(D))$ of a uniformly $\mathbb{G}$ -equivariant cycle (A, E_B, D) is exactly the descent of the bounded transform (A, E_B, F_D) . The same is true for the dual-Green–Julg map.

4.3 Conformal quantum group equivariance

To generalise Definition 4.8 to conformal (co)actions, we will consider a conformal factor μ which is an unbounded operator on $E \otimes S$, where E is a Hilbert B -module and S is a C^* -bialgebra. It is necessary to allow μ to be unbounded in the “ S direction”, as can be seen from classical group equivariance. To apply the multiplicative perturbation theory of §2.3, we will require μ to be *S -matched*, in the sense of Appendix A.3, meaning roughly that μ is locally bounded in the S -direction. We denote by K_S the Pedersen ideal of S .

Definition 4.22. Let A and B be C^* -algebras equipped with coactions of a C^* -bialgebra S . An unbounded Kasparov A - B -module (A, E_B, D) is *conformally S -equivariant* if E is an S -equivariant A - B -correspondence and there exists an (even) S -matched operator μ on $(E \otimes S)_{B \otimes S}$ whose inverse is also S -matched, such that A satisfies the following. Given $a \in \text{Lip}_0^*(D)$, let $\mathcal{S}_a$ be the set of $s \in M(S)$ such that

$$\{(a \otimes s)\mu, (a \otimes s)\mu^{-1*}\} \text{dom}(D \otimes 1)(1 \otimes K_S) \subseteq \text{dom}(D \otimes 1) \cap V_E \text{dom}(D \otimes_{\delta_B} 1)$$

and

$$V_E(D \otimes_{\delta_B} 1)V_E^*(a \otimes s) - (a \otimes s)\mu(D \otimes 1)\mu^* \quad [D \otimes 1, (a \otimes s)\mu] \quad [D \otimes 1, (a \otimes s)\mu^{-1*}]$$

extend to S -matched operators. Let $\mathcal{Q}$ be the set of $a \in \text{Lip}_0^*(D)$ such that $S \subseteq \overline{\text{span}}(S\mathcal{S}_a) \cap \overline{\text{span}}(\mathcal{S}_a S)$. Then we require that $A \subseteq \overline{\text{span}}(A\mathcal{Q}) \cap \overline{\text{span}}(\mathcal{Q}A)$.

If A and B are C^* -algebras with $\mathbb{G}$ -actions, an unbounded Kasparov module (A, E_B, D) is *conformally $\mathbb{G}$ -equivariant* if it is conformally $C_0^r(\mathbb{G})$ -equivariant.

Remarks 4.23.

1. When $\mu = 1$, Definition 4.22 reduces to Definition 4.8 of uniformly equivariant S -unbounded Kasparov modules.
2. For a discrete quantum group $\mathbb{G}$, when $C_0(\mathbb{G})$ is isomorphic as an algebra to the C^* -algebraic direct sum

$$\bigoplus_{\lambda \in \Lambda} M_{n_\lambda}(\mathbb{C})$$

of finite-dimensional matrix algebras, the Pedersen ideal $K_{C_0(\mathbb{G})}$ is the algebraic direct sum. In this case, the conformal factor and the admissible unitary would be labelled by the index set $\lambda \in \Lambda$, so that

$$V_E^\lambda \in \text{Hom}_B^*(E \otimes_{\delta_B} (B \otimes \mathbb{C}^{n_\lambda}), E \otimes \mathbb{C}^{n_\lambda}) \quad \mu^\lambda \in \text{End}_B^*(E \otimes \mathbb{C}^{n_\lambda})$$

and the equivariance conditions on $a \in \mathcal{Q}$ become that

$$(V_E^\lambda(D \otimes_{\delta_B} 1)V_E^{\lambda*} - \mu^\lambda(D \otimes 1)\mu^{\lambda*})a \otimes 1_{\mathbb{C}^{n_\lambda}} \quad [D \otimes 1, (a \otimes 1)\mu^\lambda] \quad [D \otimes 1, (a \otimes 1)(\mu^\lambda)^{-1*}]$$

be bounded for all $\lambda \in \Lambda$.

Theorem 4.24. *A conformally S -equivariant unbounded Kasparov module (A, E_B, D) , with conformal factor μ , gives rise to an S -equivariant bounded Kasparov module (A, E_B, F_D) .*

Proof. The only point of difference from the non-equivariant case is the need to prove that, for every $a \in A$ and $s \in S$, $(F_D \otimes 1 - V_E(F_D \otimes_{\delta_B} 1)V_E^*)a \otimes s$ is compact. Let c be a positive element of K_S , so that, by Proposition A.24, the restriction of μ to the $B \otimes \overline{\text{span}}(ScS)$ -module $E \otimes \overline{\text{span}}(ScS)$ is bounded. For the time being, we work on the module $E \otimes \overline{\text{span}}(ScS)$. Let $a_1, a_2, a_3, a_4 \in \mathcal{Q}$ and $s_1, s_2, s_3, s_4 \in \mathcal{S}_{a_1}, \mathcal{S}_{a_2}, \mathcal{S}_{a_3}, \mathcal{S}_{a_4}$. As in the Proof of Theorem 2.9,

$$[\mu(D \otimes 1)\mu^*, a_2^*a_3 \otimes s_2^*s_3]\mu^{-1*}\langle D \rangle^{-\alpha}$$

is bounded. We apply Theorem 2.42 to obtain that

$$(F_{\mu(D \otimes 1)\mu^*} - F_D \otimes 1)a_2^*a_3a_4^*\langle D \rangle^\beta \otimes s_2^*s_3s_4^*$$

is bounded for $\beta < 1 - \alpha$. Furthermore,

$$((D \otimes_{\delta_B} 1)V_E^*(a_1 \otimes s_1) - V_E^*(a_1 \otimes s_1)\mu(D \otimes 1)\mu^*)\mu^{-1*}(\langle D \rangle^{-\alpha} \otimes 1)$$

is bounded and, by Proposition 2.21,

$$((F_D \otimes_{\delta_B} 1)V_E^*(a_1 \otimes s_1) - V_E^*(a_1 \otimes s_1)(F_{\mu D \mu^*} \otimes 1))\mu(\langle D \rangle^\beta \otimes 1)$$

is too. Now we have

$$\begin{aligned} & (V_E(F_D \otimes_{\delta_B} 1)V_E^* - F_D \otimes 1)a_1 a_2^* a_3 a_4^* \otimes s_1 s_2^* s_3 s_4^* \\ &= V_E((F_D \otimes_{\delta_B} 1)V_E^* - V_E^*(F_D \otimes 1))a_1 a_2^* a_3 a_4^* \otimes s_1 s_2^* s_3 s_4^* \\ &= V_E((F_D \otimes_{\delta_B} 1)V_E^*(a_1 a_2^* a_3 \otimes s_1 s_2^* s_3) - V_E^*(a_1 a_2^* a_3 \otimes s_1 s_2^* s_3)(F_D \otimes 1))(a_4^* \otimes s_4^*) \\ &\quad - [F_D, a_1 a_2^* a_3]a_4^* \otimes s_1 s_2^* s_3 s_4^* \\ &= V_E((F_D \otimes_{\delta_B} 1)V_E^*(a_1 \otimes s_1) - V_E^*(a_1 \otimes s_1)F_{\mu(D \otimes 1)\mu^*})(a_2^* a_3 a_4^* \otimes s_2^* s_3 s_4^*) \\ &\quad + (a_1 \otimes s_1)(F_{\mu(D \otimes 1)\mu^*}(a_2^* a_3 \otimes s_2^* s_3) - (a_2^* a_3 \otimes s_2^* s_3)(F_D \otimes 1))(a_4^* \otimes s_4^*) \\ &\quad - [F_D, a_1 a_2^* a_3]a_4^* \otimes s_1 s_2^* s_3 s_4^* \\ &= V_E((F_D \otimes_{\delta_B} 1)V_E^*(a_1 \otimes s_1) - V_E^*(a_1 \otimes s_1)F_{\mu(D \otimes 1)\mu^*})(a_2^* a_3 a_4^* \otimes s_2^* s_3 s_4^*) \\ &\quad + (a_1 \otimes s_1)(F_{\mu(D \otimes 1)\mu^*} - F_D \otimes 1)(a_2^* a_3 a_4^* \otimes s_2^* s_3 s_4^*) \\ &\quad - [F_D, a_1]a_2^* a_3 a_4^* \otimes s_1 s_2^* s_3 s_4^* \end{aligned}$$

so that $(V_E(F_D \otimes_{\delta_B} 1)V_E^* - F_D \otimes 1)a_1 a_2^* a_3 a_4^* \langle D \rangle^\beta \otimes s_1 s_2^* s_3 s_4^*$ is bounded. Let $a_5 \in A$ and note that $c \in \overline{\text{span}}(ScS) \trianglelefteq S$. Then

$$(V_E(F_D \otimes_{\delta_B} 1)V_E^* - F_D \otimes 1)a_1 a_2^* a_3 a_4^* a_5 \otimes s_1 s_2^* s_3 s_4^* c \quad (4.25)$$

is an element of $\text{End}^0(E) \otimes \overline{\text{span}}(ScS)$. As the compacts on $E \otimes \overline{\text{span}}(ScS)_{B \otimes \overline{\text{span}}(ScS)}$ are

$$\overline{\text{span}}(EE^* \otimes ScSScS) = \text{End}^0(E) \otimes \overline{\text{span}}(ScS) \trianglelefteq \text{End}^0(E) \otimes S = \text{End}^0(E \otimes S)$$

for each $c \in K_S$, we see that (4.25) defines a compact endomorphism on $E \otimes S$. Because $S \subseteq \overline{\mathcal{S}_{a_1} \mathcal{S}_{a_2}^* \mathcal{S}_{a_3} \mathcal{S}_{a_4}^* K_S}$ and $A \subseteq \overline{\mathcal{Q}^* \mathcal{Q} \mathcal{Q}^* \mathcal{Q} A}$,

$$(V_E(F_D \otimes_{\delta_B} 1)V_E^* - F_D \otimes 1)a \otimes s$$

is compact for all $a \in A$ and $s \in S$. $\square$

Theorem 4.26. *A conformally S -equivariant unbounded Kasparov module (A, E_B, D) gives rise to a uniformly S -equivariant unbounded Kasparov module (A, E_B, L_D) via the logarithmic transform.*

Proof. By the Proof of Theorem 4.24, $(V_E(F_D \otimes_{\delta_B} 1)V_E^* - F_D \otimes 1)a_1 a_2^* a_3 a_4^* \langle D \rangle^\beta \otimes s_1 s_2^* s_3 s_4^* c$ is bounded on $E \otimes S$ for $a_1, a_2, a_3, a_4 \in \mathcal{Q}$, $s_1, s_2, s_3, s_4 \in \mathcal{S}_{a_1}, \mathcal{S}_{a_2}, \mathcal{S}_{a_3}, \mathcal{S}_{a_4}$, $c \in K_S$, and $\beta < 1$. Then

$$\left[\begin{pmatrix} V_E(F_D \otimes_{\delta_B} 1)V_E^* & \\ & F_D \otimes 1 \end{pmatrix}, \begin{pmatrix} a_1 a_2^* a_3 a_4^* \otimes s_1 s_2^* s_3 s_4^* c \\ 0 \end{pmatrix} \right] \left\langle \begin{pmatrix} V_E(F_D \otimes_{\delta_B} 1)V_E^* & \\ & F_D \otimes 1 \end{pmatrix} \right\rangle^\beta$$

is bounded and

$$\begin{pmatrix} a_1 a_2^* a_3 a_4^* \otimes s_1 s_2^* s_3 s_4^* c \\ 0 \end{pmatrix} \text{dom} \begin{pmatrix} V_E(D \otimes_{\delta_B} 1)V_E^* & \\ & D \otimes 1 \end{pmatrix} \subseteq \text{dom} \begin{pmatrix} V_E(D \otimes_{\delta_B} 1)V_E^* & \\ & D \otimes 1 \end{pmatrix}.$$

Applying Proposition 2.44,

$$\left[\begin{pmatrix} V_E(L_D \otimes_{\delta_B} 1)V_E^* & \\ & L_D \otimes 1 \end{pmatrix}, \begin{pmatrix} a_1 a_2^* a_3 a_4^* \otimes s_1 s_2^* s_3 s_4^* c \\ 0 \end{pmatrix} \right]$$

is bounded and therefore so is $(V_E(L_D \otimes_{\delta_B} 1)V_E^* - L_D \otimes 1)a_1a_2^*a_3a_4^* \otimes s_1s_2^*s_3s_4^*c$. For any $a_5 \in A$,

$$(V_E(L_D \otimes_{\delta_B} 1)V_E^* - L_D \otimes 1)a_1a_2^*a_3a_4^*a_5 \otimes s_1s_2^*s_3s_4^*c$$

is bounded. We have $S \subseteq \overline{\mathcal{S}_{a_1}\mathcal{S}_{a_2}^*\mathcal{S}_{a_3}\mathcal{S}_{a_4}^*K_S}$ and $A \subseteq \overline{\mathcal{Q}^*\mathcal{Q}\mathcal{Q}^*\mathcal{Q}A}$, as required. $\square$

Proposition 4.27. *Let G be a locally compact group. An unbounded Kasparov module is conformally $C_0(G)$ -equivariant if and only if it is conformally G -equivariant.*

Proof. Use Proposition A.28. Because $C_0(G)$ is abelian, for $a \in \mathcal{Q}$, $\mathcal{S}_a$ will always contain the Pedersen ideal $K_{C_0(G)} = C_c(G)$. $\square$

4.4 Conformal action on the Podleś sphere

The compact quantum group $SU_q(2)$ has polynomial algebra $\mathcal{O}(SU_q(2))$ generated by a, b, c, d subject to the relations

$$ab = qba \quad ac = qca \quad bd = qdb \quad cd = qdc \quad bc = cb \quad ad = 1 + qbc \quad da = 1 + q^{-1}bc$$

and with adjoints $a^* = d$, $b^* = -qc$, $c^* = -q^{-1}b$, $d^* = a$. The polynomial algebra $\mathcal{O}(SU_q(2))$ is spanned by the Peter–Weyl elements t_{ij}^l with $l \in \frac{1}{2}\mathbb{N}$ and $i, j \in \{-l, -l+1, \dots, l-1, l\}$. The generators form the fundamental representation $l = \frac{1}{2}$, that is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} t_{-\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} & t_{-\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}} \\ t_{\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} & t_{\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}} \end{pmatrix}.$$

In terms of this basis, the coproduct and counit are

$$\Delta(t_{i,j}^l) = \sum_k t_{i,k}^l \otimes t_{k,j}^l \quad \varepsilon(t_{i,j}^l) = \delta_{i,j}$$

and the adjoint is related to the antipode by $t_{i,j}^{l*} = S(t_{j,i}^l)$.

Dual to $SU_q(2)$ is the discrete quantum group $\widehat{SU_q(2)}$ [VY20, §4.2.3], whose function algebra $C_0(\widehat{SU_q(2)}) = C^*(SU_q(2))$ is the closed span of matrix elements τ_{ij}^l with $l \in \frac{1}{2}\mathbb{N}$ and $i, j \in \{-l, -l+1, \dots, l-1, l\}$, subject to

$$\tau_{i,j}^l \tau_{i',j'}^{l'} = \delta_{l,l'} \delta_{j,i'} \tau_{i,j'}^l \quad \tau_{i,j}^{l*} = \tau_{j,i}^l.$$

In particular, as C^* -algebras,

$$C_0(\widehat{SU_q(2)}) = C^*(SU_q(2)) \cong \bigoplus_{l \in \frac{1}{2}\mathbb{N}} M_{2l}(\mathbb{C}).$$

We may choose τ_{ij}^l so that the pairing between $C^*(SU_q(2))$ and $C(SU_q(2))$ is given by

$$(\tau_{ij}^l, t_{i',j'}^{l'}) = \delta_{l,l'} \delta_{i,i'} \delta_{j,j'}$$

and the multiplicative unitary $W \in M(C(SU_q(2)) \otimes C^*(SU_q(2)))$ is $W = \sum_{l,i,j} t_{i,j}^l \otimes \tau_{i,j}^l$.

The quantum universal enveloping algebra $\check{U}_q(\mathfrak{sl}(2))$ is generated by K, K^{-1}, E, F subject to

$$KK^{-1} = K^{-1}K = 1 \quad KEK^{-1} = qE \quad KFK^{-1} = q^{-1}F \quad [E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}}$$

with coproduct

$$\Delta(K) = K \otimes K \quad \Delta(E) = E \otimes K + K^{-1} \otimes E \quad \Delta(F) = F \otimes K + K^{-1} \otimes F$$

and counit and antipode

$$\varepsilon(K) = 1 \quad \varepsilon(E) = \varepsilon(F) = 0 \quad S(K) = K^{-1} \quad S(E) = -qE \quad S(F) = -q^{-1}F.$$

Note that this is not the same as $U_q(\mathfrak{sl}(2))$, although the latter is a Hopf subalgebra of $\check{U}_q(\mathfrak{sl}(2))$ [KS97, §3.1.2]. There is a nondegenerate pairing $(\cdot, \cdot)$ between $\check{U}_q(\mathfrak{sl}(2))$ and $\mathcal{O}(SU_q(2))$ [KS97, Theorem 4.21]. By this pairing, $\check{U}_q(\mathfrak{sl}(2))$ is an algebra of unbounded operators affiliated to $C^*(SU_q(2))$. We may define left and right actions of $\check{U}_q(\mathfrak{sl}(2))$ on $\mathcal{O}(SU_q(2))$ by

$$X \rightharpoonup \alpha = \alpha_{(1)}(X, \alpha_{(2)}) \quad \alpha \leftharpoonup X = (X, \alpha_{(1)})\alpha_{(2)}.$$

The left and right actions of K are automorphisms of $\mathcal{O}(SU_q(2))$ and have the properties

$$(K \rightharpoonup \alpha)^* = K^{-1} \rightharpoonup \alpha^* \quad (\alpha \leftharpoonup K)^* = \alpha^* \leftharpoonup K^{-1}.$$

In terms of the Peter–Weyl basis, $K \rightharpoonup t_{i,j}^l = q^j t_{i,j}^l$ and $t_{i,j}^l \leftharpoonup K = q^i t_{i,j}^l$. We also record the relationships $S^{-1}(\alpha) = K^2 \rightharpoonup S(\alpha) \leftharpoonup K^{-2}$ and $\phi(\alpha\beta) = \phi(\beta(K^2 \rightharpoonup \alpha \leftharpoonup K^2))$ for the left Haar state ϕ on $C(SU_q(2))$. The unitary antipode R on $C(SU_q(2))$ is then given by $R(\alpha) = K \rightharpoonup S(\alpha) \leftharpoonup K^{-1}$; on the Peter–Weyl basis, $R(t_{ij}^l) = K \rightharpoonup t_{ji}^{l*} \leftharpoonup K^{-1} = (K^{-1} \rightharpoonup t_{ji}^l \leftharpoonup K)^* = q^{-i+j} t_{ji}^{l*}$.

The Podleś sphere has polynomial algebra $\mathcal{O}(S_q^2)$, the subalgebra of $\mathcal{O}(SU_q(2))$ generated by

$$\begin{aligned} A &= -q^{-1}bc = c^*c = t_{1/2,-1/2}^{1/2*} t_{1/2,-1/2}^{1/2} = q^{-2} t_{-1/2,1/2}^{1/2} t_{-1/2,1/2}^{1/2*} = -q^{-1} [2]_q^{-1} t_{00}^1 \\ B &= ac^* = -q^{-1}ab = t_{-1/2,-1/2}^{1/2} t_{1/2,-1/2}^{1/2*} = -q^{-1/2} [2]_q^{-1/2} t_{-10}^1 \\ B^* &= cd = t_{1/2,-1/2}^{1/2} t_{1/2,1/2}^{1/2} = q^{-1/2} t_{1/2,-1/2}^{1/2} t_{-1/2,-1/2}^{1/2*} = [2]_q^{-1/2} q^{1/2} t_{10}^1. \end{aligned}$$

and is spanned by t_{i0}^l . The subspaces $S_+ = \text{span}\{t_{i,1/2}^l | l, i\}$ and $S_- = \text{span}\{t_{i,-1/2}^l | l, i\}$ of $\mathcal{O}(SU_q(2))$ are the spinor bundles of the Podleś sphere. They can be completed under the inner product on $\mathcal{O}(SU_q(2))$ given by the left Haar state. The natural Dirac operator defining a spectral triple $(C(S_q^2), L^2(S_+ \oplus S_-), D)$ is [DS03, Theorem 8]

$$D = \begin{pmatrix} & \partial_E \\ \partial_F & \end{pmatrix}$$

where $\partial_E = E \rightharpoonup$ and $\partial_F = F \rightharpoonup$ or, in terms of the Peter–Weyl basis,

$$\partial_E t_{i,j}^l = \sqrt{[l+1/2]_q^2 - [j+1/2]_q^2} t_{i,j+1}^l \quad \partial_F t_{i,j}^l = \sqrt{[l+1/2]_q^2 - [j-1/2]_q^2} t_{i,j-1}^l.$$

We abbreviate these coefficients as $\kappa_k^l = \sqrt{[l+1/2]_q^2 - [k-1/2]_q^2}$. We have the twisted derivation property

$$\partial_E(\alpha\beta) = \partial_E(\alpha)(K \rightharpoonup \beta) + (K^{-1} \rightharpoonup \alpha)\partial_E(\beta) \quad \partial_F(\alpha\beta) = \partial_F(\alpha)(K \rightharpoonup \beta) + (K^{-1} \rightharpoonup \alpha)\partial_F(\beta)$$

which shows that D has bounded commutators with elements of $\mathcal{O}(S_q^2)$. The relationships

$$\partial_E(\alpha^*) = -q\partial_F(\alpha)^* \quad \partial_F(\alpha^*) = -q^{-1}\partial_E(\alpha)^*$$

can be used to show that D is self-adjoint [Sen11, Lemma A.1].

There is an action of $SU_q(2)$ on S_q^2 given by the restriction of the coaction of $C(SU_q(2))$ on itself to $C(S_q^2)$. The spectral triple $(\mathcal{O}(S_q^2), L^2(S_+ \oplus S_-), D)$ is constructed to be isometric with respect to this action, cf. [DS03, §4]. We can phrase this in terms of a right coaction

$$\delta_\Delta : \alpha \mapsto \Sigma(R \otimes 1)\Delta\alpha \quad t_{ij}^l \mapsto \sum_k t_{kj}^l \otimes q^{-i+k} t_{ki}^{l*}$$

of $C(SU_q(2))$ on $C(S_q^2)$, where R is the unitary antipode. We can write the admissible unitary as

$$V_\Delta(t_{ij}^l \otimes t_{i'j'}^{l'}) = \sum_k t_{kj}^l \otimes q^{-i+k} t_{ki}^{l*} t_{i'j'}^{l'}.$$

We then have

$$\begin{aligned} (\partial_E \otimes 1)V_\Delta(t_{ij}^l \otimes t_{i'j'}^{l'}) &= \sum_k \kappa_{j+1}^l t_{k,j+1}^l \otimes q^{-i+k} t_{ki}^{l*} t_{i'j'}^{l'} \\ &= \kappa_{j+1}^l V_\Delta(t_{i,j+1}^l \otimes t_{i'j'}^{l'}) = V_\Delta(\partial_E \otimes 1)(t_{ij}^l \otimes t_{i'j'}^{l'}) \end{aligned}$$

and, similarly, that $\partial_F \otimes 1$ commutes with V_Δ , which means that $(C(S_q^2), L^2(S_+ \oplus S_-), D)$ is isometrically equivariant for the action of $SU_q(2)$.

In addition, there is an action of $\widehat{SU_q(2)}$ on S_q^2 given by the restriction of the adjoint action of $C(SU_q(2))$ on itself to $C(S_q^2)$ [Voi11, §4]. Together, these actions give an action of $SL_q(2) = SU_q(2) \ltimes \widehat{SU_q(2)}$, the Drinfeld double of $SU_q(2)$, which can be thought of as the quantisation of the classical Lorentz group $SL(2, \mathbb{C})$ action on the sphere S^2 . The left adjoint action of $C(SU_q(2))$ is given by

$$\text{ad}(\alpha) : \beta \rightarrow \alpha_{(1)}\beta S(\alpha_{(2)}).$$

For $z \in \mathbb{C}$, we define a slightly adjusted action

$$\omega_z(\alpha) : \beta \rightarrow \alpha_{(1)}\beta(K^{2z} \rightharpoonup S(\alpha_{(2)})).$$

For any $\alpha \in C(SU_q(2))$, $\omega_z(\alpha)$ preserves the subalgebra $C(S_q^2)$ and its spinor bundles. In terms of the Peter–Weyl basis,

$$\omega_z(t_{i,j}^l)(\beta) = \sum_k q^{-2zk} t_{i,k}^l \beta t_{j,k}^{l*} \quad \text{and} \quad \omega_z(t_{i,j}^{l*})(\beta) = \sum_k q^{2((z-1)k+j)} t_{i,k}^{l*} \beta t_{j,k}^l.$$

With respect to the inner product on $C(SU_q(2))$ given by the Haar state ϕ , ω_1 is self-adjoint; in general,

$$\langle \omega_z(\alpha)(\beta) | \gamma \rangle = \langle \beta | \omega_{-z+2}(\alpha^*)(\gamma) \rangle.$$

From the left action ω_1 of $C(SU_q(2))$ on itself, we obtain a right coaction of $C^*(SU_q(2))$ on $C(SU_q(2))$ by the formula

$$\beta_{(0)}(\beta_{(1)}, \alpha) = \omega_1(\alpha)(\beta),$$

using the Sweedler notation $\delta_{\omega_1}(\beta) = \beta_{(0)} \otimes \beta_{(1)}$ for the coaction. In particular, we obtain that

$$\delta_{\omega_1}(t_{i,j}^l) = \sum_{l', i', j'} \omega_1(t_{i',j'}^{l'})(t_{i,j}^l) \otimes \tau_{i',j'}^{l'}.$$

The admissible unitary V_{ω_1} on $L^2(S_+ \oplus S_-) \otimes C^*(SU_q(2))$ is given by

$$V_{\omega_1} = \sum_{l,i,j} \omega_1(t_{i,j}^l) \otimes \tau_{i,j}^l = \sum_k q^{-2k} t_{i,k}^l \cdot t_{j,k}^{l*} \otimes \tau_{i,j}^l = \sum_{k,k'} q^{-2k'} t_{i,k}^l \cdot t_{j,k'}^{l*} \otimes \tau_{i,k}^l \tau_{k',j}^l.$$

We claim that the spectral triple $(C(S_q^2), L^2(S_+ \oplus S_-), D)$ is conformally $\widehat{SU_q(2)}$ -equivariant. The conformal geometry of the Podleś sphere is examined at the level of bounded KK-theory in [NV10, Voi11]. Because $\widehat{SU_q(2)}$ is discrete, the conformal factor μ will be the sum of components $\mu^l \in B(L^2(S_+ \oplus S_-)) \otimes M_{2l}(\mathbb{C})$, $l \in \frac{1}{2}\mathbb{N}_{\geq 1}$ labelling the irreducible representations of $SU_q(2)$. Noting that $C(S_q^2)$ is unital, conformal equivariance will be a consequence of

$$V_{\omega_1}^l(D \otimes 1)V_{\omega_1}^{l*} - \mu^l(D \otimes 1)\mu^{l*} \quad [D \otimes 1, \mu^l]$$

being bounded for all $l \in \frac{1}{2}\mathbb{N}_{\geq 1}$.

Note that $(K \otimes K) \rightarrow (1 \otimes S)\Delta(\alpha) = (1 \otimes S)\Delta(\alpha)$ because

$$\begin{aligned} (K \otimes K) \rightarrow (1 \otimes S)\Delta(t_{i,j}^l) &= \sum_k K \rightarrow t_{i,k}^l \otimes K \rightarrow S(t_{k,j}^l) \\ &= \sum_k q^k t_{i,k}^l \otimes K \rightarrow t_{k,j}^{l*} \\ &= \sum_k q^k t_{i,k}^l \otimes (K^{-1} \rightarrow t_{k,j}^l)^* \\ &= \sum_k t_{i,k}^l \otimes t_{k,j}^{l*} \\ &= (1 \otimes S)\Delta(t_{i,j}^l). \end{aligned}$$

Then

$$\begin{aligned} \partial_E(\omega_z(\alpha)(\beta)) &= \partial_E(\alpha_{(1)}\beta(K^{2z} \rightarrow S(\alpha_{(2)}))) \\ &= \partial_E(\alpha_{(1)})(K \rightarrow \beta)(K^{2z+1} \rightarrow S(\alpha_{(2)})) + (K^{-1} \rightarrow \alpha_{(1)})\partial_E(\beta)(K^{2z+1} \rightarrow S(\alpha_{(2)})) \\ &\quad + (K^{-1} \rightarrow \alpha_{(1)}\beta)\partial_E(K^{2z} \rightarrow S(\alpha_{(2)})) \\ &= \partial_E(\alpha_{(1)})(K \rightarrow \beta)(K^{2z+1} \rightarrow S(\alpha_{(2)})) + \omega_{z+1}(\alpha)(\partial_E(\beta)) \\ &\quad + (K^{-1} \rightarrow \alpha_{(1)}\beta)\partial_E(K^{2z} \rightarrow S(\alpha_{(2)})) \end{aligned}$$

so that $\partial_E\omega_z(\alpha) - \omega_{z+1}(\alpha)\partial_E$ and $\partial_F\omega_z(\alpha) - \omega_{z+1}(\alpha)\partial_F$, similarly, are bounded on $S_+ \oplus S_-$. Furthermore,

$$\begin{aligned} \sum_j \omega_0(t_{i,j}^l)(\omega_1(t_{i',j}^{l*})(\beta)) &= \sum_{j,k} t_{i,k}^l \omega_1(t_{i',j}^{l*})(\beta) t_{j,k}^{l*} \\ &= \sum_{j,k,k'} q^{2j} t_{i,k}^l t_{i',k'}^{l*} \beta t_{j,k}^l t_{j,k}^{l*} \\ &= \sum_{j,k,k'} q^{2k'} t_{i,k}^l t_{i',k'}^{l*} \beta (K^{-2} \rightarrow t_{j,k'}^l \leftarrow K^2) S(t_{k,j}^l) \\ &= \sum_{j,k,k'} q^{2k'} t_{i,k}^l t_{i',k'}^{l*} \beta (K^{-2} \rightarrow (t_{j,k'}^l (K^2 \rightarrow S(t_{k,j}^l) \leftarrow K^{-2})) \leftarrow K^2) \\ &= \sum_{j,k,k'} q^{2k'} t_{i,k}^l t_{i',k'}^{l*} \beta (K^{-2} \rightarrow (t_{j,k'}^l S^{-1}(t_{k,j}^l)) \leftarrow K^2) \\ &= \sum_k q^{2k} t_{i,k}^l t_{i',k}^{l*} \beta (K^{-2} \rightarrow 1 \leftarrow K^2) \\ &= \sum_k q^{2k} t_{i,k}^l t_{i',k}^{l*} \beta \\ &= \omega_{-1}(t_{i,i'}^l)(1)\beta. \end{aligned}$$

Let $\mu^l = \sum_{i,j} \omega_{-1/2}(t_{i,j}^l)(1) \otimes \tau_{i,j}^l = \sum_{i,j,k} q^k t_{i,k}^l t_{j,k}^{l*} \otimes \tau_{i,j}^l$. For $l = \frac{1}{2}$,

$$\mu^{\frac{1}{2}} = q^{\frac{1}{2}} T_{\frac{1}{2}}^{\frac{1}{2}} T_{\frac{1}{2}}^{\frac{1}{2}*} + q^{-\frac{1}{2}} T_{-\frac{1}{2}}^{\frac{1}{2}} T_{-\frac{1}{2}}^{\frac{1}{2}*} = q^{\frac{1}{2}} \begin{pmatrix} q^2 A & -B \\ -B^* & 1 - A \end{pmatrix} + q^{-\frac{1}{2}} \begin{pmatrix} 1 - q^2 A & B \\ B^* & A \end{pmatrix}.$$

Thus, with $P_{\pm}$ the projections onto the positive and negative spinors, $\mu^{1/2} = q^{1/2} P_+ + q^{-1/2} P_-$. If we regard K as an unbounded operator on $C^*(SU_q(2))$ the conformal factor is

$$\mu = W(1 \otimes K)W^*$$

where W is the multiplicative unitary of $SU_q(2)$.

We remark that μ^l is positive and $(\mu^l)^z = \sum_{i,j} \omega_{-z/2}(t_{i,j}^l)(1) \otimes \tau_{i,j}^l$. Because $\mu^l \in \mathcal{O}(S_q^2) \otimes M_{2l}(\mathbb{C})$, it is clear that $[D \otimes 1, \mu^l]$ is bounded. We are now in a position to see also that

$$\begin{aligned} V_{\omega_1}^l(D \otimes 1)V_{\omega_1}^{l*} - \mu^l(D \otimes 1)\mu^{l*} \\ &= \sum_{l,i,j,i',j'} \left(\omega_1(t_{i,j}^l) D \omega_1(t_{i',j'}^{l*}) - \omega_{-1/2}(t_{i,j}^l)(1) D \omega_{-1/2}(t_{i',j'}^l)(1) \right) \otimes \tau_{i,j}^l \tau_{j',i'}^{l*} \\ &= \sum_{l,i,j,i'} \left(\omega_1(t_{i,j}^l) D \omega_1(t_{i',j}^{l*}) - \omega_{-1/2}(t_{i,j}^l)(1) D \omega_{-1/2}(t_{i',j}^l)(1) \right) \otimes \tau_{i,i'}^l \\ &= \sum_{l,i,j,i'} \left(-(D \omega_0(t_{i,j}^l) - \omega_1(t_{i,j}^l) D) \omega_1(t_{i',j}^{l*}) + [D, \omega_{-1/2}(t_{i,j}^l)(1)] \omega_{-1/2}(t_{i',j}^l)(1) \right) \otimes \tau_{i,i'}^l \end{aligned}$$

is bounded. Finally, we obtain that $(\mathcal{O}(S_q^2), L^2(S_+ \oplus S_-), D)$ is conformally $\widehat{SU_q(2)}$ -equivariant with conformal factor μ .

The locally compact quantum group $SL_q(2)$ is the Drinfeld double $SU_q(2) \bowtie \widehat{SU_q(2)}$; see e.g. [VY20, §4.4.1]. As C^* -algebras,

$$C(SL_q(2)) = C(SU_q(2)) \otimes C^*(SU_q(2)).$$

The comultiplication on $C(SL_q(2))$ is

$$\Delta_{SL_q(2)} = (1 \otimes \Sigma \otimes 1)(\text{id} \otimes \text{ad}(W) \otimes \text{id}) \circ (\Delta \otimes \hat{\Delta})$$

and the antipode is

$$S_{SL_q(2)} = \text{ad}(W^*) \circ (S \otimes \hat{S}) = (S \otimes \hat{S}) \circ \text{ad}(W).$$

By [BV05, Theorem 5.3] the unitary antipode is similarly

$$R_{SL_q(2)} = \text{ad}(W^*) \circ (R \otimes \hat{R}) = (R \otimes \hat{R}) \circ \text{ad}(W).$$

Our conventions differ from those of [NV10] in that we use right coactions rather than left ones. The translation between these is not difficult: a left coaction can be turned into a right coaction, and vice versa, by applying the unitary antipode to the C^* -bialgebra leg and then flipping the legs. Taking this into account in [NV10, Proposition 3.2] the action of $SL_q(2)$ on S_q^2 is given by the coaction

$$\begin{aligned} \delta_{\bowtie} &= (\Sigma \otimes 1)(1 \otimes \Sigma)(\text{ad}(W^*) \otimes \text{id})(R \otimes \hat{R} \otimes \text{id})(1 \otimes \Sigma)(\text{id} \otimes \text{id} \otimes \hat{R})(\text{id} \otimes \delta_{\omega_1})\Sigma(\text{id} \otimes R)\delta_{\Delta} \\ &= (\Sigma \otimes 1)(1 \otimes \Sigma)(\text{ad}(W^*) \otimes \text{id})(1 \otimes \Sigma)(\Sigma \otimes 1)(1 \otimes \Sigma)(\delta_{\omega_1} \otimes \text{id})\delta_{\Delta} \\ &= (\text{id} \otimes \text{ad}(W^*))(\Sigma \otimes 1)(\delta_{\omega_1} \otimes \text{id})\delta_{\Delta} \end{aligned}$$

of $C(SL_q(2))$. Using the standard leg-numbering notation the admissible unitary is

$$V_{\bowtie} = (1 \otimes W^*)V_{\omega_1,13}(V_{\Delta} \otimes 1)(1 \otimes W).$$

Let $\mu_{\boxtimes} = (1 \otimes W^*)\mu_{13}(1 \otimes W)$. Then

$$V_{\boxtimes}(D \otimes 1)V_{\boxtimes}^* - \mu_{\boxtimes}(D \otimes 1)\mu_{\boxtimes}^* = (1 \otimes W^*) \left(V_{\omega_1,13}(D \otimes 1 \otimes 1)V_{\omega_1,13}^* - \mu_{13}(D \otimes 1 \otimes 1)\mu_{13}^* \right) (1 \otimes W)$$

is $C(SU_q(2)) \otimes C^*(SU_q(2))$ -matched because it is bounded when restricted to each of the submodules $L^2(S_+ \oplus S_-) \otimes C(SU_q(2)) \otimes M_{2l}(\mathbb{C})$. In terms of the Peter–Weyl basis,

$$\begin{aligned} \mu_{\boxtimes} &= \sum_{i,j,k,l,i',j',i'',j''} q^k t_{i,k}^l t_{j,k}^{l*} \otimes t_{i'',j''}^l t_{i',j'}^{l*} \otimes \tau_{j'',i''}^l \tau_{i',j'}^l \\ &= \sum_{i,j,k,l,m,n} q^k t_{i,k}^l t_{j,k}^{l*} \otimes t_{i,m}^l t_{j,n}^{l*} \otimes \tau_{m,n}^l. \end{aligned}$$

This shows that the first leg of $\mu_{\boxtimes}$ is in $\mathcal{O}(S_q^2)$ so that $[D \otimes 1 \otimes 1, \mu_{\boxtimes}]$ is similarly $C(SL_q(2))$ -matched. Regarding K as an unbounded operator on $C^*(SU_q(2))$, the conformal factor is

$$\mu_{\boxtimes} = W_{23}^* W_{13}(1 \otimes 1 \otimes K)W_{13}^* W_{23}.$$

We have now demonstrated

Proposition 4.28. *The spectral triple $(C(S_q^2), L^2(S_+ \oplus S_-), D)$ is conformally $SL_q(2)$ -equivariant with conformal factor $\mu_{\boxtimes}$.*

5 Equivalence relations on unbounded KK-theory

The most restrictive natural equivalence relation in the bounded picture of KK-theory is locally compact perturbation. If (A, E_B, F) is a bounded Kasparov module and $T \in \text{End}^*(E)$ is such that $Ta \in \text{End}^0(E)$ for all $a \in A$, then $(A, E_B, F + T)$ will still be a bounded Kasparov module. The only condition which is not immediate is that $((F + T)^2 - 1)a \in \text{End}^0(E)$, demonstrated by the computation

$$((F + T)^2 - 1)a = (F^2 - 1)a + (F + T)Ta + TFa = (F^2 - 1)a + (F + T)Ta + T[F, a] + TaF.$$

It is unclear, in the unbounded picture of KK-theory, what should stand in for equivalence up to locally compact perturbation. The most immediate relation that suggests itself is equivalence up to bounded perturbation. If (A, E_B, D) is an unbounded Kasparov module and $T = T^* \in \text{End}^*(E)$, then $(A, E_B, D + T)$ will still be an unbounded Kasparov module. The local compactness of the resolvent takes a little work, see e.g. [CP98, Lemma B.6]. It is possible to consider locally bounded perturbations, at least in the presence of an adequate approximate unit [vdD18, §4].

In the bounded picture of KK-theory, there are several (combinations of) equivalence relations on Kasparov modules, each of which will give rise to the KK-theory groups. In [CS86, §3], *cobordism* is introduced as one such equivalence relation. (We remark that the similarly named equivalence relation of *bordism* [Hil10, DGM18] is unrelated.) First, we require a small Lemma.

Lemma 5.1. [CS86, §3] *If (A, E_B, F) is a bounded Kasparov module and $p \in \text{End}^*(E)$ is an even projection commuting with the representation of A such that $[F, p]a$ is compact for all $a \in A$, then (A, pE_B, pFp) is a Kasparov module.*

Definition 5.2. [CS86, Definition 3.1] Two bounded Kasparov modules (A, E'_B, F_1) and (A, E''_B, F_2) are *cobordant* if there exists a Kasparov module (A, E_B, F) and an even partial isometry $v \in \text{End}^*(E)$, such that

- v commutes with (the representation of) A ;

- $[F, v]a$ is compact for all $a \in A$;
- $(A, (1 - vv^*)E_B, (1 - vv^*)F(1 - vv^*))$ is unitarily equivalent to (A, E'_B, F_1) ; and
- $(A, (1 - v^*v)E_B, (1 - v^*v)F(1 - v^*v))$ is unitarily equivalent to (A, E''_B, F_2) .

We call $(A, E_B, F; v)$ a cobordism.

It turns out that cobordism is an equivalence relation, and is compatible with direct sums. Even though apparently much stronger than homotopy, it gives rise to the same KK-groups, provided A is separable [CS86, Theorem 3.7].

Example 5.3. Suppose that two bounded Kasparov modules (A, E'_B, F_1) and (A, E''_B, F_2) are unitarily equivalent, up to a locally compact perturbation, that is, there exists a unitary $U : E'_B \rightarrow E''_B$ such that $(U^*F_2U - F_1)a \in \text{End}^0(E)$ for all $a \in A$. Then

$$\left(A, (E' \oplus E'')_B, \begin{pmatrix} F_1 & \\ & F_2 \end{pmatrix} \right) \quad v = \begin{pmatrix} & 0 \\ U & \end{pmatrix}$$

constitute a cobordism between the two modules.

Lemma 5.4. *If two bounded Kasparov modules $(A, E_{1,B}, F_1)$ and $(A, E_{2,B}, F_2)$ are cobordant, there exists a cobordism $(A, E_B, F; v)$ such that vv^* , v^*v , and F mutually commute.*

Proof. Let $(A, E'_B, F'; v')$ be any cobordism between $(A, E_{1,B}, F_1)$ and $(A, E_{2,B}, F_2)$. Let $w_1 : E_1 \rightarrow (1 - v'v'^*)E$ and $w_2 : E_2 \rightarrow (1 - v'^*v')E$ be the unitaries of the cobordism. Then

$$(A, E_1 \oplus E' \oplus E_2, F_1 \oplus F' \oplus F_2; w_1 + v' + w_2^*)$$

is a cobordism between $(A, E_{1,B}, F_1)$ and $(A, E_{2,B}, F_2)$. We have

$$(w_1 + v' + w_2^*)(w_1 + v' + w_2^*)^* = 1 \oplus 1 \oplus 0 \quad (w_1 + v' + w_2^*)^*(w_1 + v' + w_2^*) = 0 \oplus 1 \oplus 1.$$

We can check that

$$\begin{aligned} [F_1 \oplus F' \oplus F_2, w_1 + v' + w_2^*]a &= (F'(w_1 + v') + F_2w_2^* - w_1F_1 - (v' + w_2^*)F')a \\ &= ([F', v'] + F'w_1 + w_2^*F'(1 - v'^*v) - (1 - v'v'^*)F'w_1 - w_2^*F')a \\ &= ([F', v'] + v'v'^*F'w_1 - w_2^*F'v'^*v')a \\ &= [F', v']a - v'[F', v'^*]aw_1 - w_2^*[F', v'^*]av' \end{aligned}$$

is compact for all $a \in A$, as required. $\square$

We shall make a natural generalisation to unbounded Kasparov modules but, first, a Lemma.

Lemma 5.5. *Let (A, E_B, D) be an unbounded Kasparov module and $p \in \text{End}^*(E)$ an even projection such that p commutes with A and D . Then (A, pE_B, pDp) is an unbounded Kasparov module and, furthermore, $F_{pDp} = pF_Dp$ on pE .*

A similar result to Lemma 5.5 would follow from weaker assumptions than that p and D commute.

Definition 5.6. Two unbounded Kasparov modules (A, E'_B, D_1) and (A, E''_B, D_2) are *cobordant* if there exist an unbounded Kasparov module (A, E_B, D) and an even partial isometry $v \in \text{End}^*(E)$, such that

- v commutes with (the representation of) A , and vv^* and v^*v commute with D ;

- $v^*vA \subseteq \overline{\mathcal{Q}}$, where $\mathcal{Q}$ is the set of $a \in \text{End}^*(v^*vE)$ such that

$$[D, v]a \quad a[D, v^*] \quad [D, a]$$

extend to adjointable operators;

- $(A, (1 - vv^*)E_B, (1 - vv^*)D(1 - vv^*))$ is unitarily equivalent to (A, E'_B, D_1) ; and
- $(A, (1 - v^*v)E_B, (1 - v^*v)D(1 - v^*v))$ is unitarily equivalent to (A, E''_B, D_2) .

For a dense $*$ -subalgebra $\mathcal{A} \subseteq A$, $(\mathcal{A}, E_B, D; v)$ is a cobordism between $(\mathcal{A}, E'_B, D_1)$ and $(\mathcal{A}, E''_B, D_2)$ if $v^*v\mathcal{A} \subseteq \mathcal{Q}$.

At the cost of further technicalities, we could proceed with weaker assumptions than that D commute with vv^* and v^*v . However, by a similar argument to Lemma 5.4, this would not be terribly useful.

Proposition 5.7. *cf. [CS86, Lemma 3.3] Cobordism of unbounded Kasparov modules is an equivalence relation and is compatible with direct sums.*

Proof. For symmetry, remark that, for $b = vav^* \in v\mathcal{Q}v^* \subseteq \text{End}^*(vv^*E)$,

$$[D, v^*]b = -v^*[D, v]av^* \quad b[D, v] = va[D, v^*]v \quad [D, b] = [D, v]av^* + v[D, a]v^* + va[D, v^*]$$

are bounded.

For transitivity, suppose that $(A, E_B, D; v)$ is a cobordism between the cycles $(A, E_{1,B}, D_1)$ and $(A, E_{2,B}, D_2)$, and that $(A, E'_B, D'; v')$ is a cobordism between $(A, E_{2,B}, D_2)$ and $(A, E_{3,B}, D_3)$. Let $U : (1 - v^*v)E \rightarrow (1 - v'^*v')E$ be the unitary equivalence between $(A, (1 - v^*v)E_B, (1 - v^*v)D(1 - v^*v))$ and $(A, (1 - v'^*v')E'_B, (1 - v'^*v')D'(1 - v'^*v'))$. Then

$$(A, (E \oplus E')_B, D \oplus D'; v + U + v')$$

is a cobordism between $(A, E_{1,B}, D_1)$ and $(A, E_{3,B}, D_3)$. We have

$$(v + U + v')(v + U + v')^* = vv^* \oplus 1 \quad (v + U + v')^*(v + U + v') = 1 \oplus v'^*v'.$$

Because D commutes with $(1 - v^*v)$ and D' commutes with $(1 - v'^*v')$, $D'U - UD$ is zero on $E \oplus E'$. Let $a \in \mathcal{Q}$, $a' \in \mathcal{Q}'$, and $b \in \text{Lip}_0^*((1 - v^*v)D(1 - v^*v))$. Then $(a + b) \oplus a' \in \text{End}^*(E \oplus v'^*v'E')$ and

$$\begin{aligned} [D \oplus D', v + U + v']((a + b) \oplus a') &= [D, v]a + [D', v']a' \\ ((a + b) \oplus a')[D \oplus D', v^* + U^* + v'^*] &= a^*[D, v^*] + a'^*[D', v'^*] \\ [D \oplus D', (a + b) \oplus a']b &= ([D, a] + [(1 - v^*v)D(1 - v^*v), b]) \oplus [D', a'] \end{aligned}$$

are bounded. We have $(1 \oplus v'^*v')A \subseteq \overline{\mathcal{Q} + \text{End}^*((1 - v^*v)E) \oplus \mathcal{Q}'}$ as required. Finally, it is straightforward to check that direct sums of cobordisms are cobordisms of direct sums in an obvious way. $\square$

Example 5.8. Suppose that two unbounded Kasparov modules (A, E'_B, D_1) and (A, E''_B, D_2) are unitarily equivalent, up to a locally bounded perturbation, that is there exists a unitary $U : E'_B \rightarrow E''_B$ such that A is contained in the closure of the set of $a \in \text{End}^*(E')$ such that

$$Ua \text{ dom } D_1 \subseteq \text{dom } D_2 \quad a \text{ dom } D_1 \subseteq \text{dom } D_1 \quad UaU^* \text{ dom } D_2 \subseteq \text{dom } D_2$$

and

$$(U^*D_2U - D_1)a \quad [D_1, a] \quad [D_2, UaU^*]$$

are bounded. Then

$$\left(A, (E' \oplus E'')_B, \begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix} \right) \quad v = \begin{pmatrix} & 0 \\ U & \end{pmatrix}$$

constitute a cobordism between the two cycles.

Proposition 5.9. *Given two cobordant unbounded Kasparov modules (A, E'_B, D_1) and (A, E''_B, D_2) , their bounded transforms (A, E'_B, F_{D_1}) and (A, E''_B, F_{D_2}) are cobordant and so define the same element in $KK(A, B)$.*

Proof. Let $(A, E_B, D; v)$ be a cobordism between (A, E'_B, D_1) and (A, E''_B, D_2) . By Lemma 5.5, $(A, E_B, F_D; v)$ is a bounded cobordism between (A, E'_B, F_{D_1}) and (A, E''_B, F_{D_2}) . $\square$

A natural question to ask is whether one can identify unbounded Kasparov modules cobordant to the zero module. In [vdDM20, §3–4], several notions of degenerate module are surveyed and shown to be homotopic to zero. Instead of making a similar survey, we shall make the following definition, in the safety of the knowledge that it contains as special cases the *spectrally degenerate* cycles of [vdDM20, Definition 3.5], the *spectrally symmetric* cycles of [vdDM20, Definition 4.6] (which, in turn, include the *spectrally decomposable* cycles of [Kaa20, Definition 4.1]), the *Clifford symmetric* cycles of [vdDM20, Definition 4.13], and the *weakly degenerate* cycles of [DGM18, Definition 3.1].

Definition 5.10. An unbounded Kasparov module (A, E_B, D) is *positively degenerate* if there exists an odd self-adjoint unitary $s \in \text{End}^*(E)$, preserving the domain of D , such that

- The anticommutator $Ds + sD$ is semibounded below and
- $A \subseteq \overline{\mathcal{P}}$, where $\mathcal{P}$ is the set of $a \in \text{Lip}_0^*(D)$ such that $[s, a] = 0$.

Proposition 5.11. *A positively degenerate unbounded Kasparov module (A, E_B, D) is cobordant to $(A, 0_B, 0)$.*

Proof. Let $s \in \text{End}^*(E)$ be a symmetry implementing the degeneracy. Let N be the number operator and S the unilateral shift on $\ell^2(\mathbb{N}_{\geq 0})$. Then $(A, E_B \otimes \ell^2(\mathbb{N}_{\geq 0}), D \otimes 1 + s \otimes N)$ is an unbounded Kasparov module. The main point is the compactness of the resolvent, for which we compute

$$(D \otimes 1 + s \otimes N)^2 = D^2 \otimes 1 + 1 \otimes N^2 + (Ds + sD) \otimes N.$$

Let $C \geq 0$ be a constant such that $Ds + sD + C$ is positive; then

$$D^2 \otimes 1 + 1 \otimes N^2 + (Ds + sD) \otimes N \geq D^2 \otimes 1 + 1 \otimes N(N - C).$$

For large enough $\lambda > 0$,

$$(\lambda + (D \otimes 1 + s \otimes N)^2)^{-1}a$$

is compact. The constructed Kasparov module, together with the isometry $1 \otimes S$, implements the required cobordism. Using the relation $NS = S(N + 1)$, we check that

$$(D \otimes 1 + s \otimes N)(1 \otimes S) - (1 \otimes S)(D \otimes 1 + s \otimes N) = s \otimes [N, S] = s \otimes S$$

is bounded. $\square$

We can now show that unbounded Kasparov modules, subject to the equivalence relation of cobordism, form a group under direct sum.

Corollary 5.12. *Given an unbounded Kasparov module (A, E_B, D) ,*

$$(A, E_B, D) \oplus (A, E_B^{op}, -D) = \left(A, E_B \oplus E_B^{op}, \begin{pmatrix} D & \\ & -D \end{pmatrix} \right)$$

is cobordant to $(A, 0_B, 0)$.

Proof. The symmetry $s = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ makes the direct sum module positively degenerate. $\square$

Combining Propositions 5.7, 5.9 and Corollary 5.12 proves

Theorem 5.13. *Cobordism classes of unbounded Kasparov A - B -modules form an abelian group which surjects onto $KK(A, B)$.*

5.1 Conformism of unbounded Kasparov modules

Whereas unbounded cobordism accounts for additive perturbations, we can use the multiplicative perturbation theory developed earlier to define a weaker relation, *conformism*.

Definition 5.14. Two unbounded Kasparov modules (A, E'_B, D_1) and (A, E''_B, D_2) are *conformant* if there exists an unbounded Kasparov module (A, E_B, D) an even partial isometry $v \in \text{End}^*(E)$, and (even) invertible elements $\mu_L \in \text{End}^*(vv^*E)$ and $\mu_R \in \text{End}^*(v^*vE)$ such that

- v commutes with (the representation of) A , and vv^* and v^*v commute with D ;
- $v^*vA \subseteq \overline{\text{span}}(A\mathcal{Q}) \cap \overline{\text{span}}(\mathcal{Q}A)$, where $\mathcal{Q}$ is the set of $a \in \text{End}^*(v^*vE)$ such that

$$a, a\mu_R, a\mu_R^{-1*}, vav^*, \mu_L vav^*, \mu_L^{-1} vav^* \in \text{Lip}_0^*(D),$$

$va\mu_R^{-1*} \text{dom } D \subseteq \mu_L^{-1*} \text{dom } D$, and $\mu_L D \mu_L^* va - va \mu_R D \mu_R^*$ extends to an adjointable operator;

- $(A, (1 - vv^*)E_B, (1 - vv^*)D(1 - vv^*))$ is unitarily equivalent to (A, E'_B, D_1) ; and
- $(A, (1 - v^*v)E_B, (1 - v^*v)D(1 - v^*v))$ is unitarily equivalent to (A, E''_B, D_2) .

Example 5.15. Suppose that (U, μ) is a conformal transformation from the unbounded Kasparov module (A, E'_B, D_1) to (A, E''_B, D_2) . Then

$$\left(A, (E' \oplus E'')_B, \begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix}; v = \begin{pmatrix} & 0 \\ U & \end{pmatrix}, (\mu_L, \mu_R) = (1, \mu) \right)$$

is a conformism between the two modules. We point out, however, that singular conformal transformations do not give rise to conformisms; we return to this in Example 6.21.

Unlike additive perturbations, conformal transformations are not necessarily reversible nor composable. The extra room in the definition of conformism circumvents this issue, as the next Proposition shows.

Proposition 5.16. *Conformism of unbounded Kasparov modules is an equivalence relation and is compatible with direct sums.*

Proof. For symmetry, note that, for $a \in \mathcal{Q}$,

$$-(\mu_L D \mu_L^* va - va \mu_R D \mu_R^*)^* = \mu_R D \mu_R^* v^* (va^* v^*) - v^* (va^* v^*) \mu_L D \mu_L^*$$

and similarly for the other conditions, so that making the substitution of v^* for v and (μ_R, μ_L) for (μ_L, μ_R) reverses the roles of (A, E'_B, D_1) and (A, E''_B, D_2) .

For transitivity, suppose that $(A, E_B, D; v, \mu)$ defines a conformism between $(A, E_{1,B}, D_1)$ and $(A, E_{2,B}, D_2)$, and that $(A, E'_B, D'; v', \mu')$ is a conformism between $(A, E_{2,B}, D_2)$ and $(A, E_{3,B}, D_3)$. Let $U : (1 - v^*v)E \rightarrow (1 - v'v'^*)E$ be the unitary equivalence between $(A, (1 - v^*v)E_B, (1 - v^*v)D(1 - v^*v))$ and $(A, (1 - v'v'^*)E'_B, (1 - v'v'^*)D'(1 - v'v'^*))$. Then

$$(A, (E \oplus E')_B, D \oplus D'; v + U + v', (\mu_L \oplus (1 - v'v'^* + \mu'_L), (\mu_R + 1 - v^*v) \oplus \mu'_R))$$

is a conformism between $(A, E_{1,B}, D_1)$ and $(A, E_{3,B}, D_3)$. We have

$$(v + U + v')(v + U + v')^* = vv^* \oplus 1 \quad (v + U + v')^*(v + U + v') = 1 \oplus v'^*v'.$$

Because D commutes with $(1 - v^*v)$ and D' commutes with $(1 - v'v'^*)$, $D'U = UD$ on $E \oplus E'$. Let $a \in \mathcal{Q}$, $a' \in \mathcal{Q}'$, and $b \in \text{Lip}_0^*((1 - v^*v)D(1 - v^*v))$. Then $(a + b) \oplus a' \in \text{End}^*(E \oplus v'^*v'E')$ and

$$\begin{aligned} & (\mu_L D \mu_L^* \oplus (1 - v'v'^* + \mu'_L) D' (1 - v'v'^* + \mu'_L)^*) (v + U + v') ((a + b) \oplus a') \\ & - (v + U + v') ((a + b) \oplus a') ((\mu_R + 1 - v^*v) D (\mu_R + 1 - v^*v)^* \oplus \mu'_R D' \mu_R'^*) \\ & = (\mu_L D \mu_L^* va - va \mu_R D \mu_R^*) + U[D, b] + (\mu'_L D' \mu_L'^* v' a' - v' a' \mu'_R D' \mu_R'^*) \end{aligned}$$

is bounded. We remark that

$$(v + U + v')((a + b) \oplus a')(v + U + v')^* = vav^* + UbU^* + v'a'v'^*$$

and the remaining conditions are easily verified. We have

$$(1 \oplus v'^*v')A \subseteq \overline{\text{span}}(A(\mathfrak{Q} + \text{Lip}_0^*((1 - v^*v)D(1 - v^*v))) \oplus A\mathfrak{Q}') \\ \cap \overline{\text{span}}((\mathfrak{Q} + \text{Lip}_0^*((1 - v^*v)D(1 - v^*v)))A \oplus \mathfrak{Q}'A),$$

as required. $\square$

Proposition 5.17. *Given conformant unbounded Kasparov modules (A, E'_B, D_1) and (A, E''_B, D_2) , their bounded transforms (A, E'_B, F_{D_1}) and (A, E''_B, F_{D_2}) are cobordant and so define the same element in $KK(A, B)$.*

Proof. Let $(A, E_B, D; v, \mu)$ be a conformism between (A, E'_B, D_1) and (A, E''_B, D_2) . We claim that $(A, E_B, F_D; v)$ is a bounded cobordism between (A, E'_B, F_{D_1}) and (A, E''_B, F_{D_2}) . The main point to elucidate is the local compactness of $[F_D, v]$. Let $l_1, l_2, r_1, r_2, r_3 \in \mathfrak{Q}$; then

$$[\mu_R D \mu_R^*, r_1^* r_2] = -(\mu_L D \mu_L^* v r_1 - v r_1 \mu_R D \mu_R^*)^* v r_2 + r_1^* v^* (\mu_L D \mu_L^* v r_2 - v r_2 \mu_R D \mu_R^*)$$

is bounded, as is $[\mu_L D \mu_L^*, v l_1 l_2^* v^*]$. Theorem 2.42 shows that

$$(F_{\mu_L D \mu_L^*} - F_D) v l_1 l_2^* v^* \mu_L \langle D \rangle^\beta \quad (F_{\mu_R D \mu_R^*} - F_D) r_1^* r_2^* r_3^* \langle D \rangle^\beta$$

are bounded for $\beta < 1$. With $l = l_1 l_2^*$ and $r = r_1^* r_2^* r_3^*$,

$$(F_D v l v^* - v l v^* F_{\mu_L D \mu_L^*}) \langle \mu_L D \mu_L^* \rangle^\beta \quad (F_{\mu_R D \mu_R^*} r - r F_D) \langle D \rangle^\beta$$

are hence bounded. Let $a \in \mathfrak{Q}$. By Proposition 2.21,

$$(F_{\mu_L D \mu_L^*} v a - v a F_{\mu_R D \mu_R^*}) \langle \mu_R D \mu_R^* \rangle^\beta$$

is bounded, and we have

$$[F_D, v] l a r = (F_D v l v^* - v l v^* F_{\mu_L D \mu_L^*}) v a r + v l v^* (F_{\mu_L D \mu_L^*} v a - v a F_{\mu_R D \mu_R^*}) r \\ + v l a (F_{\mu_R D \mu_R^*} r - r F_D) - v [F_D, l a r].$$

For $b \in A$, $[F_D, v] l a r b = [F_D, v] l a r \langle D \rangle^\beta \langle D \rangle^{-\beta} b$ is compact. By the inclusion of $A \subseteq \overline{(\mathfrak{Q} \mathfrak{Q}^*)^3 A}$, we are done. $\square$

The identification of inverses in Corollary 5.12 is still valid for conformism, and so we have proved

Theorem 5.18. *Conformism classes of unbounded Kasparov A - B -modules form an abelian group which surjects onto $KK(A, B)$.*

In fact, given two conformant unbounded Kasparov modules (A, E'_B, D_1) and (A, E''_B, D_2) , their logarithmic transforms (A, E'_B, L_{D_1}) and (A, E''_B, L_{D_2}) will be cobordant in the unbounded sense, which would give another proof of Proposition 5.17.

6 Conformally generated cycles and twisted spectral triples

In this section, we present a new way of guaranteeing that unbounded cycles without bounded commutators in the conventional sense have well-defined bounded transforms. In particular, our approach covers all known examples of twisted spectral triples with well-defined bounded transforms. One of the features of our approach is that no ‘twist’ or automorphism of the algebra is involved, which suggests that this structure is a red herring, at least as far as KK-theory is concerned.

So far, relatively few examples of twisted spectral triples have been described in the literature. One reason for this is the difficulty in guaranteeing that the bounded transform is well-defined. The *Lipschitz regularity* condition [CM08, Definition 3.1 (3.3)], although natural in a relatively classical situation, where a pseudodifferential calculus is available, is not so satisfactory in general. Part of the motivation for developing the technical results in this paper was the construction of twisted spectral triples for certain badly behaved dynamical systems, for which Lipschitz regularity becomes intractable.

The framework of conformally generated cycles is applicable to all examples of twisted spectral triples with topological content in the literature, as far as we are aware. Among those examples to which it can be applied are

- Conformal perturbations of spectral triples (or Kasparov modules) of the $D \rightsquigarrow kDk$ type [CM08, §2.2];
- Crossed products by groups of conformal diffeomorphisms [CM08, §2.3] [Mos10, §3.1] (and, more generally, the dual-Green–Julg map of conformally equivariant unbounded Kasparov modules);
- Cuntz–Krieger algebras, as in [Haw13, Chapter 6];
- Unbounded modular cycles, in the sense of [Kaa21, Definition 3.1]; and
- Pseudodifferential calculus on the Podleś sphere and other examples with *diagonalisable twist*, as treated in [MY19].

The multiplicative perturbation theory developed in §2.3 was partly inspired by [MY19]. In principle, the techniques here could be used to build pseudodifferential calculi, mimicking the approach in [MY19]. Examples of twisted spectral triples to which our methods do not apply are

- The quantum statistical mechanics constructions of [GMT14] which are not Lipschitz regular and, indeed, whose bounded transform is manifestly not a Fredholm module;
- The Lorentzian geometry constructions of [DFLM18], whose twist is an involution and not relevant to the topology; and
- Examples without (locally) compact resolvent, such as those in [KS12] and [IM16].

To formulate a framework sufficient to describe the examples, we will again use the notions of matched operators and compactly supported states from Appendices A.3 and A.4. Recall from Proposition A.26 the $*$ -algebra of matched operators $\text{Mtc}^*(F, C)$ on the module F with respect to the algebra C .

Definition 6.1. A *conformally generated A - B -cycle* $(A, E_B, D; C, \mu)$ is an A - B -correspondence E , a regular operator D on E , a C^* -algebra C , and a pair $\mu = (\mu_L, \mu_R)$ of (even) C -matched operators on $E \otimes C$, whose inverses are also C -matched, such that

1. D is self-adjoint;
2. $(1 + D^2)^{-1}a$ is compact for all $a \in A$; and

3. With $\mathcal{L}$ the set of $a \in \text{Mtc}^*(E \otimes C, C)$ such that

$$[D \otimes 1, a] \quad [\mu_L(D \otimes 1)\mu_L^*, a] \quad [D \otimes 1, \mu_L^*a] \quad [D \otimes 1, \mu_L^{-1}a] \quad [D \otimes 1, a\mu_L] \quad [D \otimes 1, a\mu_L^{-1*}]$$

are C -matched, with $\mathcal{R}$ the set of $a \in \text{Mtc}^*(E \otimes C, C)$ such that

$$[D \otimes 1, a] \quad [\mu_R(D \otimes 1)\mu_R^*, a] \quad [D \otimes 1, \mu_R^*a] \quad [D \otimes 1, \mu_R^{-1}a] \quad [D \otimes 1, a\mu_R] \quad [D \otimes 1, a\mu_R^{-1*}]$$

are C -matched, and with

$$\mathcal{T} = \{a \in \text{Mtc}^*(E \otimes C, C) \mid \mu_L(D \otimes 1)\mu_L^*a - a\mu_R(D \otimes 1)\mu_R^* \in \text{Mtc}^*(E \otimes C, C)\},$$

the algebra A is contained in $C^*((1 \otimes \psi)(\mathcal{L}\mathcal{T}\mathcal{R}) \mid \psi \in \mathcal{S}_c(C))$, where $\mathcal{S}_c(C)$ are the compactly supported states on C .

Remarks 6.2.

1. The spaces $\mathcal{L}$ and $\mathcal{R}$ are $*$ -algebras. The space $\mathcal{T}$ is a ternary ring of C -matched operators. We have $\mathcal{L}\mathcal{T} \subseteq \mathcal{T}$ and $\mathcal{T}\mathcal{R} \subseteq \mathcal{T}$, and $\mathcal{L}\mathcal{T}\mathcal{R}$ is also a ternary ring of C -matched operators.
2. Proposition A.34 shows that the application of a compactly supported state on C to a C -matched operator is well-defined. By Proposition A.35, $\mathcal{S}_c(C)$ in condition 3. of Definition 6.1 could be replaced with $\mathcal{S}(C)$, the set of all states on C , at least to those elements of $\mathcal{L}\mathcal{T}\mathcal{R}$ which are adjointable.
3. Any unbounded Kasparov module (A, E_B, D) can be regarded as a conformally generated cycle $(A, E_B, D; \mathbb{C}, (1, 1))$.

One should think of conformally generated cycles as having a dynamical quality, in addition to a strictly geometrical one, with the C^* -algebra C as a ‘dynamical direction’. In examples, the elements of $\mathcal{T}$ correspond to endomorphisms with bounded ‘twisted’ commutators with D , as we will see in Theorem 6.5. Elements of $\mathcal{L}, \mathcal{R}$ encode the regularity of the “conformal factors” μ_L, μ_R .

Definition 6.1 could be extended to higher-order cycles but, in the interests of readability, we do not pursue this here.

Remark 6.3. Using Proposition A.28, we may specialise Definition 6.1 to the case when $C = C_0(X)$ for a locally compact Hausdorff space X . Consider a conformally generated A - B -cycle $(A, E_B, D; C_0(X), \mu)$. We may interpret $\mu = (\mu_L, \mu_R)$ as a pair of $*$ -strongly continuous families $(\mu_{L,x})_{x \in X}$ and $(\mu_{R,x})_{x \in X}$ of (even) invertible adjointable operators over X . Condition 3. of Definition 6.1 becomes:

3'. With $\mathcal{L}$ the set of $*$ -strongly continuous maps $a : X \rightarrow \text{End}^*(E)$ such that the maps

$$\begin{aligned} x \mapsto [D, a_x] & \quad x \mapsto [\mu_{L,x}D\mu_{L,x}^*, a_x] \\ x \mapsto [D, \mu_{L,x}^*a_x] & \quad x \mapsto [D, \mu_{L,x}^{-1}a_x] \quad x \mapsto [D, a_x\mu_{L,x}] \quad x \mapsto [D, a_x\mu_{L,x}^{-1*}] \end{aligned}$$

are $*$ -strongly continuous to $\text{End}^*(E)$, with $\mathcal{R}$ the set of $*$ -strongly continuous maps $a : X \rightarrow \text{End}^*(E)$ such that the maps

$$\begin{aligned} x \mapsto [D, a_x] & \quad x \mapsto [\mu_{R,x}D\mu_{R,x}^*, a_x] \\ x \mapsto [D, \mu_{R,x}^*a_x] & \quad x \mapsto [D, \mu_{R,x}^{-1}a_x] \quad x \mapsto [D, a_x\mu_{R,x}] \quad x \mapsto [D, a_x\mu_{R,x}^{-1*}] \end{aligned}$$

are $*$ -strongly continuous to $\text{End}^*(E)$, and with

$$\mathcal{T} = \{a \in C(X, \text{End}^*(E)_{*-s}) \mid x \mapsto \mu_{L,x}D\mu_{L,x}^*a_x - a_x\mu_{R,x}D\mu_{R,x}^* \in C(X, \text{End}^*(E)_{*-s})\},$$

the algebra A is contained in $C^*((1 \otimes m)(\mathcal{L}\mathcal{T}\mathcal{R}) \mid m \in \mathcal{M}_c(X))$, where $\mathcal{M}_c(X)$ is the set of compactly supported Radon measures on X .

An important special case is when X is a discrete set (and, in particular, when X is a point). In this case, Condition 3. of Definition 6.1 becomes:

3". With $\mathcal{L}_x$ the set of $a \in \text{End}^*(E)$ such that

$$[D, a] \quad [\mu_{L,x} D \mu_{L,x}^*, a] \quad [D, \mu_{L,x}^* a] \quad [D, \mu_{L,x}^{-1} a] \quad [D, a \mu_{L,x}] \quad [D, a \mu_{L,x}^{-1*}]$$

are adjointable, with $\mathcal{R}_x$ the set of $a \in \text{End}^*(E)$ such that

$$[D, a] \quad [\mu_{R,x} D \mu_{R,x}^*, a] \quad [D, \mu_{R,x}^* a] \quad [D, \mu_{R,x}^{-1} a] \quad [D, a \mu_{R,x}] \quad [D, a \mu_{R,x}^{-1*}]$$

are adjointable, and with

$$\mathcal{T}_x = \{a \in \text{End}^*(E) \mid \mu_{L,x} D \mu_{L,x}^* a - a \mu_{R,x} D \mu_{R,x}^* \in \text{End}^*(E)\},$$

the algebra A is contained in the C^* -algebra $C^*(\mathcal{L}_x \mathcal{T}_x \mathcal{R}_x \mid x \in X)$.

Theorem 6.4. *Let $(A, E_B, D; C, \mu)$ be a conformally generated A - B -cycle. Then (A, E_B, F_D) is a bounded Kasparov module.*

Proof. The main point to check is that $[F_D, a]$ is compact for all $a \in A$. Let c be a positive element of the Pedersen ideal K_C , so that, by Proposition A.24, the restriction of μ to the $B \otimes \overline{\text{span}}(CcC)$ -module $E \otimes \overline{\text{span}}(CcC)$ is bounded. From now on, we work on the module $E \otimes \overline{\text{span}}(CcC)$. Let $l_1, l_2 \in \mathcal{L}$ and $r_1, r_2, r_3 \in \mathcal{R}$. Omitting instances of $\otimes 1$ for simplicity, Theorem 2.42 shows that

$$(F_{\mu_L D \mu_L^*} - F_D) l_1 l_2 \langle \mu_L D \mu_L^* \rangle^\beta \quad (F_{\mu_R D \mu_R^*} - F_D) r_1 r_2 r_3 \langle D \rangle^\beta$$

are bounded for $\beta < 1$. With $l = l_1 l_2$ and $r = r_1 r_2 r_3$,

$$(F_D l - l F_{\mu_L D \mu_L^*}) \langle \mu_L D \mu_L^* \rangle^\beta \quad (F_{\mu_R D \mu_R^*} r - r F_D) \langle D \rangle^\beta$$

are hence bounded. Let $t \in \mathcal{T}$. By Proposition 2.21,

$$(F_{\mu_L D \mu_L^*} t - t F_{\mu_R D \mu_R^*}) \langle \mu_R D \mu_R^* \rangle^\beta$$

is bounded and we have

$$[F_D, ltr] = (F_D l - l F_{\mu_L D \mu_L^*}) tr + l (F_{\mu_L D \mu_L^*} t - t F_{\mu_R D \mu_R^*}) r + lt (F_{\mu_R D \mu_R^*} r - r F_D).$$

We see that $[F_D \otimes 1, ltr] \langle D \rangle^\beta \otimes 1$ is bounded on the module $E \otimes \overline{\text{span}}(CcC)$. This is the case for every positive $c \in K_C$ so, by Proposition A.24, $[F_D \otimes 1, ltr] \langle D \rangle^\beta \otimes 1$ is a C -matched operator on $E \otimes C$. Let ψ be a compactly supported state on C . By Proposition A.34, we may apply $1 \otimes \psi$ to $[F_D \otimes 1, ltr] \langle D \rangle^\beta \otimes 1$ to obtain the bounded operator

$$(1 \otimes \psi)([F_D \otimes 1, ltr] \langle D \rangle^\beta \otimes 1) = [F_D, (1 \otimes \psi)(ltr)] \langle D \rangle^\beta.$$

For $a \in A$ the operator

$$[F_D, 1 \otimes \psi(ltr)] a = [F_D, (1 \otimes \psi)(ltr)] \langle D \rangle^\beta \langle D \rangle^{-\beta} a$$

is compact. Using the Leibniz rule, $[F_D, b]$ is compact for all $b \in C^*((1 \otimes \psi)(\mathcal{L} \mathcal{T} \mathcal{R}) \mid \psi \in \mathcal{S}_c(C))$, which includes A . $\square$

We now consider conformal perturbations of unbounded Kasparov modules, which include the conformal perturbations of noncommutative tori [CM08, §2.2].

Theorem 6.5. *Let (A, E_B, D) be an unbounded Kasparov module. Let k be an invertible normal element of $\text{End}^*(E)$. Suppose that $\overline{\text{span}}(\mathcal{M}A\mathcal{M}) \supseteq A$ where $\mathcal{M}$ is the set of $a \in \text{End}^*(E)$ such that*

$$[kDk^*, a] \quad [D, a] \quad [D, k^*]a \quad [D, k^*k]a \quad a[D, k] \quad a[D, k^*k]$$

are bounded. Then $(A, E_B, kDk^; \mathbb{C}, (k^{-1}, k^{-1}))$ is a conformally generated cycle. In particular, if k is normal and invertible and (A, E_B, D) is an unbounded Kasparov module with $[D, k]$ bounded then the data $(A, E_B, kDk^*; \mathbb{C}, (k^{-1}, k^{-1}))$ define a conformally generated cycle. Hence (A, E_B, F_{kDk^*}) is a Kasparov module and $[(A, E_B, F_{kDk^*})] = [(A, E_B, F_D)] \in KK(A, B)$.*

Proof. It is straightforward to check that, for all $a \in \mathcal{M}$,

$$[kDk^*, a] \quad [D, a] \quad [kDk^*, k^{-1*}a] \quad [kDk^*, ka] \quad [kDk^*, ak^{-1}]$$

are bounded so that $\mathcal{M} \subseteq \mathcal{L} \cap \mathcal{R}$ where $\mathcal{L}, \mathcal{R}$ are as in Definition 6.1. As $A \subseteq \overline{\mathcal{T}} = \overline{\text{Lip}_0^*(D)}$, we are done. For the final statements, if $[D, k]$ is bounded then $\mathcal{M}$ contains scalar multiples of the identity and so $\overline{\text{span}}(\mathcal{M}A\mathcal{M}) \supseteq A$. An application of Theorem 2.42 gives the equality of the Kasparov classes. $\square$

Example 6.6. We recall the noncommutative torus $C(\mathbb{T}_\alpha^2)$ from Example 2.18 and the spectral triple

$$\left(C(\mathbb{T}_\alpha^2), L^2(\mathbb{T}_\alpha^2) \otimes \mathbb{C}^2, D := \begin{pmatrix} \delta_1 + \tau \delta_2 & \\ & \delta_1 + \bar{\tau} \delta_2 \end{pmatrix} \right).$$

As in Example 2.18, choose a positive invertible element $k \in C(\mathbb{T}_\alpha^2)$ in the domains of δ_1 and δ_2 . Using left multiplication by k yields a conformally generated cycle

$$\left(C(\mathbb{T}_\alpha^2), L^2(\mathbb{T}_\alpha^2) \otimes \mathbb{C}^2, kDk; \mathbb{C}, (k^{-1}, k^{-1}) \right).$$

Thus the classes defined by F_D and F_{kDk} in $KK(C(\mathbb{T}_\alpha^2), \mathbb{C})$ coincide.

The unbounded Kasparov module $(C(\mathbb{T}_\alpha^2), L^2(C(\mathbb{T}_\alpha^2), \Phi)_{C(S^1)}, \delta_2)$ also gives rise to a conformally generated cycle

$$(C(\mathbb{T}_\alpha^2), L^2(C(\mathbb{T}_\alpha^2), \Phi)_{C(S^1)}, k\delta_2k; \mathbb{C}, (k^{-1}, k^{-1}))$$

where $k \in C(\mathbb{T}_\alpha^2)$ is now a positive invertible element in the domain of δ_2 . Thus the classes defined by F_{δ_2} and $F_{k\delta_2k}$ in $KK(C(\mathbb{T}_\alpha^2), C(S^1))$ coincide.

Next we consider unbounded modular cycles in the sense of [Kaa21, Definition 3.1] [Kaa24, Definition 8.1]. Using our methods the bounded transform can be achieved in greater generality. Compare Proposition 2.51.

Proposition 6.7. *Let E be an A - B correspondence. Let D be a self-adjoint regular operator and Δ_+ and Δ_- a pair of commuting positive adjointable operators on E such that*

- *For all $a \in A$, $(1 + D^2)^{-1}a$ is compact and the sequence $(a(\Delta_+ + \Delta_-)(\Delta_+ + \Delta_- + \frac{1}{n})^{-1})_{n=1}^\infty$ converges in norm to (the representation of) a ;*
- *$\{\Delta_+, \Delta_-\} \text{ dom } D \subseteq \text{dom } D$ and $[D, \Delta_+], [D, \Delta_-]$ are bounded; and*
- *$A \subseteq \overline{\mathcal{N}}$, where $\mathcal{N}$ is the set of $a \in \text{End}^*(E)$ such that $\Delta_- Da \Delta_+ - \Delta_+ a D \Delta_-$ is bounded.*

Let $(h_n)_{n \in \mathbb{N}_{\geq 1}} \subseteq C_b^\infty(\mathbb{R}_+^\times)$ be any sequence of positive functions with bounded reciprocals which agree with the function $x \mapsto x^{-1/2}$ on the interval $[\frac{1}{n}, n]$. Then, with $\mu_{L,n} = \mu_{R,n} = h_n(\Delta_+)h_n(\Delta_-)^{-1}$, the data $(A, E_B, D; C_0(\mathbb{N}_{\geq 1}), \mu)$ define a conformally generated cycle.

Proof. First, by the smooth functional calculus of Theorem 2.52, $[D, h_n(\Delta_+)h_n(\Delta_-)^{-1}]$ is bounded so $1 \in \mathcal{L}_n, \mathcal{R}_n$ for every $n \in \mathbb{N}_{\geq 1}$. Second, $\mathcal{T}_n$ consists of those $b \in \text{End}^*(E)$ such that

$$[h_n(\Delta_+)h_n(\Delta_-)^{-1}Dh_n(\Delta_+)h_n(\Delta_-)^{-1}, b]$$

extends to an adjointable operator. Let $f_1, f_2, f_3, f_4 \in C_c^\infty((\frac{1}{n}, n))$ and $a \in \mathcal{N}$ and define $b \in \text{End}^*(E)$ to be the product

$$f_1(\Delta_+)f_2(\Delta_-)af_3(\Delta_+)f_4(\Delta_-) \in C_0((\frac{1}{n}, n))(\Delta_+)C_0((\frac{1}{n}, n))(\Delta_-)\mathcal{N}C_0((\frac{1}{n}, n))(\Delta_+)C_0((\frac{1}{n}, n))(\Delta_-).$$

Then $bh_n(\Delta_+)h_n(\Delta_-)^{-1} = b\Delta_+^{-1/2}\Delta_-^{1/2}$ and, again using the smooth functional calculus,

$$\begin{aligned} & [h_n(\Delta_+)h_n(\Delta_-)^{-1}Dh_n(\Delta_+)h_n(\Delta_-)^{-1}, b] \\ &= f_1(\Delta_+)f_2(\Delta_-)\Delta_+^{-1}(\Delta_-Da\Delta_+ - \Delta_+aD\Delta_-)\Delta_+^{-1}f_3(\Delta_+)f_4(\Delta_-) \\ & \quad + h_n(\Delta_+)h_n(\Delta_-)^{-1} \left[D, \Delta_+^{-1/2}\Delta_-^{1/2}f_1(\Delta_+)f_2(\Delta_-) \right] af_3(\Delta_+)f_4(\Delta_-) \\ & \quad + f_1(\Delta_+)f_2(\Delta_-)a \left[D, \Delta_-^{1/2}\Delta_+^{-1/2}f_3(\Delta_+)f_4(\Delta_-) \right] h_n(\Delta_+)h_n(\Delta_-)^{-1} \end{aligned}$$

is bounded. The closure of $C_0((\frac{1}{n}, n))(\Delta_+)C_0((\frac{1}{n}, n))(\Delta_-)$ is $C^*(\Delta_+, \Delta_-)$. By Lemma 2.53, $A \subseteq \overline{AC^*(\Delta_+, \Delta_-)}$ so that

$$\overline{\mathcal{L}\mathcal{T}\mathcal{R}} \supseteq \overline{C^*(\Delta_+, \Delta_-)\mathcal{N}C^*(\Delta_+, \Delta_-)} \supseteq \overline{C^*(\Delta_+, \Delta_-)AC^*(\Delta_+, \Delta_-)} \subseteq A$$

and we are done. $\square$

As a last application we consider again the relation to the logarithmic transform.

Proposition 6.8. *cf. [GMR19, Corollary 1.20] Let (A, E_B, D) consist of a C^* -algebra A represented on a Hilbert B -module E and a regular operator D on E , such that*

- D is self-adjoint;
- $(1 + D^2)^{-1/2}a$ is compact for all $a \in A$; and
- There is a dense subset of $a \in A$ such that $a \text{ dom } D \subseteq \text{dom } D$ and $[F_D, a] \log \langle D \rangle$ is bounded.

Then, with $L_D = F_D \log \langle D \rangle$, the triple (A, E_B, L_D) is an unbounded Kasparov module whose bounded transform is equal to (A, E_B, F_D) up to a locally compact difference.

Theorem 6.9. *Let $(A, E_B, D; C, (\mu_L, \mu_R))$ be a conformally generated cycle. Then (A, E_B, L_D) is an unbounded Kasparov module.*

Proof. By the Proof of Theorem 6.4, $[F_D, (1 \otimes \psi)(ltr)] \langle D \rangle^\beta$ is bounded for $\psi \in \mathcal{S}_c(C)$, $l \in \mathcal{L}^2$, $t \in \mathcal{T}$, $r \in \mathcal{R}^3$, and $\beta < 1$. We have

$$\begin{aligned} ltr \text{ dom}(D \otimes 1)(1 \otimes K_C) &\subseteq l\mu_R^{-1*} \text{ dom}(D \otimes 1)(1 \otimes K_C) \subseteq l\mu_L^{-1*} \text{ dom}(D \otimes 1)(1 \otimes K_C) \\ &\subseteq \text{ dom}(D \otimes 1)(1 \otimes K_C). \end{aligned}$$

Hence $(\langle D \rangle \otimes 1)ltr(\langle D \rangle^{-1} \otimes 1)$ is C -matched. Applying Proposition A.34,

$$(1 \otimes \psi)(\langle D \rangle \otimes 1)ltr(\langle D \rangle^{-1} \otimes 1) = \langle D \rangle(1 \otimes \psi)(ltr)\langle D \rangle^{-1}$$

is an adjointable operator on E , and so $(1 \otimes \psi)(ltr) \text{ dom } D \subseteq \text{ dom } D$. By Proposition 2.44, the commutator $[L_D, (1 \otimes \psi)(ltr)]$ is bounded. By the Leibniz rule, $[L_D, b]$ is bounded for all b in the $*$ -algebra generated by $\{(1 \otimes \psi)(\mathcal{L}\mathcal{T}\mathcal{R}) | \psi \in \mathcal{S}_c(C)\}$. This is dense in $C^*((1 \otimes \psi)(\mathcal{L}\mathcal{T}\mathcal{R}) | \psi \in \mathcal{S}_c(C))$, which includes A . $\square$

In principle, the logarithmic transform, if carried out piece-by-piece, could be used to produce KK-classes from “multi-twisted” spectral triples which have appeared in the literature, such as for quantum groups [KK20] and dynamical systems [KK22]. (See also [DS22], where an approach similar to that of [Sit15] is used to obtain ordinary spectral triples from partial conformal rescalings.) The development of such a framework would be beyond the scope of this paper.

6.1 Descent and the dual-Green–Julg map for conformal equivariance

In the conformally equivariant setting, the descent map and the dual-Green–Julg map produce conformally generated cycles.

Proposition 6.10. *Let G be a locally compact group and let (A, E_B, D) be a $(\mu_g)_{g \in G}$ -conformally G -equivariant unbounded Kasparov module. Then, for $t \in \{u, r\}$,*

$$(A \rtimes_t G, (E \rtimes_t G)_{B \rtimes_t G}, \tilde{D}; C_0(G), (1, \tilde{\mu}_g)_{g \in G})$$

is a conformally generated cycle, where $\tilde{D}$ is the regular operator given on $\xi \in C_c(G, E) \subseteq E \rtimes_t G$ by $(\tilde{D}\xi)(h) = D(\xi(h))$ and $(\tilde{\mu}_g)_{g \in G}$ are given by $(\tilde{\mu}_g\xi)(h) = \mu_g(\xi(h))$.

Proof. The local compactness of the resolvent is the same as in the uniform case, Proposition 3.14. Recall the spaces $\mathcal{L}$, $\mathcal{T}$, and $\mathcal{R}$ of Remark 6.3. It is straightforward to verify that the constant families $(\tilde{d})_{g \in G} \in \mathcal{L}$ and $(\tilde{b}^*\tilde{c})_{g \in G} \in \mathcal{R}$ for all $d \in \text{Lip}_0^*(D)$ and $b, c \in \mathcal{Q}$. Let $(u_g)_{g \in G} \subseteq \text{End}_{B \rtimes_t G}^*(E \rtimes_t G)$ be the canonical unitaries implementing the group action, given by

$$(u_h\xi)(g) = U_h\xi(h^{-1}g)$$

on $\xi \in C_c(G, E)$ (where we recall the notation of Definition 3.6). A family of operators t is in $\mathcal{T}$ if $g \mapsto \tilde{D}t_g - t_g\tilde{\mu}_g\tilde{D}\tilde{\mu}_g^*$ is $*$ -strongly continuous into bounded operators. Using the condition for conformal equivariance that for $a \in \mathcal{Q}$ the map

$$g \mapsto U_gDU_g^*a - a\mu_gD\mu_g^*$$

is $*$ -strongly continuous into bounded operators, we see that $g \mapsto u_g^*\tilde{a}$ is in $\mathcal{T}$. So, $g \mapsto \tilde{d}u_g^*\tilde{a}\tilde{b}^*\tilde{c}$ is in $\mathcal{L}\mathcal{T}\mathcal{R}$.

We now evaluate $\mathcal{L}\mathcal{T}\mathcal{R}$ on compactly supported Radon measures on G and ask if this generates $A \rtimes_t G$. It will suffice to integrate the paths $g \mapsto \tilde{d}u_g^*\tilde{a}\tilde{b}^*\tilde{c}$, which are constant apart from u_g^* , against compactly supported continuous functions on G . Proceeding step-by-step,

$$\begin{aligned} \overline{\text{span}}(\text{Lip}_0^*(D)C_c^*(G)\mathcal{Q}\mathcal{Q}^*\mathcal{Q}) &\supseteq \overline{\text{span}}(AC_t^*(G)\mathcal{Q}\mathcal{Q}^*\mathcal{Q}) \\ &= \overline{\text{span}}((A \rtimes_t G)\mathcal{Q}\mathcal{Q}^*\mathcal{Q}) \\ &= \overline{\text{span}}(C_t^*(G)A\mathcal{Q}\mathcal{Q}^*\mathcal{Q}) \\ &\supseteq \overline{\text{span}}(C_t^*(G)A) \\ &= A \rtimes_t G \end{aligned}$$

as required. $\square$

Proposition 6.11. *Let (A, E_B, D) be a $(\mu_g)_{g \in G}$ -conformally G -equivariant unbounded Kasparov module, with G acting trivially on B . Then*

$$(A \rtimes_u G, E_B, D; C_0(G), (1, \mu_g)_{g \in G})$$

is a conformally generated cycle, with the integrated representation of $A \rtimes_u G$.

Proof. The local compactness of the resolvent is the same as in the uniform case, Proposition 3.15. Recall the spaces $\mathcal{L}$, $\mathcal{T}$, and $\mathcal{R}$ of Remark 6.3. It is straightforward to verify that the constant families $(d)_{g \in G} \in \mathcal{L}$ and $(b^*c)_{g \in G} \in \mathcal{R}$ for all $d \in \text{Lip}_0^*(D)$ and $b, c \in \mathcal{Q}$. A path of operators t is in $\mathcal{T}$ if

$$g \mapsto Dt_g - t_g\mu_gD\mu_g^*$$

is $*$ -strongly continuous into bounded operators. Using the condition for conformal equivariance that $g \mapsto U_gDU_g^*a - a\mu_gD\mu_g^*$ is $*$ -strongly continuous into bounded operators for $a \in \mathcal{Q}$, we see that $g \mapsto U_g^*a$ is in $\mathcal{T}$. So, $g \mapsto dU_g^*ab^*c$ is in $\mathcal{L}\mathcal{T}\mathcal{R}$. As in the Proof of Proposition 6.10, the closed span of $\text{Lip}_0^*(D)C_c^*(G)\mathcal{Q}\mathcal{Q}^*\mathcal{Q}$ includes $A \rtimes_u G$. $\square$

Remark 6.12. It is clear that the bounded transform $(A \rtimes_t G, (E \rtimes_t G)_{B \rtimes_t G}, F_{\tilde{D}} = \tilde{F}_D)$ of the descent

$$(A \rtimes_t G, (E \rtimes_t G)_{B \rtimes_t G}, \tilde{D}; C_0(G), (1, \tilde{\mu}_g)_{g \in G})$$

of a conformally G -equivariant cycle (A, E_B, D) is exactly the descent of the bounded transform (A, E_B, F_D) . The same is true for the dual-Green–Julg map.

We recall the identity

$$2A^*CB = (A+B)^*C(A+B) - i(A+iB)^*C(A+iB) + (-1+i)(B^*CB + A^*CA)$$

for elements A , B , and C of a $*$ -algebra, which implies that

$$\text{span}\{x^*Cx | x \in \text{span}\{A, B\}\} = \text{span}\{x^*Cy | x, y \in \text{span}\{A, B\}\}.$$

Proposition 6.13. *Let $\mathbb{G}$ be a locally compact quantum group and let (A, E_B, D) be a μ -conformally $\mathbb{G}$ -equivariant unbounded Kasparov module. For $t \in \{u, r\}$, let ι be the inclusion $\text{End}^0(E) \rightarrow M(\text{End}^0(E) \rtimes_t \mathbb{G}) \cong \text{End}_{B \rtimes_t \mathbb{G}}^*(E \rtimes_t \mathbb{G})$. Then*

$$(A \rtimes_t \mathbb{G}, (E \rtimes_t \mathbb{G})_{B \rtimes_t \mathbb{G}}, \iota(D); C_0^r(\mathbb{G}), (1, (\iota \otimes \text{id})(\mu)))$$

is a conformally generated cycle.

Proof. The compactness of the resolvent is as in the Proof of Proposition 4.18. Recall the spaces $\mathcal{L}$, $\mathcal{T}$, and $\mathcal{R}$ of Definition 6.1. It is straightforward to verify that $\iota(d) \otimes 1 \in \mathcal{L}$ and $\iota(b^*c) \otimes s_2^*s_3 \in \mathcal{R}$ for all $d \in \text{Lip}_0^*(D)$, $b, c \in \mathcal{D}$, and $s_2, s_3 \in \mathcal{S}_b, \mathcal{S}_c$.

By the universality of the crossed product, see [Ver02, §4.1] [Vae05, §2.3], the morphism

$$\text{End}^0(E) \rtimes_u \mathbb{G} \rightarrow \text{End}^0(E) \rtimes_t \mathbb{G}$$

gives rise both to the morphism

$$\iota : \text{End}^0(E) \rightarrow M(\text{End}^0(E) \rtimes_t \mathbb{G}) \cong \text{End}^*(E \rtimes_t \mathbb{G})$$

and a unitary $X \in M((\text{End}^0(E) \rtimes_t \mathbb{G}) \otimes C_0^r(\mathbb{G})) \cong \text{End}^*((\text{End}^0(E) \rtimes_t \mathbb{G}) \otimes C_0^r(\mathbb{G}))$ such that

$$X(\iota(T) \otimes 1)X^* = (\iota \otimes \text{id})\delta_{\text{End}^0(E)}(T)$$

for $T \in \text{End}^0(E)$. Let $a \in \mathcal{D}$ and $s_1 \in \mathcal{S}_a$; then $X^*(\iota(a) \otimes s_1) \in \mathcal{T}$ because

$$\begin{aligned} & (\iota(D) \otimes 1)X^*(\iota(a) \otimes s_1) - X^*(\iota(a) \otimes s_1)(\iota \otimes \text{id})(\mu)(\iota(D) \otimes 1)(\iota \otimes \text{id})(\mu)^* \\ &= X^*(X(\iota(D) \otimes 1)X^*(\iota(a) \otimes s_1) - (\iota \otimes \text{id})((a \otimes s_1)\mu(D \otimes 1)\mu^*)) \\ &= X^*(\iota \otimes \text{id})\left(\delta_{\text{End}^0(E)}(D)(a \otimes s_1) - (a \otimes s_1)\mu(D \otimes 1)\mu^*\right) \\ &= X^*(\iota \otimes \text{id})\left(V_E(D \otimes_{\delta_B} 1)V_E^*(a \otimes s_1) - (a \otimes s_1)\mu(D \otimes 1)\mu^*\right) \end{aligned}$$

is $C_0^r(\mathbb{G})$ -matched. So, $(d \otimes 1)X^*(\iota(ab^*c) \otimes s_1s_2^*s_3)$ is in $\mathcal{L}\mathcal{T}\mathcal{R}$.

We need to show that $A \rtimes_t \mathbb{G}$ is contained in $C^*((1 \otimes \omega)(\mathcal{L}\mathcal{T}\mathcal{R})|\omega \in \mathcal{S}_c(C))$. Proceeding

step-by-step,

$$\begin{aligned}
& C^*((1 \otimes \omega)(\mathcal{L}\mathcal{T}\mathcal{R})|_{\omega \in \mathcal{S}_c(C)}) \\
& \supseteq \overline{\text{span}} \left\{ (1 \otimes \omega) ((\iota(d) \otimes 1)X^*(\iota(ab^*c) \otimes s_1 s_2^* s_3)) = \iota(d)(1 \otimes \omega) ((1 \otimes s_3^* s_2 s_1^*)X)^* \iota(ab^*c) \right. \\
& \quad \left. | a, b, c \in \mathcal{Q}; d \in \text{Lip}_0^*(D); s_1 \in \mathcal{S}_a; s_2 \in \mathcal{S}_b; s_3 \in \mathcal{S}_c; \omega \in \mathcal{S}_c(C_0^r(\mathbb{G})) \right\} \\
& = \overline{\text{span}} \left(\iota(\text{Lip}_0^*(D)) \{ (1 \otimes \omega) ((1 \otimes s_3^* s_2 s_1^*)X) | s_1 \in \mathcal{S}_a; s_2 \in \mathcal{S}_b; s_3 \in \mathcal{S}_c; \omega \in \mathcal{S}_c(C_0^r(\mathbb{G})) \}^* \iota(\mathcal{Q}) \right) \\
& \supseteq \overline{\text{span}} \left(\iota(\text{Lip}_0^*(D)) \{ (1 \otimes \eta^* s_4^* s_3^* s_2 s_1^*)X(1 \otimes s_4 \eta) \right. \\
& \quad \left. | s_1 \in \mathcal{S}_a; s_2 \in \mathcal{S}_b; s_3 \in \mathcal{S}_c; s_4 \in K_{C_0^r(\mathbb{G})}; \eta \in L^2(C_0^r(\mathbb{G})) \}^* \iota(\mathcal{Q}) \right) \\
& = \overline{\text{span}} \left(\iota(\text{Lip}_0^*(D)) \{ (1 \otimes \eta_1^* s_4^* s_3^* s_2 s_1^*)X(1 \otimes s_5 \eta_2) \right. \\
& \quad \left. | s_1 \in \mathcal{S}_a; s_2 \in \mathcal{S}_b; s_3 \in \mathcal{S}_c; s_4, s_5 \in K_{C_0^r(\mathbb{G})}; \eta_1, \eta_2 \in L^2(C_0^r(\mathbb{G})) \}^* \iota(\mathcal{Q}) \right) \\
& = \overline{\text{span}} \left(\iota(\text{Lip}_0^*(D)) \{ (1 \otimes \eta_1^*)X(1 \otimes \eta_2) | \eta_1, \eta_2 \in L^2(C_0^r(\mathbb{G})) \}^* \iota(\mathcal{Q}) \right) \\
& = \overline{\text{span}} \left(\iota(\text{Lip}_0^*(D)) \{ (1 \otimes \omega)(X) | \omega \in L^1(\mathbb{G}) \}^* \iota(\mathcal{Q}) \right) \\
& = \overline{\text{span}}(\iota(\text{Lip}_0^*(D))C_u^*(\mathbb{G})\iota(\mathcal{Q})) \\
& \supseteq \overline{\text{span}}(\iota(A)C_u^*(\mathbb{G})\iota(\mathcal{Q})) = \overline{\text{span}}((A \rtimes_u \mathbb{G})\iota(\mathcal{Q})) = \overline{\text{span}}(C_u^*(\mathbb{G})\iota(A\mathcal{Q})) \\
& \supseteq \overline{\text{span}}(C_u^*(\mathbb{G})\iota(A)) = A \rtimes_u \mathbb{G}
\end{aligned}$$

by the density of $L^2(\mathbb{G})K_{C_0^r(\mathbb{G})}\mathcal{S}_c^*\mathcal{S}_b\mathcal{S}_a^* \subseteq L^2(\mathbb{G})$ and the inclusion $A \subseteq \overline{\text{span}}(A\mathcal{Q})$. $\square$

Proposition 6.14. *Let $\mathbb{G}$ be a locally compact quantum group and let (A, E_B, D) be a conformally $\mathbb{G}$ -equivariant unbounded Kasparov module, with $\mathbb{G}$ acting trivially on B . Then*

$$(A \rtimes_u \mathbb{G}, E_B, D; C_0^r(\mathbb{G}), (1, \mu))$$

is a conformally generated cycle, with the integrated representation of $A \rtimes_u \mathbb{G}$.

Proof. Recall the spaces $\mathcal{L}$, $\mathcal{T}$, and $\mathcal{R}$ of Definition 6.1. It is straightforward to verify that $d \otimes 1 \in \mathcal{L}$ and $b^*c \otimes s_2^*s_3 \in \mathcal{R}$ for all $d \in \text{Lip}_0^*(D)$, $b, c \in \mathcal{Q}$, and $s_2, s_3 \in \mathcal{S}_b, \mathcal{S}_c$. Let $a \in \mathcal{Q}$ and $s_1 \in \mathcal{S}_a$; then, by Definition 4.22,

$$(D \otimes 1)V_E^*(a \otimes s) - V_E^*(a \otimes s)\mu(D \otimes 1)\mu^*$$

is $C_0^r(\mathbb{G})$ -matched and $V_E^*(a \otimes s_1) \in \mathcal{T}$. So $(d \otimes 1)V_E^*(ab^*c \otimes s_1 s_2^* s_3)$ is in $\mathcal{L}\mathcal{T}\mathcal{R}$.

We need to show that $A \rtimes_u \mathbb{G}$ is contained in $C^*((1 \otimes \omega)(\mathcal{L}\mathcal{T}\mathcal{R})|_{\omega \in \mathcal{S}_c(C)})$. The same manipulations as in the proof of Proposition 6.13, with V_E in place of X , show that

$$C^*((1 \otimes \omega)(\mathcal{L}\mathcal{T}\mathcal{R})|_{\omega \in \mathcal{S}_c(C)}) \supseteq \overline{\text{span}}(\text{Lip}_0^*(D)C_u^*(\mathbb{G})\mathcal{Q}) \supseteq A \rtimes_u \mathbb{G},$$

as required. $\square$

Remark 6.15. It is again clear that the bounded transform $(A \rtimes_t \mathbb{G}, (E \rtimes_t \mathbb{G})_{B \rtimes_t \mathbb{G}}, F_{\iota(D)} = \iota(F_D))$ of the descent

$$(A \rtimes_t \mathbb{G}, (E \rtimes_t \mathbb{G})_{B \rtimes_t \mathbb{G}}, \iota(D); C_0^r(\mathbb{G}), (1, (\iota \otimes \text{id})(\mu)))$$

of a conformally $\mathbb{G}$ -equivariant cycle (A, E_B, D) is exactly the descent of the bounded transform (A, E_B, F_D) . The same is true for the dual-Green–Julg map.

6.2 Equivalence relations on conformally generated cycles

In this section, we consider an equivalence relation on conformally generated cycles making the equivalence classes an abelian group, following §5.

Remark 6.16. The direct sum of two conformally generated cycles $(A, E_{1,B}, D_1; C_1, \mu_1)$ and $(A, E_{2,B}, D_2; C_2, \mu_2)$ is

$$(A, E_{1,B} \oplus E_{2,B}, D_1 \oplus D_2; C_1 \oplus C_2, \mu_1 \oplus 1 \oplus 1 \oplus \mu_2)$$

where $\mu_1 \oplus 1 \oplus 1 \oplus \mu_2 \in (E_1 \otimes C_1 \oplus E_2 \otimes C_1 \oplus E_1 \otimes C_2 \oplus E_2 \otimes C_2)^2$. If $C_1 = C_2$ or, more generally, if C_1 and C_2 have a common ideal J , one could write the direct sum in a smaller way. In practice, also, it is often possible to change C and μ without affecting the validity of a cycle $(A, E_B, D; C, \mu)$. One should therefore think of conformally generated cycles $(A, E_B, D; C_1, \mu_1)$ and $(A, E_B, D; C_2, \mu_2)$ as equivalent.

The external product of conformally generated cycles is not well-defined.

Definition 6.17. Two conformally generated cycles $(A, E_{1,B}, D_1; C_1, \mu_1)$ and $(A, E_{2,B}, D_2; C_2, \mu_2)$ are *conformant* if there exists a conformally generated cycle $(A, E_B, D; C, \mu)$ and an even partial isometry $v \in \text{End}^*(E)$ such that

1. v commutes with (the representation of) A , and vv^* and v^*v commute with D ;
2. $vA \subseteq C^*((1 \otimes \psi)(\mathcal{L}\mathcal{T}\mathcal{R})|\psi \in \mathcal{S}_c(C))$;
3. $(A, (1 - vv^*)E_B, (1 - vv^*)D(1 - vv^*))$ is unitarily equivalent to $(A, E_{1,B}, D_1)$; and
4. $(A, (1 - v^*v)E_B, (1 - v^*v)D(1 - v^*v))$ is unitarily equivalent to $(A, E_{2,B}, D_2)$.

Example 6.18. Let $(A, E_B, D; v)$ be a cobordism between unbounded Kasparov modules (A, E'_B, D_1) and (A, E''_B, D_2) . Then

$$(A, E_B, D; \mathbb{C}, (1, 1); v)$$

is a conformism between $(A, E'_B, D_1; \mathbb{C}, (1, 1))$ and $(A, E''_B, D_2; \mathbb{C}, (1, 1))$.

Example 6.19. Let $(A, E_B, D; v, \mu)$ be a conformism of unbounded Kasparov modules (A, E'_B, D_1) and (A, E''_B, D_2) . Then

$$(A, E_B, D; \mathbb{C}^2, (1 \oplus \mu_L, 1 \oplus \mu_R); v)$$

is a conformism between $(A, E'_B, D_1; \mathbb{C}, (1, 1))$ and $(A, E''_B, D_2; \mathbb{C}, (1, 1))$. Furthermore, $\mathcal{L}$, $\mathcal{T}$, and $\mathcal{R}$ all contain $\text{Lip}^*(D) \oplus 0$. Hence

$$vA \subseteq (v\mathcal{Q}\mathcal{Q}^*v^*)v\mathcal{Q}(\mathcal{Q}^*\mathcal{Q})\text{Lip}_0^*(D)^3 \subseteq (\mathcal{L}\mathcal{T}\mathcal{R})^2$$

and we are done.

Example 6.20. We pick up from the setting of Theorem 6.5, adopting the notation there. We will show that the conformally generated cycles

$$(A, E_B, D; \mathbb{C}, (1, 1)) \quad (A, E_B, kDk^*; \mathbb{C}, (k^{-1}, k^{-1}))$$

are conformant. A suitable conformism is

$$\left(A, (E \oplus E)_B, \begin{pmatrix} D & \\ & kDk^* \end{pmatrix}; \mathbb{C}, \left(\begin{pmatrix} 1 & \\ & k^{-1} \end{pmatrix}, \begin{pmatrix} 1 & \\ & k^{-1} \end{pmatrix} \right); \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right).$$

We check that

$$\begin{pmatrix} 1 & \\ & k^{-1} \end{pmatrix} \begin{pmatrix} D & \\ & kDk^* \end{pmatrix} \begin{pmatrix} 1 & \\ & k^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \\ & k^{-1} \end{pmatrix} \begin{pmatrix} D & \\ & kDk^* \end{pmatrix} \begin{pmatrix} 1 & \\ & k^{-1} \end{pmatrix}^* = 0$$

so that $\begin{pmatrix} 1 & 0 \\ & \end{pmatrix} \in \mathcal{T}$. Both $\mathcal{L}$ and $\mathcal{R}$ contain $\mathbb{C}1 \oplus \mathcal{M}$. We remark that $\mathcal{M}$ is a $*$ -algebra of operators, so $\overline{\text{span}}(\mathcal{M}^2) = \overline{\mathcal{M}}$. We have

$$\begin{pmatrix} 1 & 0 \\ & \end{pmatrix} A \subseteq \overline{\text{span}}\left(\begin{pmatrix} 1 & 0 \\ & \end{pmatrix} \begin{pmatrix} 1 & 0 \\ & \end{pmatrix} \begin{pmatrix} 0 & \\ & \mathcal{M}^2 \end{pmatrix} A \begin{pmatrix} 0 & \\ & \mathcal{M} \end{pmatrix}\right) \subseteq \overline{\text{span}}(\mathcal{L}\mathcal{T}\mathcal{R}\mathcal{L}\mathcal{T}\mathcal{R})$$

and we are done.

Example 6.21. Let $(U, (\mu_i)_{i \in I})$ be a singular conformal transformation from one unbounded Kasparov module, (A, E_B, D_1) , to another, (A, E'_B, D_2) , as in Definition 2.47. We will show that

$$\left(A, (E \oplus E')_B, \begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix}; C_0(\{\text{pt}\} \sqcup I), \left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}_{i \in I}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \oplus \begin{pmatrix} \mu_i & \\ & 1 \end{pmatrix}_{i \in I}\right); \begin{pmatrix} U & 0 \\ & \end{pmatrix}\right)$$

is a conformism between $(A, E_B, D_1; \mathbb{C}, (1, 1))$ and $(A, E'_B, D_2; \mathbb{C}, (1, 1))$. Here, I is treated as a discrete set. For $a \in \mathcal{M}_i$, we can check that

$$\begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix} \begin{pmatrix} Ua & 0 \\ & \end{pmatrix} - \begin{pmatrix} Ua & 0 \\ & \end{pmatrix} \begin{pmatrix} \mu_i & \\ & 1 \end{pmatrix} \begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix} \begin{pmatrix} \mu_i & \\ & 1 \end{pmatrix}^* = \begin{pmatrix} U(U^* D_2 U a - a \mu_i D_1 \mu_i^*) & 0 \\ & \end{pmatrix}$$

is bounded, so that $\begin{pmatrix} U & 0 \\ & \end{pmatrix} \begin{pmatrix} \mathcal{M} & 0 \\ & \end{pmatrix} \in \mathcal{T}_i$. One can check that $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \in \mathcal{L}_i$ and that $\mathcal{R}_i$ contains $\begin{pmatrix} \mathcal{M}_i^* \mathcal{M}_i & \\ & 0 \end{pmatrix}$. Furthermore, $\mathcal{L}_{\text{pt}}$, $\mathcal{T}_{\text{pt}}$, and $\mathcal{R}_{\text{pt}}$ all contain $\begin{pmatrix} \text{Lip}_0^*(D_1) & \\ & \text{Lip}_0^*(D_2) \end{pmatrix}$. Hence

$$\begin{aligned} \begin{pmatrix} U & 0 \\ & \end{pmatrix} A &\subseteq \overline{\text{span}}_{i \in I} \left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \begin{pmatrix} U & 0 \\ & \end{pmatrix} \begin{pmatrix} \mathcal{M} & 0 \\ & \end{pmatrix} \begin{pmatrix} \mathcal{M}_i^* \mathcal{M}_i & \\ & 0 \end{pmatrix} \begin{pmatrix} \text{Lip}_0^*(D_1) & \\ & 0 \end{pmatrix} \right) \\ &\subseteq \overline{\text{span}}_{i \in I} (\mathcal{L}_i \mathcal{T}_i \mathcal{R}_i \mathcal{L}_{\text{pt}} \mathcal{T}_{\text{pt}} \mathcal{R}_{\text{pt}}) \subseteq C^*(\mathcal{L}_x \mathcal{T}_x \mathcal{R}_x | x \in \{\text{pt}\} \sqcup I) \end{aligned}$$

and we are done.

Proposition 6.22. *Conformism of conformally generated cycles is an equivalence relation and is compatible with direct sums.*

Proof. For symmetry, note that $v^* A = (vA)^* \subseteq C^*((1 \otimes \psi)(\mathcal{L}\mathcal{T}\mathcal{R}) | \psi \in \mathcal{S}_c(C))$ so that making the substitution of v^* for v reverses the roles of (A, E'_B, D_1) and (A, E''_B, D_2) .

For transitivity, suppose that $(A, E_B, D; C, \mu; v)$ is a conformism between $(A, E_{1,B}, D_1; C_1, \mu_1)$ and $(A, E_{2,B}, D_2; C_2, \mu_2)$, and $(A, E'_B, D'; C', \mu'; v')$ is a conformism between $(A, E_{2,B}, D_2; C_2, \mu_2)$ and $(A, E_{3,B}, D_3; C_3, \mu_3)$. Let $U : (1 - v^* v)E \rightarrow E_2$ and $U' : (1 - v' v'^*)E \rightarrow E_2$ be the unitary equivalences between the cycles

$$(A, (1 - v^* v)E_B, (1 - v^* v)D(1 - v^* v)) \quad \text{and} \quad (A, (1 - v' v'^*)E'_B, (1 - v' v'^*)D'(1 - v' v'^*))$$

and the cycle $(A, E_{2,B}, D_2)$. Then

$$(A, (E \oplus E')_B, D \oplus D'; C \oplus C_2 \oplus C', \mu \oplus 1 \oplus v^* v + U^* \mu_2 U \oplus v' v'^* + U'^* \mu_2 U' \oplus 1 \oplus \mu'; v + U'^* U + v'),$$

is a conformism between $(A, E_{1,B}, D_1; C_1, \mu_1)$ and $(A, E_{3,B}, D_3; C_3, \mu_3)$, where

$$\begin{aligned} \mu \oplus 1 \oplus v^* v + U^* \mu_2 U \oplus v' v'^* + U'^* \mu_2 U' \oplus 1 \oplus \mu' \\ \in (E \otimes C) \oplus (E' \otimes C) \oplus (E \otimes C_2) \oplus (E' \otimes C_2) \oplus (E \otimes C') \oplus (E' \otimes C'). \end{aligned}$$

We have

$$(v + U'^* U + v')(v + U'^* U + v')^* = vv^* \oplus 1 \quad (v + U'^* U + v')^*(v + U'^* U + v') = 1 \oplus v'^* v'.$$

Let $\mathcal{L}''$, $\mathcal{T}''$, and $\mathcal{R}''$ be the spaces of Definition 6.1, corresponding to this cycle. We have

$$\mathcal{L} \oplus \mathcal{L}' \subseteq \mathcal{L}'' \quad \mathcal{T} \oplus \mathcal{T}' \subseteq \mathcal{T}'' \quad \mathcal{R} \oplus \mathcal{R}' \subseteq \mathcal{R}'',$$

so that $(v + v')A \subseteq C^*((1 \otimes \psi)(\mathcal{L}''\mathcal{T}''\mathcal{R}'')|\psi \in \mathcal{S}_c(C \oplus C_2 \oplus C'))$. Because D commutes with $(1 - v^*v)$ and D' commutes with $(1 - v'v^*)$, $D'U'^*U = U'^*D_2U = U'^*UD$ on $E \oplus E'$. Hence

$$U'^*\mathcal{L}_2\mathcal{T}_2\mathcal{R}_2U \subseteq \mathcal{L}''\mathcal{T}''\mathcal{R}''$$

and

$$\begin{aligned} U'^*UA &= U'^*AU \subseteq U'^*C^*((1 \otimes \psi)(\mathcal{L}_2\mathcal{T}_2\mathcal{R}_2)|\psi \in \mathcal{S}_c(C_2))U \\ &\subseteq C^*((1 \otimes \psi)(\mathcal{L}''\mathcal{T}''\mathcal{R}'')|\psi \in \mathcal{S}_c(C \oplus C_2 \oplus C')) \end{aligned}$$

as required. $\square$

Proposition 6.23. *Given two conformant conformally generated cycles $(A, E_{1,B}, D_1; C_1, \mu_1)$ and $(A, E_{2,B}, D_2; C_2, \mu_2)$, their bounded transforms $(A, E_{1,B}, F_{D_1})$ and $(A, E_{2,B}, F_{D_2})$ are cobordant and so define the same element in $KK(A, B)$.*

Proof. Let $(A, E_B, D; C, \mu; v)$ be a conformism between the two cycles $(A, E_{1,B}, D_1; C_1, \mu_1)$ and $(A, E_{2,B}, D_2; C_2, \mu_2)$. By Theorem 6.4, (A, E_B, F_D) is a bounded Kasparov module and $[F_D, vA] \subseteq \text{End}^0(E)$. By Lemma 5.5, $F_{(1-vv^*)D(1-vv^*)} = (1 - vv^*)F_D(1 - vv^*)$ on the module $(1 - vv^*)E$ and $F_{(1-v^*v)D(1-v^*v)} = (1 - v^*v)F_D(1 - v^*v)$ on the module $(1 - v^*v)E$. Hence $(A, E_B, F_D; v)$ is a bounded cobordism between (A, E'_B, F_{D_1}) and (A, E''_B, F_{D_2}) . $\square$

In the following, we use the notation $Z_{\mathcal{X}}(T) = \{x \in \mathcal{X} \mid [T, x] = 0\}$ for the centraliser in a subspace $\mathcal{X} \subseteq \text{Mtc}^*(E \otimes C, C)$ of an adjointable operator T on $E \otimes C$.

Definition 6.24. A conformally generated cycle $(A, E_B, D; C, \mu)$ is *positively degenerate* if there exists an odd self-adjoint unitary $s \in \text{End}^*(E)$, preserving the domain of D , such that

- The anticommutator $Ds + sD$ is semibounded below;
- $[\mu, s \otimes 1] = 0$; and
- $A \subseteq C^*((1 \otimes \psi)(Z_{\mathcal{L}}(s \otimes 1)Z_{\mathcal{T}}(s \otimes 1)Z_{\mathcal{R}}(s \otimes 1))|\psi \in \mathcal{S}_c(C))$.

Proposition 6.25. *A positively degenerate conformally generated cycle $(A, E_B, D; C, \mu)$ is cobordant to the zero cycle $(A, 0_B, 0; 0, 0)$.*

Proof. Let $s \in \text{End}^*(E)$ be a symmetry implementing the degeneracy. Let N be the number operator and S the unilateral shift on $\ell^2(\mathbb{N}_{\geq 0})$. Then

$$(A, E_B \otimes \ell^2(\mathbb{N}_{\geq 0}), D \otimes 1 + s \otimes N; C \oplus \mathbb{C}, (\mu_L \otimes 1 \oplus 1 \otimes 1, \mu_R \otimes 1 \oplus 1 \otimes 1); 1 \otimes S) \quad (6.26)$$

is a cobordism from $(A, E_B, D; C, \mu)$ to $(A, 0_B, 0; 0, 0)$. The compactness of the resolvent is as in Proposition 5.11.

Let $\mathcal{L}'$, $\mathcal{T}'$, and $\mathcal{R}'$ be the spaces of Definition 6.1, corresponding to the cycle (6.26). Using the relation $NS = S(N + 1)$, we check that

$$(D \otimes 1 + s \otimes N)(1 \otimes S) - (1 \otimes S)(D \otimes 1 + s \otimes N) = s \otimes [N, S] = s \otimes S$$

is bounded. Hence, noting that $[\mu, s \otimes 1] = 0$,

$$\begin{aligned} \mathcal{L}' &\supseteq Z_{\mathcal{L}}(s \otimes 1) \oplus \mathbb{C}1 \otimes \text{span}\{1, S\} \\ \mathcal{R}' &\supseteq Z_{\mathcal{R}}(s \otimes 1) \oplus \mathbb{C}1 \otimes \text{span}\{1, S\} \\ \mathcal{T}' &\supseteq Z_{\mathcal{T}}(s \otimes 1) \oplus \mathbb{C}1 \otimes \text{span}\{1, S\} \end{aligned}$$

and $(1 \otimes S)A \subseteq C^*((1 \otimes \psi)(\mathcal{L}'\mathcal{T}'\mathcal{R}')|\psi \in \mathcal{S}_c(C \oplus \mathbb{C}))$, as required. $\square$

Corollary 6.27. *Given a conformally generated cycle $(A, E_B, D; C, \mu)$,*

$$(A, E_B, D; C, \mu) \oplus (A, E_B^{op}, -D; C, \mu) = (A, E_B \oplus E_B^{op}, \begin{pmatrix} D & \\ & -D \end{pmatrix}; C \oplus C, \mu \oplus 1 \oplus 1 \oplus \mu)$$

is conformant to $(A, 0_B, 0; 0, 0)$.

Proof. Using the observations of Remark 6.16, we may replace the direct sum cycle with

$$(A, E_B \oplus E_B^{op}, \begin{pmatrix} D & \\ & -D \end{pmatrix}; C, \mu)$$

and the symmetry $s = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ makes this positively degenerate. $\square$

We thereby obtain:

Theorem 6.28. *Conformism classes of conformally generated A-B-cycles form an abelian group which surjects onto $KK(A, B)$.*

A Appendix

A.1 Fractional powers of positive operators on Hilbert modules

The proof of the following Theorem can be found for the Hilbert space case in [KZPS76, Theorem 12.5]. We include a proof in the generality of Hilbert modules, beginning with a few basic Lemmas.

Lemma A.1. *Let A and B be closed densely defined operators on a Banach space X . If the product AB with domain $\text{dom}(AB) = \{\xi \in \text{dom } B \mid B\xi \in \text{dom } A\}$ is densely defined then AB is closed if either*

- *A has everywhere defined and bounded inverse, or*
- *B is everywhere defined and bounded.*

Proof. Take the case that A is invertible, so that $\text{dom } A = A^{-1}X$. Suppose that $(\xi_n)_{n \in \mathbb{N}} \subseteq \text{dom}(AB) = \{x \in \text{dom } B \mid Bx \in A^{-1}X\}$ such that $\xi_n \rightarrow \xi$ and $AB\xi_n \rightarrow \eta$ as $n \rightarrow \infty$. Because A^{-1} is bounded, $B\xi_n = A^{-1}AB\xi_n \rightarrow A^{-1}\eta$. As B is closed, $\xi \in \text{dom } B$ and $B\xi = A^{-1}\eta$. So $\xi \in \text{dom}(AB)$ and $AB\xi = AA^{-1}\eta = \eta$ and we conclude that AB is closed.

Take the case that B is bounded. Suppose that $(\xi_n)_{n \in \mathbb{N}} \subseteq \text{dom}(AB) = \{x \in X \mid Bx \in \text{dom } A\}$ such that $\xi_n \rightarrow \xi$ and $AB\xi_n \rightarrow \eta$ as $n \rightarrow \infty$. Because B is bounded, $B\xi_n \rightarrow B\xi$. As A is closed, $B\xi \in \text{dom } A$ (meaning that $\xi \in \text{dom}(AB)$) and $AB\xi = \eta$. Hence, AB is closed. $\square$

Lemma A.2. *Let A and B be closed densely defined operators on Banach spaces X_1 and X_2 . Let T be a bounded operator from X_2 to X_1 with $T \text{dom } B \subseteq \text{dom } A$. Suppose that B is invertible (so B^{-1} is everywhere-defined and bounded). Then ATB^{-1} is everywhere-defined and bounded.*

Proof. By construction, ATB^{-1} is defined everywhere. By the closed graph theorem, it is bounded if and only if it is closed, which it is by Lemma A.1. $\square$

Theorem A.3. *cf. [KZPS76, Theorem 12.5] Let A and B be positive regular operators on Hilbert B -modules E_1 and E_2 respectively. Let T be an adjointable operator from E_2 to E_1 . If $T \text{dom}(B) \subseteq \text{dom}(A)$, then $T \text{dom}(B^\alpha) \subseteq \text{dom}(A^\alpha)$ for any $0 < \alpha \leq 1$. If, in addition, there exists an $M \geq 0$ such that, for all $\xi \in \text{dom}(B)$,*

$$\|AT\xi\| \leq M\|B\xi\|, \tag{A.4}$$

then

$$\|A^\alpha T\xi\| \leq M^\alpha \|T\|^{1-\alpha} \|B^\alpha \xi\|.$$

In particular, if B is invertible,

$$\|A^\alpha TB^{-\alpha}\| \leq \|ATB^{-1}\|^\alpha \|T\|^{1-\alpha}.$$

Proof. We will begin with the case of A bounded and adjointable and B invertible. In this case, a bound of the form (A.4) always holds, the best available bound being given by $M = \|ATB^{-1}\|$. For any $0 < \alpha \leq 1$, A^α is adjointable and B^α is invertible. Define the function

$$f : z \mapsto \|A^z T B^{-z}\|$$

on the strip where $0 < \Re(z) \leq 1$. For $\beta \in \mathbb{R}$,

$$f(1 + \beta i) = \|A^{\beta i} A T B^{-1} B^{-\beta i}\| \leq \|ATB^{-1}\|.$$

By considering the inclusion of $\text{End}^*(E)$ into its enveloping von Neumann algebra $\text{End}^*(E)''$, we can extend the function f to the imaginary line, $\Re(z) = 0$, using the Borel functional calculus. Then

$$f(\beta i) = \|A^{\beta i} T B^{-\beta i}\|_{\text{End}^*(E)''} \leq \|T\|.$$

By Hadamard's three-line theorem, we obtain that

$$f(\alpha) \leq \|ATB^{-1}\|^\alpha \|T\|^{1-\alpha}$$

for $0 \leq \alpha \leq 1$. Restricting to $0 < \alpha \leq 1$, for $\xi \in \text{dom}(B^\alpha)$,

$$\|A^\alpha T \xi\| \leq \|A^\alpha T B^{-\alpha}\| \|B^\alpha \xi\| \leq \|ATB^{-1}\|^\alpha \|T\|^{1-\alpha} \|B^\alpha \xi\|$$

as required.

Now consider the case of general A and B when the bound (A.4) applies. As in the previous section, let $(\varphi_n)_{n \in \mathbb{N}} \subset C_c(\mathbb{R})$ be a sequence of positive functions, bounded by 1 and converging uniformly on compact subsets to the constant function 1. Let

$$A_n = A\varphi_n(A) \quad B_n = B + \frac{1}{n} \quad (n > 0).$$

The operators A_n are bounded and adjointable and B_n are invertible. For $\eta \in \text{dom } A$ and $\xi \in \text{dom } B$,

$$\|A_n \eta\| \leq \|A \eta\| \quad \|B \xi\| \leq \|B_n \xi\|$$

and so

$$\|A_n T \xi\| \leq M \|B_n \xi\|.$$

As we have seen, for $0 < \alpha \leq 1$,

$$\|A_n^\alpha T \xi\| \leq M^\alpha \|T\|^{1-\alpha} \|B_n^\alpha \xi\| \quad (\xi \in \text{dom}(B_n^\alpha) = \text{dom}(B^\alpha)).$$

The sequence $\varphi_n(A)^\alpha T \xi \rightarrow T \xi$ as $n \rightarrow \infty$ by Theorem 2.25. The bounded functions

$$x \mapsto (x + 1/n)^\alpha - x^\alpha$$

converge uniformly to zero as $n \rightarrow \infty$, hence $B_n^\alpha \xi \rightarrow B^\alpha \xi$, again by Theorem 2.25. Then

$$\sup_n \|A^\alpha \varphi_n(A)^\alpha T \xi\| = \sup_n \|A_n^\alpha T \xi\| \leq \sup_n M^\alpha \|T\|^{1-\alpha} \|B_n^\alpha \xi\| < \infty.$$

Because A^α is a closed operator, $T \xi \in \text{dom}(A^\alpha)$ and $A_n^\alpha T \xi = A^\alpha \varphi_n(A)^\alpha T \xi \rightarrow A^\alpha T \xi$ as $n \rightarrow \infty$. Taking the limit as $n \rightarrow \infty$, we find that for $\xi \in \text{dom}(B^\alpha)$

$$\|A^\alpha T \xi\| \leq M^\alpha \|T\|^{1-\alpha} \|B^\alpha \xi\|.$$

For the case of general A and B with $T \text{dom}(B) \subseteq \text{dom}(A)$ but without the bound (A.4), we let $B_1 = B + 1$. As B_1 is invertible, for $\xi \in \text{dom}(B)$

$$\|AT\xi\| \leq \|ATB_1^{-1}\| \|B_1\xi\|.$$

We have shown that $T \text{dom}(B_1^\alpha) \subseteq \text{dom}(A^\alpha)$ and, as $\text{dom}(B_1) = \text{dom}(B)$, we are done. $\square$

A.2 Hilbert C*-modules over topological spaces

We review and extend some known facts about Hilbert modules built from functions $X \rightarrow E_B$ for a fixed Hilbert module E_B and a locally compact Hausdorff space X .

Definition A.5. e.g. [RW98, §B.2] Let A be a C*-algebra and X a locally compact Hausdorff space. Define $C_0(X, A)$ to be the C*-algebra of norm-continuous functions $f : X \rightarrow A$ such that $x \mapsto \|f(x)\|_A$ vanishes at infinity, equipped with the supremum norm. Let E be a right Hilbert A -module. Define $C_0(X, E)$ to be the set of continuous functions $f : X \rightarrow E$ such that $x \mapsto \|f(x)\|_E$ vanishes at infinity.

Lemma A.6. cf. [RW98, Example 2.13] Let E be a right Hilbert A -module and X a locally compact Hausdorff space. Then $C_0(X, E)$ is a right Hilbert $C_0(X, A)$ -module with inner product and right action defined pointwise in X .

Proof. The algebraic conditions on a Hilbert module are satisfied for $C_0(X, E)$ since they are satisfied pointwise for E . The norm on an element $f \in C_0(X, E)$ arising from the inner product is

$$\left\| (f|f)_{C_0(X, A)} \right\|_{C_0(X, A)}^{1/2} = \sup_{x \in X} \left\| (f|f)_{C_0(X, A)}(x) \right\|_A^{1/2} = \sup_{x \in X} \|(f(x)|f(x))_A\|_A^{1/2} = \sup_{x \in X} \|f(x)\|_E$$

which is the supremum norm. Hence, $C_0(X, E)$ is complete as Hilbert module. $\square$

Lemma A.7. Let E be a right Hilbert B -module and X a locally compact Hausdorff space. Let $J = \overline{\text{span}}(E|E)_B$ be the ideal of A generated by inner products on E . There is an equality

$$\overline{\text{span}}(C_0(X, E)|C_0(X, E))_{C_0(X, B)} = C_0(X, J)$$

of ideals of $C_0(X, B)$.

Proof. Consider $f_1, f_2 \in C_0(X, E)$. Their inner product is given at $x \in X$ by

$$(f_1|f_2)_{C_0(X, B)}(x) = (f_1(x)|f_2(x))_B \in J.$$

Noting that

$$\|(f_1(x)|f_2(x))_B\|_B \leq \|(f_1(x)|f_1(x))_B\|_B^{1/2} \|(f_2(x)|f_2(x))_B\|_B^{1/2} = \|f_1(x)\|_E \|f_2(x)\|_E,$$

we see that $(f_1|f_2)_{C_0(X, B)} \in C_0(X, J)$. Hence

$$(C_0(X, E)|C_0(X, E))_{C_0(X, B)} \subseteq C_0(X, J).$$

Label the ideal $I = \overline{\text{span}}(C_0(X, E)|C_0(X, E))_{C_0(X, B)}$ of $C_0(X, B)$. By e.g. [Fel61, §1.2], I must have the form

$$\{s \in C_0(X, B) | \forall x \in X, s(x) \in I_x\}$$

where each $I_x = \{s(x) | s \in I\}$ is an ideal of B . We must have $I_x \subseteq J$ for every $x \in X$. Suppose that $I_{x_0} \neq J$ for some $x_0 \in X$. Since $(E|E)_B$ is linearly dense in J , it is not contained in I_{x_0} , and there must be a pair $e_1, e_2 \in E$ such that $(e_1|e_2)_B \in J \setminus I_{x_0}$. Choose a function $h \in C_0(X)$ for which $h(x_0) = 1$ and define $f_1, f_2 \in C_0(X, E)$ on $x \in X$ by $f_i(x) = e_i h(x)$. Then

$$(f_1|f_2)_{C_0(X, B)}(x_0) = (f_1(x_0)|f_2(x_0))_B = (e_1|e_2)_B$$

is not in I_{x_0} , so $(f_1|f_2)_{C_0(X, B)}$ is not in I , which is a contradiction. In other words, $I_x = J$ for every $x \in X$ and $I = C_0(X, J)$. $\square$

Lemma A.8. *Let E be a Morita equivalence A - B -bimodule and X a locally compact Hausdorff space. Then $C_0(X, E)$ is a Morita equivalence $C_0(X, A)$ - $C_0(X, B)$ -bimodule.*

Proof. The left and right norms on E agree by [RW98, Lemma 2.30], so there is no ambiguity in the continuity used to define $C_0(X, E)$. The algebraic properties of a Morita equivalence bimodule are satisfied for $C_0(X, E)$ because they are satisfied pointwise for E . The fullness of $C_0(X, E)$ as a right and left Hilbert module follows from Lemma A.7 and the fullness of E . $\square$

Lemma A.9. *Let E be a right Hilbert B -module and X a locally compact Hausdorff space. Then*

$$\text{End}^*(C_0(X, E)) = C_b(X, \text{End}^*(E)_{*-s})$$

the C^ -algebra of $*$ -strong-continuous functions $f : X \rightarrow \text{End}^*(E)$ such that $\sup_{x \in X} \|f(x)\|_{\text{End}^*(E)} < \infty$. Furthermore, $\text{End}^0(C_0(X, E)) = C_0(X, \text{End}^0(E))$.*

Proof. Let $A = \text{End}^0(E)$, so that E is a Morita equivalence A - B -bimodule. By [RW98, Corollary 2.54], $\text{End}^*(E) = M(A)$, the multiplier algebra of A . The equality

$$\text{End}^0(C_0(X, E)) = C_0(X, \text{End}^0(E)) = C_0(X, A)$$

is a consequence of Lemma A.8. Again by [RW98, Corollary 2.54],

$$\text{End}^*(C_0(X, E)) = M(\text{End}^0(C_0(X, E))) = M(C_0(X, A)).$$

Let $M(A)_\beta$ be $M(A)$ equipped with the strict topology. By [APT73, Corollary 3.4],

$$M(C_0(X, A)) = C_b(X, M(A)_\beta),$$

the C^* -algebra of strictly continuous and norm-bounded functions. By [RW98, Proposition C.7], the strict topology on $M(A) = \text{End}^*(E)$ agrees with the $*$ -strong topology on norm-bounded subsets. Hence

$$C_b(X, M(A)_\beta) = C_b(X, \text{End}^*(E)_{*-s}).$$

To be clear, the elements in both algebras are canonically identified and the topology on both algebras is the operator norm on E composed with the supremum norm over X , so they are really equal. Finally, we obtain

$$\text{End}^*(C_0(X, E)) = C_b(X, \text{End}^*(E)_{*-s}),$$

as required. $\square$

Definition A.10. e.g. [Wil70, Definition 43.8] A topological space X is a k -space if a subset Y of X is open if, and only if, for every compact subset K of X , $Y \cap K$ is open in K . Conditions on X which imply that it is a k -space include local compactness and first-countability [Wil70, Theorem 43.9].

Lemma A.11. e.g. [Wil70, Lemma 43.10] *Let $f : X \rightarrow Y$ be a map between topological spaces with X a k -space. Then the continuity of f is equivalent to the continuity of f restricted to K for all compact subsets $K \subseteq X$.*

Lemma A.12. *Let E be a right Hilbert A -module and X a locally compact Hausdorff space. The norm-continuity of a function $f : X \rightarrow \text{End}^0(E)$ is equivalent to the condition that $f|_K \in \text{End}^0(C(K, E))$ for all compact subsets $K \subseteq X$.*

Proof. By Lemma A.11, the norm-continuity of a function $f : X \rightarrow \text{End}^0(E)$ is equivalent to the norm-continuity of $f|_K$ for every compact subset $K \subseteq X$. By Lemma A.9, the norm-continuous functions from a given K to $\text{End}^0(E)$ can be identified with the elements of $\text{End}^0(C(K, E))$. $\square$

Theorem A.13. (*Banach–Steinhaus or uniform boundedness principle*) e.g. [RS80, Theorem III.9] Let V be a Banach space and W a normed linear space. Let $\mathcal{F} \subset B(V, W)$ be a family of bounded operators from V to W with $\sup_{T \in \mathcal{F}} \|Tv\|_W < \infty$ for each $v \in V$. Then $\sup_{T \in \mathcal{F}} \|T\|_{B(V, W)} < \infty$.

Corollary A.14. Let V be a Banach space and X be a compact space. Let $f : X \rightarrow B(V)$ be a strongly continuous map. Then f is bounded in operator norm; in other words, $\sup_{x \in X} \|f(x)\|_{B(V)} < \infty$.

Proof. We have a family $\mathcal{F} = (f(x))_{x \in X} \subset B(V)$ of bounded operators. The strong continuity of f implies that $x \mapsto f(x)v$ is continuous for every $v \in V$. Since X is compact, its image $f(X)v \subseteq V$ is compact and thus bounded. Hence, for a fixed $v \in V$,

$$\sup_{T \in \mathcal{F}} \|Tv\|_V = \sup_{x \in X} \|f(x)v\|_V < \infty.$$

Applying Theorem A.13, we obtain that

$$\sup_{x \in X} \|f(x)\|_{B(V)} = \sup_{T \in \mathcal{F}} \|T\|_{B(V)} < \infty,$$

as required. $\square$

Lemma A.15. Let E be a right Hilbert A -module and X a compact Hausdorff space. The $*$ -strong-continuity of a function $f : X \rightarrow \text{End}^*(E)$ is equivalent to the condition that $f \in \text{End}^*(C(X, E))$.

Proof. By Lemma A.9, $\text{End}^*(C(X, E)) = C_b(X, \text{End}^*(E)_{*-s})$, the C^* -algebra of $*$ -strongly continuous functions $f : X \rightarrow \text{End}^*(E)$ such that $\sup_{x \in X} \|f(x)\|_{\text{End}^*(E)} < \infty$. If $f \in \text{End}^*(C(X, E))$, then it is $*$ -strongly continuous as a function $f : X \rightarrow \text{End}^*(E)$. On the other hand, if we assume $f : X \rightarrow \text{End}^*(E)$ is $*$ -strongly continuous, we may apply Corollary A.14. Thereby, $\sup_{x \in X} \|f(x)\|_{\text{End}^*(E)} < \infty$ and so $f \in \text{End}^*(C(X, E))$. $\square$

Lemma A.16. Let E be a right Hilbert A -module and X a locally compact Hausdorff space. The $*$ -strongly continuity of a function $f : X \rightarrow \text{End}^*(E)$ is equivalent to the condition that $f|_K \in \text{End}^*(C(K, E))$ for all compact subsets $K \subseteq X$.

Proof. By Lemma A.11, the $*$ -strong-continuity of a function $f : X \rightarrow \text{End}^*(E)$ is equivalent to the $*$ -strong continuity of $f|_K$ for every compact subset $K \subseteq X$. By Lemma A.15, the $*$ -strong continuity of $f|_K : K \rightarrow \text{End}^*(E)$ for a given K is equivalent to the condition that $f|_K \in \text{End}^*(C(K, E))$. $\square$

A.3 Matched operators

Definition A.17. Let E be a Hilbert B -module and C a C^* -algebra represented on the right of E by a nondegenerate C^* -homomorphism $\rho : C \rightarrow M(B)$. A regular operator T on E is C -matched if those $c \in C$ for which

$$E\rho(c) \subseteq \text{dom}(T)$$

are dense in C .

Remark A.18. The condition that $E\rho(c) \subseteq \text{dom}(T)$ combined with Lemma A.2 implies that the $\mathbb{C}$ -linear map

$$E \rightarrow E \quad \xi \mapsto T\xi c$$

is bounded.

Lemma A.19. Let E be a Hilbert B -module and C a C^* -algebra represented on the right of E by a C^* -homomorphism $\rho : C \rightarrow M(B)$. Let T be a regular operator on E . The set of $c \in C$ for which

$$E\rho(c) \subseteq \text{dom}(T)$$

form a (not necessarily closed) two-sided ideal in C .

Proof. This follows from a general statement about rings and modules. Suppose that we have $E\rho(c) \subseteq \text{dom}(T)$ for some $c \in C$. If $c_1, c_2 \in C$, then

$$E\rho(c_1 c_2) = E\rho(c_1)\rho(c)\rho(c_2) \subseteq E\rho(c)\rho(c_2) \subseteq \text{dom}(T)\rho(c_2) \subseteq \text{dom}(T)$$

and we are done. $\square$

Recall that the Pedersen ideal K_C of a C^* -algebra C is the minimal dense two-sided ideal of C ; see e.g. [Bla06, §II.5.2].

Proposition A.20. *Let T be a regular operator on E_B which is C -matched. Then*

$$E\rho(c) \subseteq \text{dom}(T)$$

for all $c \in K_C$, the Pedersen ideal of C . Furthermore, $E\rho(K_C)B$ is a core for T .

Proof. As those $c \in C$ for which $E\rho(c) \subseteq \text{dom}(T)$ form a dense two-sided ideal, they must include the Pedersen ideal. For an element $c \in K_C$, there exists an element $d \in K_C$ such that $dc = c$. Hence

$$E\rho(c) = E\rho(d)\rho(c) \subseteq \text{dom}(T)\rho(c) \subseteq E\rho(c)$$

and $E\rho(K_C) = \text{dom}(T)\rho(K_C) = (1 + T^*T)^{-1/2}E\rho(K_C)$. Next, note that $\rho(K_C)$ is dense in $\rho(C)$. By the continuity of multiplication, $E\rho(K_C)B$ is dense in $E\rho(C)B$. By nondegeneracy of ρ , $B\rho(C)$ is dense in B and, again, by the continuity of multiplication, $EB\rho(C)B = E\rho(C)B$ is dense in $EB = E$. Hence $E\rho(K_C)B$ is dense in E and $E\rho(K_C)B = (1 + T^*T)^{-1/2}E\rho(K_C)B$ is consequently a core for T . $\square$

Remark A.21. In [Web04], the multiplier algebra $\Gamma(K_B)$ of the Pedersen ideal of B is shown to consist of exactly those unbounded operators affiliated with B , in the sense of [Wor91], whose domains include K_B . A similar characterisation is given in [Pie06, Théorème 1.30]. The previous Proposition can be used to show that, if $\rho(C) = B$, the eventually C -bounded operators on E_B are exactly the multipliers $\Gamma(K_{\text{End}^0(E)})$ of the Pedersen ideal of $\text{End}^0(E)$. See [Ara01, Proposition 1.7] for the details of passing through the Morita equivalence bimodule ${}_{\text{End}^0(E)}E_B$.

Lemma A.22. *Let E be a Hilbert B -module and C a C^* -algebra represented on the right of E by a C^* -homomorphism $\rho : C \rightarrow M(B)$. A regular operator T on E is C -matched if and only if, for all $c \in K_C$, the restriction $T|_{\overline{E\rho(c)}}$ of T to the Hilbert submodule $\overline{E\rho(c)}$ over the hereditary C^* -subalgebra $\overline{\rho(c)^*B\rho(c)}$ of B is bounded.*

Proof. Assume that $E\rho(c) \subseteq \text{dom}(T)$ for $c \in K_C$. Choose $d \in K_C$ such that $dc = c$. As $E\rho(d) \subseteq \text{dom}(T)$, the $\mathbb{C}$ -linear map $\xi \mapsto T\xi\rho(d)$ on E is bounded by Lemma A.2. On $\overline{E\rho(c)}$, $\rho(d)$ acts as the identity, meaning T restricts to a bounded operator on $\overline{E\rho(c)}$.

On the other hand, assume that $T|_{\overline{E\rho(c)}}$ is bounded for $c \in K_C$. Then $\text{dom}(T) \supseteq \overline{E\rho(c)} \supseteq E\rho(c)$, as required. $\square$

The following is well-known.

Lemma A.23. *Let a be an element of the multiplier algebra of a C^* -algebra A . Then the closed right ideal $\overline{aA}$ is a Morita equivalence bimodule between the hereditary C^* -subalgebra $\overline{aAa^*}$ of A and the (closed two-sided) ideal $\overline{\text{span}}(Aa^*aA) \trianglelefteq A$.*

Proposition A.24. *Let E be a Hilbert B -module and C a C^* -algebra represented on the right of E by a C^* -homomorphism $\rho : C \rightarrow M(B)$. A regular operator T on E is C -matched if and only if, for all positive $c \in K_C$, the restriction $T|_{\overline{\text{span}}(E\rho(c)B)}$ of T to the Hilbert submodule $\overline{\text{span}}(E\rho(c)B)$ over the ideal $\overline{\text{span}}(B\rho(c)B) \trianglelefteq B$ is bounded.*

Proof. Assume that $E\rho(c) \subseteq \text{dom}(T)$ for $c \in K_C$. Then the restriction of T to $\overline{E\rho(c)}_{\overline{\rho(c)^*B\rho(c)}}$ is bounded. The closed right ideal $\overline{\rho(c)^*B}$ of B is a Morita equivalence $\overline{\rho(c)^*B\rho(c)}$ - $\overline{\text{span}(B\rho(cc^*)B)}$ -bimodule. We have a natural isomorphism

$$\overline{\text{span}(E\rho(cc^*)B)}_{\overline{\text{span}(B\rho(cc^*)B)}} \cong \overline{E\rho(c)}_{\overline{\rho(c)^*B\rho(c)}} \otimes_{\overline{\rho(c)^*B\rho(c)}} \overline{\rho(c)^*B}_{\overline{\text{span}(B\rho(cc^*)B)}}$$

of Hilbert $\overline{\text{span}(B\rho(cc^*)B)}$ -modules, under which $T|_{\overline{E\rho(cc^*)B}} \cong T|_{\overline{E\rho(c)}} \otimes_{\overline{\rho(c)^*B\rho(c)}} 1$. Hence the restriction $T|_{\overline{\text{span}(E\rho(cc^*)B)}}$ is bounded. Since every positive element of K_C is of the form cc^* , we conclude this direction of the argument.

On the other hand, assume that $T|_{\overline{\text{span}(E\rho(c)B)}}$ is bounded for $c \in K_C$. Recall that the product of (two-sided) closed ideals in a C^* -algebra is again a closed ideal, so that $\overline{B\rho(c)B} = \overline{BM(B)\rho(c)M(B)}$. Then

$$\text{dom}(T) \supseteq E \overline{\text{span}(B\rho(c)B)} = E \overline{\text{span}(M(B)\rho(cc^*)M(B))} \supseteq E\rho(c),$$

as required. $\square$

Lemma A.25. *cf. [LT76, Proof of Proposition 4.5] Let π be an irreducible representation of a C^* -algebra A on a Hilbert space H . Then $K_A H = H$.*

Proof. Let $\xi \in H$ be a cyclic vector and choose $a \in K_A$ such that $\|\pi(a)\xi\| = 1$. (Such an $a \in K_A$ can always be found; otherwise the density of K_A in A would imply that $\xi = 0$.) Let $\eta \in H$ be any non-zero vector. The finite rank operator $|\eta\rangle\langle\pi(a)\xi|$ takes $a\xi$ to η . By [Dix77, Theorem 2.8.3(i)], there exists an element $b \in A$ such that

$$\eta = |\eta\rangle\langle\pi(a)\xi|\pi(a)\xi = \pi(b)\pi(a)\xi \in K_A H$$

as required. $\square$

Proposition A.26. *The C -matched operators on E_B form a $*$ -algebra $\text{Mtc}_B^*(E, C)$.*

Proof. Let T be a regular operator on E_B which is C -matched. By Lemma A.22, T restricts to a bounded operator on $\overline{E\rho(c)}_{\overline{\rho(c)B\rho(c)}}$ for all $c \in K_C$. The restrictions $(T|_{E\rho(c)})^* = T^*|_{E\rho(c)}$ of the adjoint T^* of T are consequently bounded, and so T^* is also C -matched, again by Lemma A.22.

Let T_1 and T_2 be C -matched operators. For an element $c \in K_C$, we have

$$T_2 E\rho(c) = T_2 \text{dom}(T_2)\rho(c) \subseteq E\rho(c) \subseteq \text{dom}(T_1)$$

so that $T_1 T_2$ is well-defined on $E\rho(K_C)B$. Similarly, $T_2^* T_1^*$ is also well-defined on $E\rho(K_C)B$ so that $T_1 T_2$ is semiregular. The localisation of $E K_B \subseteq E\rho(K_C)B$ to any irreducible $\pi \in \hat{B}$ is equal to

$$E_B K_B \otimes_\pi H_\pi = E_B \otimes_\pi \pi(K_B) H_\pi = E_B \otimes_\pi H_\pi$$

by Lemma A.25. Hence, $\text{dom}((T_1 T_2)^\pi) = E_B \otimes_\pi H_\pi$ and $(T_1 T_2)^\pi$ is bounded. As the same is true for $(T_2^* T_1^*)^\pi$, we may apply the local-global principle [Pie06, Théorème 1.18(2)] to obtain that the closure of $T_1 T_2$ is a regular operator on E . By similar reasoning, we conclude that the closure of the sum $T_1 + T_2$, defined on the common core $E\rho(K_C)B$, is a regular operator on E . $\square$

Remark A.27. Combined with Proposition A.24, Proposition A.26 could be used to show that $\text{Mtc}_B^*(E, C)$ is a pro- C^* -algebra (or locally C^* -algebra) [Phi88], [Fra05, Chapter II].

Proposition A.28. *Let X be a locally compact Hausdorff space and E a Hilbert B -module. Then the $C_0(X)$ -matched operators on $C_0(X, E)$ are exactly the elements of $C(X, \text{End}^*(E)_{*-s})$, the (not necessarily bounded) $*$ -strongly continuous functions from X to $\text{End}^*(E)$.*

Proof. Suppose that T is a $C_0(X)$ -matched operator on $C_0(X, E)$. Because $T(1 + T^*T)^{-1/2} \in \text{End}^*(C_0(X, E)) = C_b(X, \text{End}^*(E)_{*-s})$ uniquely determines T , we may say that T is given by a function from X to regular operators on E . Let K be a compact subset of X . The Pedersen ideal of $C_0(X)$ is $C_c(X)$, the compactly supported functions on X . Let f be a positive element of $C_c(X)$ which is nonzero on K . We have

$$\text{dom}(T) \supseteq C_0(X, E)f = C_0(\text{supp } f, E)$$

so that T restricts to a bounded operator on $C_0(\text{supp } f, E)_{C_0(\text{supp } f, B)}$. By Lemma A.9,

$$\text{End}^*(C_0(\text{supp } f, E)) = C_b(\text{supp } f, \text{End}^*(E)_{*-s}).$$

Furthermore, the localisation of T to $C(K, E)_{C(K, B)}$ must also be bounded and so an element of $C_b(K, \text{End}^*(E)_{*-s})$. Given that T is a $*$ -strongly continuous function on every compact subset K of the k -space X , by Lemma A.11, T is a $*$ -strongly continuous function on X .

Let $T \in C(X, \text{End}^*(E)_{*-s})$. Then $T(1 + T^*T)^{-1/2} \in C_b(X, \text{End}^*(E)_{*-s})$ and

$$(1 + T^*T)^{-1/2}C_0(X, E) \supseteq C_c(X, E)$$

so that T is a regular operator on $C_0(X, E)$. (For a more detailed argument, cf. [Pal99, §4].) Furthermore, for an element $f \in K_{C_0(X)} = C_c(X)$, $C_0(X, E)f \subseteq C_c(X, E) \subseteq \text{dom}(T)$ and T is $C_0(X)$ -matched. $\square$

A.4 Compactly supported states

Definition A.29. [Har23, Definition 6.11] A state ψ on a C^* -algebra A is *compactly supported* if there exists an $a \in A$ such that $\psi(a) = \|a\|$. We denote the set of compactly supported states on A by $\mathcal{S}_c(A)$.

Proposition A.30. *For a state ψ of a C^* -algebra A , the following are equivalent:*

- (1) ψ is compactly supported, i.e. there exists an $a \in A$ such that $\psi(a) = \|a\|$.
- (2) There exists an $a \in K_A$ such that $\psi(a) = \|a\|$.
- (3) There exists a positive $a \in K_A$ such that $\psi(a) = 1 = \|a\|$ and $\psi(ab) = \psi(b)$ for all $b \in A$.
- (4) ψ is given by $b \mapsto \frac{\phi(a^*ba)}{\phi(a^*a)}$ for a state ϕ of A and an $a \in K_A$.

Proof. (2) clearly implies (1). (4) implies (2) almost by definition of the Pedersen ideal. If $\psi : b \mapsto \frac{\phi(a^*ba)}{\phi(a^*a)}$ for $a \in K_A$, there exists positive $c \in A$ such that $ca = a$. Let $f \in C_c(\mathbb{R}_+^\times)$ be a compactly supported continuous function which is equal to 1 on the spectrum of c . By the continuous functional calculus, we obtain $f(c) \in K_A$ such that $f(c)a = a$ and $\|f(c)\| = 1$, and therefore

$$\psi(f(c)) = \frac{\phi(a^*f(c)a)}{\phi(a^*a)} = 1 = \|f(c)\|.$$

To see that (1) implies (3), let $a \in A$ be such that $\psi(a) = 1 = \|a\|$. By the Kadison inequality, $\psi(a^*a) \geq |\psi(a)|^2 = 1$ and since $\|a^*a\| = \|a\|^2 = 1$, we must have $\psi(a^*a) = 1$. We may assume, without loss of generality, that a is positive. Let $\tilde{A}$ be the minimal unitisation of A and $\tilde{\psi}$ the unique extension of ψ . Let $H_{\tilde{\psi}}$ be the Hilbert space of the corresponding GNS representation and $\xi_{\tilde{\psi}}$ the cyclic vector. Then

$$\|\xi_{\tilde{\psi}} - a\xi_{\tilde{\psi}}\| = \langle (1 - a)^2 \xi_{\tilde{\psi}} | \xi_{\tilde{\psi}} \rangle = \psi(1 - 2a + a^2) = 0$$

and so $a\xi_{\tilde{\psi}} = \xi_{\tilde{\psi}}$. Let $f \in C_c(\mathbb{R}_+^\times)$ be a compactly supported continuous function such that $f(1) = 1$ and $\|f\|_\infty = 1$. By the continuous functional calculus, $f(a)$ is an element of the Pedersen ideal of A such that $f(a)\xi_{\tilde{\psi}} = \xi_{\tilde{\psi}}$ and $\psi(f(a)) = \langle f(a)\xi_{\tilde{\psi}} | \xi_{\tilde{\psi}} \rangle = 1 = \|f(a)\|$. Hence ψ satisfies

$$\psi(f(a)b) = \langle f(a)b\xi_{\tilde{\psi}} | \xi_{\tilde{\psi}} \rangle = \langle b\xi_{\tilde{\psi}} | f(a)\xi_{\tilde{\psi}} \rangle = \langle b\xi_{\tilde{\psi}} | \xi_{\tilde{\psi}} \rangle = \psi(b) \quad (\text{A.31})$$

for all $b \in B$.

To see that (3) implies (4), let positive $a \in A$ be such that $\psi(a) = 1 = \|a\|$. As before, we must have $\psi(a^2) = 1$. For all $b \in A$, as in (A.31) we have

$$\frac{\psi(aba)}{\psi(a^2)} = \psi(aba) = \langle aba\xi_{\tilde{\psi}} | \xi_{\tilde{\psi}} \rangle = \langle b\xi_{\tilde{\psi}} | \xi_{\tilde{\psi}} \rangle = \psi(b)$$

so we may simply choose $\phi = \psi$. □

Remarks A.32.

1. In [LT76, Chapter 3], a topology κ on $\Gamma(K_A)$, the multipliers of the Pedersen ideal of A , is introduced. In [LT76, Proposition 6.5], condition (4) of Proposition A.30 is shown to be equivalent to ψ being a norm-1 positive κ -continuous functional on $\Gamma(K_A)$.
2. For a locally compact Hausdorff space X , recall that the states on $C_0(X)$ are exactly given by the Radon probability measures on X [Bla98, II.6.2.3(ii)]. The compactly supported states on $C_0(X)$ are then exactly given by the compactly supported Radon probability measures on X .

Proposition A.33. *cf. [Har23, Lemma 6.12] The compactly supported states $\mathcal{S}_c(A)$ on a C^* -algebra A are weak- $*$ -dense in $\mathcal{S}(A)$.*

Proof. Let ψ be a state on A . Using [Bla98, II.4.1.4], let $(h_\lambda)_{\lambda \in \Lambda}$ be an approximate unit for A contained in the Pedersen ideal K_A . Consider the net of states $(\psi_\lambda)_{\lambda \in \Lambda}$ given by

$$\psi_\lambda : a \mapsto \frac{\psi(h_\lambda a h_\lambda)}{\psi(h_\lambda^2)}.$$

Each of these is compactly supported by Proposition A.30(4). The net $(\psi(h_\lambda^2))_{\lambda \in \Lambda}$ converges to 1 by [Bla98, II.6.2.5(i)]. To see that the net $(\psi(h_\lambda a h_\lambda))_{\lambda \in \Lambda}$ converges to $\psi(a)$, observe that

$$\begin{aligned} \|\psi(a) - \psi(h_\lambda a h_\lambda)\| &= \|\psi((1 - h_\lambda)a) + \psi(h_\lambda a (1 - h_\lambda))\| \\ &\leq (\|(1 - h_\lambda)a\| + \|a(1 - h_\lambda)\|) \\ &\rightarrow 0, \end{aligned}$$

where we have used the bounds $\|\psi\| = 1$ and $\|h_\lambda\| \leq 1$. □

Proposition A.34. *Let E be a Hilbert B -module and C a C^* -algebra. Let T be a regular operator on $(E \otimes C)_{B \otimes C}$ which is C -matched. Then, for any compactly supported state ψ on C , $(1 \otimes \psi)(T)$ is well-defined and a bounded operator on E .*

Proof. The state ψ extends to a completely positive map $1 \otimes \psi$ from $\text{End}^0(E \otimes C) = \text{End}^0(E) \otimes C$ to $\text{End}^0(E)$. Being nondegenerate, this completely positive map further extends to a map from $M(\text{End}^0(E) \otimes C) = \text{End}^*(E \otimes C)$ to $M(\text{End}^0(E)) = \text{End}^*(E)$ [Lan95, Corollary 5.7].

Let a be a positive element of K_C such that $\psi(a) = 1 = \|a\|$ and $\psi(c) = \psi(ac) = \psi(ca)$ for all $c \in C$. As $(E \otimes C)K_C \subseteq \text{dom } T$, $1 \otimes a(E \otimes C) \subseteq \text{dom } T$. By Lemma A.2, $T(1 \otimes a)$ is a bounded operator on $E \otimes C$. Hence we may apply $1 \otimes \psi$ to $T(1 \otimes a)$ to obtain an element of $\text{End}^*(E)$. To see that the choice of a does not affect the value of $(1 \otimes \psi)(T(1 \otimes a))$, let $b \in K_C$ be another positive

element such that $\psi(b) = 1 = \|b\|$ and $\psi(c) = \psi(bc) = \psi(cb)$ for all $c \in C$. We note that, because T^* is C -matched, $T^*(1 \otimes a)$ is also a bounded operator. We have a series of equalities

$$\begin{aligned}
(1 \otimes \psi)(T(1 \otimes b)) &= (1 \otimes \psi)((1 \otimes a)T(1 \otimes b)) \\
&= (1 \otimes \psi)((1 \otimes b)T^*(1 \otimes a))^* \\
&= (1 \otimes \psi)(T^*(1 \otimes a))^* \\
&= (1 \otimes \psi)((1 \otimes a)T^*(1 \otimes a))^* \\
&= (1 \otimes \psi)((1 \otimes a)T(1 \otimes a)) \\
&= (1 \otimes \psi)(T(1 \otimes a))
\end{aligned}$$

so that $(1 \otimes \psi)(T)$ has a unique meaning. $\square$

Proposition A.35. *Let E be a Hilbert B -module and C a C^* -algebra. Then $1 \otimes \mathcal{S}_c(C)$ is dense in $1 \otimes \mathcal{S}(C)$ in the pointwise-norm topology on completely positive maps from $\text{End}^0(E) \otimes C$ to $\text{End}^0(E)$. That is, for $\psi \in \mathcal{S}(C)$, there exists a net $(\psi_\lambda)_{\lambda \in \Lambda} \subseteq \mathcal{S}_c(C)$ such that, for all $y \in \text{End}^0(E) \otimes C$, $(1 \otimes \psi)(y) \in \text{End}^0(E)$ is the norm limit of $(1 \otimes \psi_\lambda)(y)$. As a consequence, $1 \otimes \mathcal{S}_c(C)$ is dense in $1 \otimes \mathcal{S}(C)$ in the pointwise-norm topology on completely positive maps from $\text{End}^*(E \otimes C)$ to $\text{End}^*(E)$.*

Proof. Let $(h_\lambda)_{\lambda \in \Lambda}$ be an approximate unit for C contained in the Pedersen ideal K_C . Let

$$\psi_\lambda : a \mapsto \frac{\psi(h_\lambda a h_\lambda)}{\psi(h_\lambda^2)}.$$

By [Fra05, Lemma 29.8], $(1 \otimes h_\lambda)_{\lambda \in \Lambda}$ is an approximate unit for $\text{End}^*(E) \otimes C$. For $y \in \text{End}^0(E) \otimes C$,

$$\begin{aligned}
\|(1 \otimes \psi)(y) - (1 \otimes \psi_\lambda)(y)\| &= \|(1 \otimes \psi)((1 \otimes (1 - h_\lambda))y) + (1 \otimes \psi)((1 \otimes h_\lambda)y(1 \otimes (1 - h_\lambda)))\| \\
&\leq \|1 \otimes \psi\| (\|(1 \otimes (1 - h_\lambda))y\| + \|y(1 \otimes (1 - h_\lambda))\|) \\
&\rightarrow 0,
\end{aligned}$$

as required.

For the second statement, let H_ψ be the Hilbert space of the GNS representation of C corresponding to ψ . One can check that the KSGNS construction [Lan95, Chapter 5] gives

$$(\text{End}^0(E) \otimes C) \otimes_{1 \otimes \psi} E \cong H_\psi \otimes E.$$

Let ξ_ψ be the cyclic vector of the GNS construction. Then, by [Lan95, Theorem 5.6],

$$(1 \otimes \psi)(y) = (1 \otimes \xi_\psi)^* y (1 \otimes \xi_\psi)$$

for $y \in \text{End}^0(E) \otimes C$. By [Lan95, Corollary 5.7], $1 \otimes \psi$ is extended to a completely positive map from $\text{End}^*(E \otimes C)$ to $\text{End}^*(E)$ by the same formula, viz.

$$(1 \otimes \psi)(y) = (1 \otimes \xi_\psi^*) y (1 \otimes \xi_\psi)$$

for $y \in \text{End}^*(E \otimes C)$. We have

$$\begin{aligned}
\|(1 \otimes \psi)(y) - (1 \otimes \psi_\lambda)(y)\| &= \|(1 \otimes \psi)((1 \otimes (1 - h_\lambda))y) + (1 \otimes \psi)((1 \otimes h_\lambda)y(1 \otimes (1 - h_\lambda)))\| \\
&= \|(1 \otimes \xi_\psi^*)(1 \otimes (1 - h_\lambda))y(1 \otimes \xi_\psi) \\
&\quad + (1 \otimes \xi_\psi^*)(1 \otimes h_\lambda)y(1 \otimes (1 - h_\lambda))(1 \otimes \xi_\psi)\| \\
&\leq 2\|y\| \|(1 - h_\lambda)\xi_\psi\| \\
&\rightarrow 0,
\end{aligned}$$

as required. $\square$

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