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Groups and Their Actions in Unbounded Kasparov Theory

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Statement of originality

The greater part of the contents of §§I.1–3, Chapter III, and §A.1 has appeared in [MR25]. The greater part of the contents of Chapter IV has appeared in [FGM25].

Declaration

I, *Ada Masters*, declare that this thesis, submitted in fulfilment of the requirements for the conferral of the degree *Doctor of Philosophy (Mathematics)* from the University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. This document has not been submitted for qualifications at any other academic institution.

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Contents

Abstract

This thesis presents a number of new approaches to the treatment of group actions in unbounded Kasparov theory. Its results are motivated by the desire to incorporate into spectral noncommutative geometry several formerly problematic examples. We extend unbounded Kasparov theory to encompass conformal group and quantum group equivariance. We use this, along with tools from geometric group theory, to study the geometry of group C*-algebras and Fell bundles. We prove a nontriviality result for Kasparov modules built from group actions on CAT(0) spaces. We also study the geometry of group extensions using the unbounded Kasparov product.

We introduce a new multiplicative perturbation theory that enables us to treat conformal actions on both manifolds and noncommutative spaces. As examples, we present unbounded representatives of Kasparov's γ-element for the real and complex Lorentz groups and display the conformal $SL_q(2)$-equivariance of the standard spectral triple of the Podleś sphere. In pursuing descent for conformally equivariant cycles, we are led to a new framework for representing Kasparov classes. Our new representatives, *conformally generated cycles*, are unbounded, possess a dynamical quality, and also include known twisted spectral triples. We define an equivalence relation on these new representatives whose classes form an abelian group surjecting onto KK-theory.

We also develop a new framework for the treatment of parabolic features in noncommutative geometry, in the form of the notion of *tangled cycle*. Tangled cycles incorporate anisotropy by replacing the unbounded operator in a higher order cycle that mimics a Dirac operator with several unbounded operators mimicking directional Dirac operators, allowing for varying and dependent orders in different directions, controlled by a weighted graph. Our main examples of tangled cycles fit into two classes: hypoelliptic spectral triples constructed from Rockland complexes on parabolic geometries and Kasparov product spectral triples for nilpotent group C*-algebras and crossed product C*-algebras of parabolic dynamical systems.

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Introduction

In this thesis we take as our starting point Connes's programme for spectral noncommutative geometry [Con94]. We focus on the unbounded picture of Kasparov's bivariant K-theory, which we regard as the backbone of noncommutative geometry. We resolve a number of previously outstanding conundrums relating mainly to group actions. In pursuit of this we generalise the unbounded picture [BJ83] of Baaj and Julg in a number of ways. An underlying motivation for the work in this thesis is the unbounded Kasparov product [Kuc97] on which great progress has been made in approximately the past decade by a number of authors, e.g. [Mes12, KL13, MR16, LM19, Dun22]. In particular, a starting aim of this work was to understand the Kasparov product for badly behaved dynamical systems, a problem arising in [BMR10]. In this we have been partially successful and we turn to it in the final section of Chapter IV. Many of the techniques used in this thesis are applicable to the understanding of the problem and we hope they will contribute to its broader solution.

Let G be a locally compact group acting on a C*-algebra A. A primary goal of this thesis is to understand the noncommutative geometry of the crossed product $A \rtimes G$. One means of doing this is to study the G-equivariant KK-theory of A and then to apply Kasparov's descent map [Kas88] or the Green–Julg or dual Green–Julg maps. In following this thread we are led to a new understanding of group equivariance in the unbounded picture. Equivariant unbounded KK-theory was studied by Kucerovsky in his thesis [Kuc94] in the mid-1990s but has remained largely unexplored in the intervening years. One reason for this is that Kucerovsky's definition, although natural, fails to capture all the degrees of freedom available in Kasparov's bounded picture. Perhaps the easiest illustration of the discrepancy is the Dirac spectral triple on a Riemannian manifold, equipped with the action of a group. If the action is isometric, the Dirac operator is invariant. If the action is a conformal one, the Fredholm module defined by the bounded transform yields a bounded equivariant Fredholm module, but the corresponding spectral triple fails to be equivariant in the sense of Kucerovsky. The allowance of conformal actions is a crucial feature of Kasparov's equivariant KK-theory; for example, it was used by Kasparov [Kas84], Chen [Che96], and Julg and Kasparov [JK95] to study the γ-element of the real and complex Lorentz groups.

We address this puzzle in Chapter III with a new general framework which we refer to as *conformal equivariance*. We do this by means of a novel multiplicative perturbation theory for abstract differential operators. We also define conformal equivariance for the actions of locally compact quantum groups, lifting the bounded picture due to Baaj and Skandalis [BS89]. This allows us to display the $SL_q(2)$-equivariance of the Podleś sphere, lifting a construction of Nest and Voigt [NV10]. Our techniques lead to a new class of representatives of Kasparov classes which we call *conformally generated cycles*. These new representatives include known examples of twisted spectral triples. In particular, the descent map, when applied to conformally equivariant unbounded Kasparov modules, in general produces conformally generated cycles instead of unbounded Kasparov modules. This applies to both group and quantum group equivariance.

For the case of a compact manifold, conformally equivalent Dirac operators have been addressed in the context of noncommutative geometry by Bär [Bär07]. A conformal change of metric has the effect $\not{D} \rightsquigarrow k^{-1/2}\not{D}k^{-1/2}$ on the Atiyah–Singer Dirac operator. By considering principal symbols, the bounded transform $\not{D}(1+\not{D}^2)^{-1/2}$ changes only by a compact operator. In §III.1, we give new tools to identify two self-adjoint regular operators as having 'close' bounded transforms in much more general circumstances. One interpretation of conformal actions and changes of metric is via Connes and Moscovici's *twisted spectral triples* [CM08]. One of the two main examples [CM08, §2.2] of twisted spectral triples given by Connes and Moscovici is built from a multiplicative perturbation $D \rightsquigarrow kDk$. The other main example [CM08, §2.3] [Mos10, §3.1] is built from a Dirac spectral triple $(C_0(X), L^2(X,S), \not{D})$ on a Riemannian manifold X, equipped with the conformal action of a discrete group G. One extends the algebra $C_0(X)$ to the crossed product $C_0(X) \rtimes G$ and

$$(C_0(X) \rtimes G, L^2(X,S), \not{D})$$

becomes a Lipschitz regular twisted spectral triple. In §III.4.1, we will interpret this as the dual Green–Julg map of a *conformally equivariant* unbounded cycle and show that such examples possess well-defined bounded transforms without recourse to the Lipschitz regularity condition of [CM08, Definition 3.1].

Another way of analysing the crossed product $A \rtimes G$ is to view it as a quantum principal bundle over the dual quantum group $\hat{G}$. The vertical geometry of the bundle can be given as an unbounded Kasparov $A \rtimes G$-A-module. A case where this is well understood is that of the group $\mathbb{Z}$ for which the cross-product is a quantum circle bundle. Quantum circle bundles have been studied by a number of authors and fitted into unbounded KK-theory in generality by Carey, Neshveyev, Nest, and Rennie [CNNR11]. At the level of KK-theory, the vertical geometry of $A \rtimes \mathbb{Z}$ is given by the Pimsner–Voiculescu extension class. For other discrete groups G there is a well-known method for constructing a vertical geometry for $A \rtimes G$, originating in an idea of Connes. In [Con89], Connes builds a spectral triple for the group C*-algebra $C^*(G)$ from the data of a length function on the group. Such a spectral triple can be upgraded to an unbounded Kasparov module for the cross product $A \rtimes G$. Although it has been studied by many authors, this construction suffers from the serious drawback that because the length function is defined to be positive, any resulting Kasparov module will always have trivial class in KK-theory. Further, except perhaps in the case of the Connes–Thom isomorphism, the construction has not been generalised to non-discrete groups. These are problems we resolve in considerable generality in Chapter II. We work with matrix-valued weights on locally compact group G. Let $\mathscr{B}$ be a Fell bundle over G which is *fissured*, a weakening of the saturation condition, generalising the spectral subspace assumption of [CNNR11]. From a weight which is *self-adjoint*, *proper*, and *translation-bounded*, we obtain a vertical geometry for $\mathscr{B}$ in the form of an unbounded Kasparov module for the cross-sectional C*-algebra $C^*(\mathscr{B})$ over the unit fibre B_e.

In Chapter II we also provide a general method for constructing weights for a locally compact group G from its action on a CAT(0) space. The weight is given by a directed length function. The CAT(0) condition is a generalisation of non-positive curvature to geodesic metric spaces. Examples of CAT(0) spaces include simply connected Riemannian manifolds of non-positive sectional curvature and trees, buildings, and certain other cell complexes. The appearance of non-positive curvature in equivariant index theory is credited to Miščenko [Miš74]; another early appearance is in the work of Luke [Luk77]. Kasparov made use of this idea in his construction of the dual Dirac and γ-elements for an almost connected group [Kas88, Kas95]. An analogous construction was made for groups acting on trees and buildings by Julg and Valette [JV84, JV87, Jul89] and Kasparov and Skandalis [KS91]. We discuss the relationship of our construction of weights to this earlier work. We prove a quite general nontriviality result for the Kasparov modules obtained from such weights which is applicable to manifolds, trees, and complexes.

In Chapter II we also consider the building of a weight for a group extension from weights on the constituent groups. This is a microcosm of the problem of the unbounded Kasparov product. For some group extensions such as the universal cover of $SL(2,\mathbb{R})$, the constructive unbounded Kasparov product works well. Generically, however, the constructive product fails. We give two examples where this occurs, in different ways, one a family of semidirect products, the other the Heisenberg group H_3. With some effort, for both of these the Kasparov product can be represented, using different techniques. In the case of the Heisenberg group, we encounter a phenomenon reminiscent of sub-Riemannian geometry: the Kasparov product can be repaired by 'squaring' in one direction and thereby obtaining a second order operator. This turns out to be an instance of a much more general phenomenon, which we explore in Chapter IV. This order-2 spectral triple for the Heisenberg group H_3 plays the rôle of Chekov's rifle in this thesis. We study its conformal properties in Example III.2.10, contextualise it in Example IV.1.13, and finally generalise it to all nilpotent Lie groups in §IV.3.1.

In Chapter IV we extend noncommutative geometry to situations which have parabolic features. We use the term *parabolic* to encompass both parabolic geometry of filtered manifolds [ČS09] and parabolic dynamical systems [HK02, Chapter 8]. The key concept introduced in Chapter IV is the *strictly tangled cycle* which is a new kind of representative for KK-theory. The idea is to replace the Dirac operator in an unbounded Kasparov module with a finite collection of mutually anticommuting self-adjoint regular operators. This allows us to treat situations where there may be different directions with drastically different kinds of behaviour. We replace the usual bounded (or relatively bounded) commutator condition with a collection of conditions determined by a bounding matrix ϵ. Providing that the bounding matrix satisfies the decreasing cycle condition, a higher order Kasparov module can be constructed by adding the operators according to certain powers, giving the strictly tangled cycle a well-defined class in KK-theory. Our definition of strictly tangled cycles was motivated in the first instance by our desire to incorporate the Rumin complex of a contact manifold into spectral noncommutative geometry, in which we have been successful.

Our examples of strictly tangled cycles come from two main sources: from Hilbert complexes, which include Rockland complexes on filtered manifolds and the aforementioned Rumin complex, and from unbounded Kasparov products, including for nilpotent groups, generalising the construction for H_3 already mentioned, and for crossed products of parabolic dynamical systems.

In parabolic geometry [ČS09], the tangent bundle is filtered and the different tangent directions capture different geometric features. One encodes the geometry through the structure of a graded nilpotent Lie group on each tangent space. Analytically one can study a parabolic geometry through a BGG complex [ČSS01, DH22] that replaces the de Rham complex. While the de Rham complex and associated Dirac operators are well understood, and even form prototypical examples in noncommutative geometry, BGG complexes are still not well understood analytically. The study of BGG complexes is motivated by recent work [GH25] implying that the known natural candidates for general classes of Heisenberg elliptic differential operators with interesting spectral noncommutative geometry have trivial index theory. The analytic foundations for BGG complexes were developed by Dave and Haller [DH19, DH22] building on ideas of Rumin [Rum94] on contact manifolds. At the level of noncommutative topology, i.e. index theory, BGG complexes were recently studied by Goffeng [Gof24]. Understanding the spectral noncommutative geometry of parabolic geometries is of interest in order to organise efficiently the differential geometric machinery into a global theory well adapted for studying global invariants. A problem motivating such a machinery is that of finding non-trivial global invariants of parabolic geometries. In fact, already for CR-manifolds this problem is non-trivial; see the prominent work of Fefferman [Fef79]. For more general parabolic geometries, Haller [Hal22] has studied analytic torsion building on the work of Rumin and Seshadri [RS12] for contact manifolds.

The unbounded Kasparov product was studied by Kucerovsky [Kuc94, Kuc97] and later phrased constructively by Mesland [Mes12]. We give further details in §I.4. In somewhat technical terms, the unbounded Kasparov product of an unbounded A-B-cycle $(A, E_{1,B}, S)$ with a B-C-cycle $(A, E_{2,B}, T)$ along a connection ∇ is the data $(A, (E_1 \otimes_B E_2)_C, S \otimes 1 + 1 \otimes_\nabla T)$ which under favourable circumstances form an unbounded A-C-cycle. There are functional analytic issues with $S \otimes 1 + 1 \otimes_\nabla T$ forming a self-adjoint operator, which additionally needs to be regular in the Hilbert C*-module sense. Such questions were addressed in [Mes12] under some technical restrictions which have since matured in the important work of Kaad and Lesch [KL12, KL13] and Lesch and Mesland [LM19]. An issue that is more delicate and has evaded a proper axiomatisation in unbounded KK-theory concerns the condition of bounded commutators in the unbounded Kasparov product. There are natural examples arising from dynamics [GRU19, GMR19] where $1 \otimes_\nabla T$ does not have bounded commutators with a dense subspace of A. Rather $1 \otimes_\nabla T$ ends up being of 'higher order' in contrast to $S \otimes 1$ in the sense that commutators with $1 \otimes_\nabla T$ are relatively bounded by $(1+S^2)^{-1/2+1/2m}$ for an $m \geq 1$ playing the role of an order. This phenomenon occurs for Kasparov products arising from parabolic dynamics. An ad hoc solution would be to inflate the spectrum of $S \otimes 1$ or dampen the spectrum of $1 \otimes_\nabla T$ to compensate. The aim of Chapter IV is to widen our view on spectral noncommutative geometry to allow for varying orders of operators and potential anisotropies to persevere as a feature rather than as a bug.

A related issue stems from the early years of noncommutative geometry, when there was optimism that quantum groups would be particularly well suited for noncommutative geometry [Con04, KRS12, NT10]. The Podleś sphere, for example, is a quantum homogenous space with a well-studied standard spectral triple [DS03]. We show it to be conformally equivariant in §III.3.1, building on the work of Nest and Voigt [NV10]. While much progress has been made in low dimension, little is known in higher dimension despite algebraic versions of BGG complexes [HK07] that have been studied in a noncommutative geometry context by Wagner, Díaz-García, and O'Buachalla [WDGO22] and Voigt and Yuncken [VY15, Yun18]. The fundamental problem lies precisely in the complications found in the algebraic relations between the various 'directions' in a quantum group, a statement made precise in the work of Krähmer, Rennie, and Senior [KRS12]. In fact, the problems arising in Krähmer, Rennie, and Senior's work relate to the Kasparov product, as discussed above. The methods of Chapter IV do not directly apply since the above alluded to parabolic behaviour does not capture the wild, hyperbolic features seen for quantum groups. We mention the connection, nevertheless, since our main definition drew inspiration from the noncommutative geometry of quantum groups in the work of Kaad and Kyed [KK25], and as a source for future investigations.

In a number of examples, strictly tangled cycles have a conformally equivariant behaviour. Unfortunately, we have been unable to develop a satisfactory abstract formulation of conformal equivariance for strictly tangled cycles. In particular, a refinement of the multiplicative perturbation theory of Chapter III would be necessary to allow for power rescaling of abstract differential operators. Nonetheless, we give a partial result and consider conformal equivariance in a number of examples.

The technical innovation which underpins Chapter III is a multiplicative perturbation theory for self-adjoint regular operators on Hilbert modules. This perturbation theory relates the bounded transforms $D(1+D^2)^{-1/2}$ and $\mu D\mu^*(1+(\mu D\mu^*)^2)^{-1/2}$ of D and its multiplicative perturbation $\mu D\mu^*$, for suitable μ. Together with the well-known additive perturbation theory $D \rightsquigarrow D + A$ for (relatively) bounded A, Theorem III.1.34 says, roughly, that any perturbation preserving the KK-class of the bounded transform takes the form $\mu D\mu^* + A$. We introduce several concepts making use of this multiplicative perturbation theory, among which are:

- *Conformal transformations* between unbounded Kasparov modules, Definition III.1.2, and a singular version, Definition III.1.38;
- *Conformal group equivariance* for unbounded Kasparov modules, Definition III.2.2;
- *Conformal quantum group equivariance* for unbounded Kasparov modules, Definition III.3.1;
- *Conformally generated cycles*, Definition III.4.1, providing a new picture of KK-theory, generalising unbounded KK-theory.

Conformally generated cycles have a dynamical aspect in addition to a geometrical one. To capture this, we use the idea of *matched operators* on Hilbert C*-modules, defined and studied in §A.1.2. We show that this framework is adapted to all known examples of twisted spectral triples with well-defined bounded transforms. Key features of our approach are the lack of a 'twist', in the sense of an algebra automorphism, and a bounded transform which does not depend on any additional smoothness condition such as Lipschitz regularity. We show in §III.4.1 that Kasparov's descent map (and the dual Green–Julg map) applied to group and quantum group conformally equivariant unbounded Kasparov modules give rise to conformally generated cycles whose bounded transforms define the same classes as the descent map (dual Green–Julg map) applied to the bounded transforms of the original modules.

A theme hovering in the background of this thesis, although not completely fulfilled, is the building of spectral triples for dynamical systems in full generality. We do however make a significant step in this direction in §IV.3.2. A miniature version of this problem is presented and solved, if not in the most elegant way, in §II.4.1. An aim for future work is to employ the machinery of conformally generated cycles to this end.

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The building blocks of spectral noncommutative geometry are unbounded Kasparov modules, including spectral triples. We begin Chapter I with the definition of bounded and unbounded Kasparov modules, which latter we give in the generality of higher order Kasparov modules. In §I.1, we generalise cobordism of bounded Kasparov modules, as defined by Cuntz and Skandalis [CS86], to unbounded Kasparov modules. We show in Theorem I.1.15 that cobordism classes of unbounded Kasparov modules form a $\mathbb{Z}/2\mathbb{Z}$-graded abelian group which surjects onto the usual KK-groups. In §I.2, we outline group equivariance in the bounded and unbounded pictures, the former due to Kasparov [Kas88] and the latter to Kucerovsky [Kuc94]. We mildly generalise Kucerovsky's definition to our setting, and refer to it as *uniform* equivariance; see Definition I.2.7. The terminology is to contrast with the *conformal* equivariance of Chapter III. We show how the descent and dual Green–Julg maps work in the setting of uniform equivariance. In §I.3 we study C*-bialgebra equivariance, following the treatment in the bounded picture by Baaj and Skandalis [BS89]. We give a definition for uniform equivariance of unbounded Kasparov modules which, to our knowledge, has not previously appeared in the literature (except in the *isometric* case [GB16]). We again show how the descent and dual Green–Julg maps work in the setting of uniform equivariance. In §I.4, we discuss the Kasparov product, presenting the Connes–Skandalis conditions [CS84] for the bounded picture as well as the current state of the art of the Kucerovsky conditions [Kuc97] for the unbounded picture. We also point out that the same conditions suffice for the product in equivariant KK-theory.

In Chapter II, we build and analyse unbounded Kasparov modules from matrix-valued weights on locally compact groups. In §II.1, as a preparation, we study the KK-groups $KK^G(A, C_0(G, B))$, $KK^{\hat{G}}(A \rtimes_r G, B)$, and $KK(A \rtimes_r G, B)$ for a locally compact group G and G-C*-algebras A and B. We review Kasparov's Dirac and dual Dirac elements for almost connected groups as well as Pimsner's six-term exact sequences for groups acting on trees. We also examine KK-theoretic Frobenius reciprocity for cocompact subgroups. Finally, we point out a contrast between almost connected groups and groups whose identity component is compact in how KK-theory behaves under restriction to a compact subgroup. As a consequence, the groups $KK^{\hat{G}}(A \rtimes_r G, B)$ and $KK(A \rtimes_r G, B)$ can behave very differently.

In §II.2, we present our construction of spectral triples for group C*-algebras from matrix-valued weights. After an initial study of such weights, we give an introduction to Fell bundles. For the construction of unbounded Kasparov modules for Fell bundles, we introduce the *fissuration* condition, which generalises saturation and the spectral subspace condition of [CNNR11]. We then exhibit two constructions of Kasparov modules using these weights, related to one another by Baaj–Skandalis duality. In Theorems II.2.24 and II.2.25, we prove

Theorem 1. *Let G be a locally compact group, V a finite-dimensional complex vector space, and $\ell : G \to \operatorname{End} V$ a self-adjoint, proper, translation-bounded weight. Let $\mathscr{B}$ be a Fell bundle over G. If $\mathscr{B}$ is fissured,*

$$(C_r^*(\mathscr{B}), L^2(\mathscr{B}) \otimes V, M_\ell)$$

is an isometrically $\hat{G}$-equivariant unbounded Kasparov $C_r^*(\mathscr{B})$-B_e-module. Let A be a G-C^*-algebra. Then

$$(A, C_0(G, A \otimes V)_{C_0(G,A)}, \ell)$$

is a uniformly G-equivariant unbounded Kasparov A-$C_0(G, A)$-module.

In particular, for $\mathscr{B}$ the group bundle, we obtain a spectral triple for $C_r^*(G)$. In §II.2.4, we explain how weights can be restricted to or induced from cocompact subgroups, partly in preparation for §II.4.

In §II.3, we give an explicit construction of weights for CAT(0) groups. For points x and y of a geodesic metric space X, we denote by $v(x, y) \in S_y(X)$ the direction of the geodesic from x to y as it reaches y, where $S_y(X)$ is the space of directions. In Proposition II.3.4, we prove

Theorem 2. *Let G be a locally compact group acting isometrically on a CAT(0) space (X, d). Suppose that at a point $x_0 \in X$, the space of directions $S_{x_0}(X)$ is isometric to a sphere $\mathbf{S}^{n-1} \subseteq \mathbb{R}^n$. Let V be a Clifford module for the Clifford algebra $\mathscr{C}\ell_n$. Define the function $\ell : G \to \operatorname{End} V$ by*

$$\ell(g) = d(g^{-1} \cdot x_0, x_0) v(g^{-1} \cdot x_0, x_0)$$

where $v(g^{-1} \cdot x_0, x_0) \in S_{x_0}(X) \cong \mathbf{S}^{n-1} \subseteq \mathbb{R}^n \subseteq \mathscr{C}\ell_n$ acts by Clifford multiplication on V. Then ℓ is self-adjoint and translation bounded. If G acts properly on X, ℓ is proper.

We also prove a nontriviality result for the resulting KK-classes by pairing them with a suitable Dirac class. In Theorem II.3.7 we prove

Theorem 3. *Let G be a locally compact group acting properly and isometrically on a CAT(0) space (X, d). Let A be a G-C^*-algebra. Suppose that there is a complete subspace Y of X such that*

- *every path component of Y is a convex subset of X (Y may have infinitely many path components.);*
- *Y is isometric to a spin^c Riemannian n-manifold; and*
- *Y contains a neighbourhood of a point $x_0 \in X$.*

Let $x_1 \in X$ be a point not in Y but with $S_{x_1}(X)$ isometric to a sphere $\mathbf{S}^{m-1} \subseteq \mathbb{R}^m$. Let V_0 and V_1 be Clifford modules for $\mathscr{C}\ell_n$ and $\mathscr{C}\ell_m$ respectively, with V_0 irreducible. Define the weights

$$\begin{aligned} \ell_0 : G &\to \operatorname{End} V_0 & \qquad \ell_1 : G &\to \operatorname{End} V \\ g &\mapsto d(g^{-1}x_0, x_0) v(g^{-1}x_0, x_0) & g &\mapsto d(g^{-1} \cdot x_1, x_1) v(g^{-1} \cdot x_1, x_1), \end{aligned}$$

representing classes $\sigma_A([\ell_0]), \sigma_A([\ell_1]) \in KK^G_*(A, C_0(G, A))$ and $[M_{\ell_0}], [M_{\ell_1}] \in KK^{\hat{G}}_*(A \rtimes_r G, A)$.

For any closed subgroup H of G preserving Y, let $\eta_H : C_0(Y, A)^H \to A$ be the $*$-homomorphism given by evaluating at x_0, giving a class $[\eta_H] \in KK_0(C_0(Y, A)^H, A)$. For $A = \mathbb{C}$, $[\eta_H] \in KK_0(C_0(Y/H), \mathbb{C})$ is nonzero if and only if H acts cocompactly on Y.

If there exists a closed subgroup H of G such that H preserves Y and acts by pin^c automorphisms and $[\eta_H]$ is nonzero then $\sigma_A([\ell_0]) \in KK^G_n(\mathbb{C}, C_0(G))$ is nonzero and not equal to $\sigma_A([\ell_1])$.

If G itself preserves Y, acts by spin^c automorphisms, and $[\eta_G]$ is nonzero then $r^{\hat{G},1}([M_{\ell_0}]) \in KK_n(A \rtimes_r G, A)$ is nonzero and not equal to $r^{\hat{G},1}([M_{\ell_1}])$.

We treat several examples, including Hadamard manifolds, trees and CAT(0) cell complexes, illustrating the scope of our result. We relate our construction to Kasparov's dual Dirac element for almost connected groups and to the extension classes of Pimsner's exact sequences for groups acting on trees.

In §II.4, we discuss the problem of generalising the above constructions to group extensions. We prove a general result and present one setting in which the unbounded Kasparov product succeeds immediately, the universal cover of $SL(2, \mathbb{R})$. We also exhibit two cases where the naïve unbounded product fails but can nevertheless be repaired by different technical manoeuvres: the semidirect product $\mathbb{R}^n \rtimes \mathbb{R}$ and the three-dimensional Heisenberg group H_3. This latter example we return to in both Chapters III and IV.

We begin Chapter III by considering conformal transformations between (higher order) unbounded Kasparov modules in §III.1. The motivation for such a framework is conformal changes of metric of Riemannian manifolds and the noncommutative torus, of which we give some details in §III.1.1. In the simplest instance for unbounded Kasparov modules (A, E, D_1) and (A, E', D_2), these transformations are a pair (U, μ) with $U : E \to E'$ unitary and μ a bounded invertible endomorphism (which is even if the module is graded) such that, for all a in a dense subset of A,

$$U^* D_2 U a - a \mu D_1 \mu^* \tag{1}$$

is bounded. The Leibniz rule shows that those a for which (1) is bounded naturally form a (not norm-closed) ternary ring of operators, rather than a $*$-algebra. The implicit presence of ternary rings of operators will be a feature of many of our definitions. For the technical results in §Ⅲ.1.3, we require that the 'conformal factor' μ be a bounded and invertible operator, although it need not have a globally bounded derivative. We prove the following as Theorem Ⅲ.1.4.

Theorem 4. *Let (U, μ) be a conformal transformation from the order-$\frac{1}{1-\alpha}$ cycle (A, E_B, D_1) to the order-$\frac{1}{1-\alpha}$ cycle (A, E'_B, D_2). Then the bounded transforms (A, E_B, F_{D_1}) and (A, E'_B, F_{D_2}) are unitarily equivalent up to locally compact perturbation via the unitary U; that is*

$$(U^* F_{D_2} U - F_{D_1}) a \in \operatorname{End}^0(E)$$

for all $a \in A$. Hence $[(A, E_B, F_{D_1})] = [(A, E'_B, F_{D_2})] \in KK(A, B)$.

On a noncompact manifold, this is not sufficient to describe all conformal changes of metric. One technical issue which arises is that a complete Riemannian manifold, such as the hyperbolic plane, may be conformally equivalent to an incomplete manifold, such as the unit disc, and therefore the self-adjointness of a Dirac operator may not be preserved. With this caveat, we give in §Ⅲ.1.5 a framework modelled abstractly on the idea of an open cover extending the idea in (1).

We also show in §Ⅲ.1.4 that the *logarithmic transform* $D \to L_D = F_D \log((1 + D^2)^{1/2})$, due to Goffeng, Mesland, and Rennie [GMR19], turns multiplicative perturbations into additive ones. In Theorem Ⅲ.1.37 we prove

Theorem 5. *Let (U, μ) be a conformal transformation from the order-$\frac{1}{1-\alpha}$ cycle (A, E_B, D_1) to the order-$\frac{1}{1-\alpha}$ cycle (A, E'_B, D_2). Then the logarithmic transforms (A, E_B, L_{D_1}) and (A, E'_B, L_{D_2}) are related by the unitary U, up to locally bounded perturbation; in particular, A is contained in the closure of the set of $a \in \operatorname{End}^*(E)$ such that*

$$(U^* L_{D_2} U - L_{D_1}) a \qquad [L_{D_1}, a]$$

is bounded.

We then, in §Ⅲ.2 extend the uniform group equivariance of §I.2 to encompass conformal actions, based on the idea of conformal transformation in (1). This is necessary to include the full range of equivariance encoded for bounded Kasparov modules, as indicated by the results of Bär [Bär07] and explained using the example of the $ax + b$ group acting on $\mathbb{R}$. In Theorem Ⅲ.2.4 we prove

Theorem 6. *The bounded transform of a conformally equivariant higher order Kasparov module is an equivariant bounded Kasparov module.*

The logarithmic transform again changes multiplicative perturbations coming from conformal actions to additive perturbations. In Theorem Ⅲ.2.11 we prove

Theorem 7. *The logarithmic transform of a conformally equivariant higher order Kasparov module is a uniformly equivariant unbounded Kasparov module.*

These results allow us to represent the γ-elements of Kasparov and Chen for the Lorentz groups and of Julg and Kasparov for the complex Lorentz groups, in §Ⅲ.2.1. We also give a genuinely noncommutative example, the second order spectral triple for the C*-algebra of the Heisenberg group, mentioned earlier, is equivariant for the dilation action.

In §Ⅲ.3 we define conformal quantum group equivariance for unbounded Kasparov modules. The main example to which we apply this framework is the action of $SL_q(2)$ on the Podleś sphere. In Theorems Ⅲ.3.3 and Ⅲ.3.5 we prove

Theorem 8. *The bounded transform of a conformally quantum group equivariant unbounded Kasparov module is a quantum group equivariant bounded Kasparov module.*

Theorem 9. *The logarithmic transform of a conformally quantum group equivariant unbounded Kasparov module is a uniformly quantum group equivariant unbounded Kasparov module.*

All of the generalisations we have considered so far are brought together in §Ⅲ.4 wherein we introduce conformally generated cycles. These unbounded representatives of Kasparov classes are general enough to include known examples of twisted spectral triples, as we outline at the beginning of §Ⅲ.4, as well as the result of applying descent and dual Green–Julg maps to group and quantum group conformally equivariant Kasparov modules as we see in §Ⅲ.4.1, generalising the constructions for uniform equivariance given in §I.2.2, in the group case, and §I.3.2, in the quantum group case.

Finally, in §Ⅲ.4.2, we show that cobordism extends to an equivalence relation on conformally generated cycles, and the cobordism classes of such cycles form an abelian group which surjects onto the usual KK-group. As a special case, we define *conformism* of unbounded Kasparov modules, using the framework of cobordism to turn the conformal transformations of §Ⅲ.1 and singular conformal transformations of §Ⅲ.1.5 into an equivalence relation. We show also that conformism classes of unbounded Kasparov modules are an abelian group which surjects onto the usual KK-group.

In Chapter Ⅳ, we generalise the unbounded picture of KK-theory in a different direction. We extend the notion of a higher order Kasparov module to that of a *strictly tangled cycle* in Definition Ⅳ.1.7 where the Dirac operator is replaced with a finite collection $\boldsymbol{D} = (D_j)_{j\in I}$ of self-adjoint operators which satisfies an analogue of a mild ellipticity condition and an anticommutation relation. Our generalisation of spectral triples we refer to as *strictly tangled spectral triples* or ST^2s, and we focus mainly on this case. The adjective *strictly* is to indicate that we assume the elements in the collection to anticommute. We expect our results to hold under more general assumptions, e.g. when the anticommutators are relatively small (see Remarks Ⅳ.1.9 and Ⅳ.1.19), but to reduce the technical burden in Chapter Ⅳ we focus on the simpler case, which already enables the treatment of a number of interesting examples. As mentioned above, a similar idea has appeared in the work of Kaad and Kyed [KK20, KK25]. The these works respectively describe the metric geometry of crossed products by $\mathbb{Z}$ and of $SU_q(2)$. Our main results are the following.

Theorem 10. *Let $(A, H, \boldsymbol{D})$ be an ST^2 with $\boldsymbol{D} = (D_j)_{j\in I}$ the finite collection of self-adjoint operators and bounding matrix $\boldsymbol{\epsilon} \in M_I([0,\infty))$. Consider the non-empty set*

$$\Omega(\boldsymbol{\epsilon}) := \{\boldsymbol{t} = (t_j) \in (0,\infty)^n : \epsilon_{ij} t_i < t_j \ \forall i,j\}.$$

For $\boldsymbol{t} \in \Omega(\boldsymbol{\epsilon})$, we define the operator

$$\overline{D}_{\boldsymbol{t}} := \sum_{j=1}^{n} \operatorname{sgn}(D_j)|D_j|^{t_j}.$$

If $\boldsymbol{t} \in \Omega(\boldsymbol{\epsilon}) \cap (0,1]^n$, the triple $(A, H, \overline{D}_{\boldsymbol{t}})$ defines a higher order spectral triple. If additionally the ST^2 is $(\infty)_{j\in J}$-preserving, then the same holds for any $\boldsymbol{t} \in \Omega(\boldsymbol{\epsilon})$.

The reader can find Theorem 10 as Theorem Ⅳ.1.16 below. We provide a number of examples of ST^2s throughout Chapter Ⅳ and study the role of the transform $(A, H, \boldsymbol{D}) \mapsto (A, H, \overline{D}_{\boldsymbol{t}})$. In §Ⅳ.1.1, we give a flavour of our main examples, the Rumin complex on the Heisenberg group, and two 'bad Kasparov products' involving the group C*-algebra of the Heisenberg group and a dynamical system on the torus. These examples are revisited in further detail and generality in §§Ⅳ.2.4, Ⅳ.3.1, and Ⅳ.3.2. We then proceed to study the finer analytical properties of ST^2s, for instance finite summability and equivariance properties. A number of interesting examples carry conformal actions. In the absence of a well-behaved general framework, we discuss a 'guess-and-check' method for conformal equivariance of ST^2 in §Ⅳ.1.4, which we later see in play in §§Ⅳ.2, Ⅳ.2.4, and Ⅳ.3.1.

Strictly tangled spectral triples also arise from Hilbert complexes [BL92]. We study ST^2s arising from Hilbert complexes in some detail in §IV.2, where the main example is that of Rockland complexes on filtered manifolds. Describing the noncommutative geometry of filtered manifolds is a non-trivial problem [Has14]. Of particular interest is to associate higher order spectral triples possessing further properties with Rockland complexes. By choosing $\boldsymbol{t}$ in Theorem 10 appropriately we can produce higher order spectral triples from Rockland complexes either that are H-elliptic elements in the Heisenberg calculus or that are differential operators. We summarize the results of §IV.2 in a Theorem.

Theorem 11. *Consider a compact filtered manifold X equipped with a volume density and hermitian vector bundles $E_j \to X$, $j = 0, ..., n$. Assume that $(C^\infty(X; E_\bullet), \mathrm{d}_\bullet)$ is a Rockland complex with all differentials being differential operators. Then there is an associated ST^2 $(C^\infty(X), L^2(X; \oplus_j E_j), \boldsymbol{D} = (\mathrm{d}_j + \mathrm{d}_j^*)_j)$ as in Theorem IV.2.17. Moreover, for any $\tau > 0$, $\boldsymbol{D}$ assembles into an H-elliptic pseudodifferential operator $\overline{D}_\tau$ on $\bigoplus_j E_j$ of order τ, defining a higher order spectral triple $(C^\infty(X), L^2(X; \oplus_j E_j), \overline{D}_\tau)$.*

In fact, the reader can find a version of Theorem 11 stated with conformally equivariant actions as Proposition IV.2.25 below. To be somewhat more precise, assume that G is a locally compact group acting as filtered automorphisms on X and that $(C^\infty(X; E_\bullet), \mathrm{d}_\bullet)$ is Rockland and G-equivariant with the action of G on each E_j being conformal (with respect to the volume density on X and the hermitian structure on E_j). In Proposition IV.2.25 below we show that if the conformal factors in the different degrees are multiplicatively dependent (with respect to powers from $\Omega(\epsilon)$) then we can assemble the associated ST^2 $(C^\infty(X), L^2(X; \oplus_j E_j), \boldsymbol{D})$ into a conformally equivariant higher order spectral triple.

A sobering observation is that, in practice, Rockland complexes equivariant for semisimple Lie groups of rank > 1 will not have a scalar conformal factor for the action on each degree in the complex. Our framework cannot be applicable to semisimple Lie groups G of rank > 1. Indeed, by Theorem III.2.12, Proposition IV.2.25 would give a G-equivariant finitely summable bounded Fredholm module, which is impossible for a Lie group of rank > 1, as shown by Puschnigg [Pus11]. The obstructions in higher rank are discussed in further detail in Remarks III.2.12 and IV.2.27.

Let us also mention another natural example of an ST^2 built from the dual Dirac element of a nilpotent group. If G is a simply connected nilpotent Lie group, the image of the dual Dirac element under the descent map $KK_*^G(\mathbb{C}, C_0(G)) \to KK_*(C^*(G), \mathbb{C})$ produces a K-homology class on the group C*-algebra. We discuss in §IV.3.1 how computing this element explicitly at the unbounded level produces an ST^2. We summarize the result as follows.

Theorem 12. *Let G be a simply connected nilpotent Lie group of depth s and H be a cocompact, closed subgroup (possibly G itself). Choose a Malcev basis $((e_{j,k})_{k=1}^{\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}})_{j=1}^s$ of $\mathfrak{g}$ through the lower central series $\mathfrak{g}_1 = \mathfrak{g}, \mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}], \dots, \mathfrak{g}_s$. Let E be an irreducible Clifford module for $\mathscr{C}\ell_{\dim \mathfrak{g}}$, whose generators we label $((\gamma_{j,k})_{k=1}^{\dim \mathfrak{g}_j})_{j=1}^s$. Then the collection $(\ell_j)_{j=1}^s : G \to \mathrm{End}_\mathbb{C}(E)$ of matrix-valued weights given by*

$$\ell_j : \exp_\mathfrak{g}\left(\sum_{i=1}^{s} \sum_{k=1}^{\dim \mathfrak{g}_i/\mathfrak{g}_{i+1}} x_{i,k} e_{i,k}\right) \mapsto \sum_{k=1}^{\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}} x_{j,k}\gamma_{j,k}$$

gives rise to a strictly tangled spectral triple

$$\left(C^*(H), L^2(H, E), (M_{\ell_n})_{n=1}^s\right)$$

with nontrivial class in $KK_{\dim \mathfrak{g}}(C^*(H), \mathbb{C})$ and bounding matrix $\epsilon_{ij} = \max\{i - j, 0\}$. Moreover, the dual Dirac element of a cocompact closed subgroup of a nilpotent Lie group can be realized the Baaj–Skandalis dual of a strictly tangled spectral triple of the form above.

If the group G is Carnot, it is possible to obtain a higher order spectral triple for $C^*(G)$ which is conformally equivariant under the dilation action.

In §Ⅳ.3.2, we show that parabolic dynamical systems [HK02, Chapter 8] give rise to crossed product ST^2s, generalising the constructions of [CMRV08, BMR10, HSWZ13, Pat14] for elliptic dynamical systems. The following result appears as Corollary Ⅳ.3.18.

Theorem 13. *Let $(C_c^\infty(X), L^2(X,S), D)$ be the Atiyah–Singer or Hodge–de Rham Dirac spectral triple on a complete Riemannian manifold $(X, \mathbf{g})$. Let φ be an action of a locally compact group G by diffeomorphisms on X. Let $\ell : G \to \operatorname{End} E$ be a self-adjoint, proper, translation-bounded weight where E is some finite-dimensional vector space. Suppose that φ is parabolic in the sense that for some $s \geq 0$, the matrix inequality*

$$\|d\varphi_g\|_\infty \leq C(1 + |\ell(g)|^s)$$

holds for some constant $C > 0$. Then

$$(C_c^\infty(X) \rtimes G, L^2(G,E) \tilde{\otimes} L^2(X,S), (M_\ell \tilde{\otimes} 1, 1 \tilde{\otimes} D)$$

is a strictly tangled spectral triple representing the Kasparov product of

$$(C_c^\infty(X) \rtimes G, L^2(G,E) \otimes C_0(X)_{C_0(X)}, M_\ell \otimes 1)$$

and $(C_c^\infty(X), L^2(X,S), D)$.

For group equivariance, we require certain identifications of Hilbert modules over locally compact Hausdorff spaces and their operators, which we cover in §A.1.1, based on the approach of Kucerovsky [Kuc94]. For conformal quantum group equivariance and conformally generated cycles, we use the ideas of *matched operators* and *compactly supported states*. These generalise the multipliers of the Pedersen ideal of a C*-algebra and their positive continuous dual. Given a C*-algebra C acting on the right of a Hilbert B-module via a nondegenerate $*$-homomorphism $C \to M(B)$, the C-matched operators on E are a subset of the regular operators which form a $*$-algebra (in fact, a pro-C*-algebra), as we show in §A.1.2. In §A.1.3, we characterise compactly supported states [Har23] on a C*-algebra in terms of the Pedersen ideal and show that they are weak-$*$-dense in all states.

For the multiplicative perturbation theory of §Ⅲ.1.3, we require certain bounds and domain relationships involving fractional powers of positive regular operators on Hilbert modules. Although these are well known in the Hilbert space case, we provide a complete proof in the Hilbert module case in §A.3. In order to obtain a higher order cycle from a strictly tangled cycle, we require an understanding of how power scaling a self-adjoint regular operator affects commutators with it; in §A.3.1, we provide a general result, formalised in terms of the idea of a nearly convex set. In §A.3.2, we give a form condition for relatively bounded commutators on Hilbert C*-modules, generalising the well-known characterisation [BR87, Proposition 3.2.55] in the Hilbert space case.

In §A.4, we show how the holomorphic and continuous functional calculi interact with higher order Kasparov modules.

Conventions

The Clifford algebras $\mathscr{C}\ell_n$ are complex $\mathbb{Z}/2\mathbb{Z}$-graded algebras, whose generators are self-adjoint and square to 1.

Chapter I

Pictures of KK-theory: what is known and a little more

In this Chapter, we set the technical stage for this thesis. We include a combination of existing and new results; most of these latter are mild generalisations of known results.

In 1957, Grothendieck proved an extensive generalisation of the Riemann–Roch theorem and, in doing so, invented K-theory. He constructed an abelian group $K(X)$ to count the locally free sheaves on an algebraic variety X. In 1959, Atiyah and Hirzebruch [AH59, AH61] began the study of *topological* K-theory, defining $K^0(X)$ to count the vector bundles on a compact Hausdorff space X. Generalising topological K-theory, the K-theory of unital C*-algebras counts finitely generated projective modules, informed by Serre–Swan duality [Swa62]. The search for a dual theory, K-homology, began with Atiyah's introduction in 1969 of the *Fredholm module*, defined below, which gives an abstract definition of an elliptic operator [Ati69]. In the early 1970s, Brown, Douglas, and Fillmore [BDF73] encountered the odd K-homology of certain spaces by classifying extensions of C*-algebras. In the succeeding years, Kasparov [Kas75] took both threads and wove them into the modern theory of K-homology. A standard reference on K-homology is [HR00].

KK-theory was introduced into the world by Kasparov in 1980, with the publication of [Kas81] and the distribution of the Conspectus (later published as [Kas95]). KK-theory is bivariant, taking as inputs two C*-algebras, and including K-theory and K-homology as special cases. A standard reference on KK-theory is [Bla98]. *Unbounded* KK-theory, although in some sense implicit in Kasparov's bounded KK-theory, was formally introduced by Baaj and Julg in 1983 [BJ83]. Recently, a number of technical refinements of the formalism of unbounded KK-theory have been made [DM20, Kaa20]. Connes and Moscovici [CM95] introduced the term *spectral triple*, to refer to unbounded cycles for K-homology, with the view to encapsulating other geometrical information, such as a metric or measure structure or even a physics. A recent survey on unbounded KK-theory is [Mes24].

For us, Kasparov cycles and their generalisations will be over ungraded $\mathbb{C}$-algebras. When we consider Kasparov classes, we will often write KK generically for classes of even or odd cycles, and unless mentioned all C*-algebras will be trivially graded. We refer to Appendix A for our conventions for Hilbert C*-modules; we write $\mathrm{End}^0(E)$ to refer to the compact operators on a Hilbert module E.

Definition I.0.1. [Kas81, Definition 4.1] [Kas88, Definition 2.2] A bounded Kasparov A-B-module consists of an A-B-correspondence E and a bounded operator F on E such that, for all $a \in A$, the operators

$$(F^* - F)a \qquad (1 - F^2)a \qquad [F, a]$$

are compact. If E is a $\mathbb{Z}/2\mathbb{Z}$-graded A-B-correspondence (that is, with A acting by even operators), we require that F be an odd operator and call (A, E_B, F) an *even* bounded Kasparov module. If E is ungraded, (A, E_B, F) is *odd.* If $B = \mathbb{C}$, so that E is a Hilbert space, (A, E, F) is a *Fredholm module.*

We will mostly work in the generality of higher order unbounded Kasparov modules, due to Wahl [Wah07]. We refer to [Wor91, Lan95] for the theory of regular operators on Hilbert C*-modules. Throughout we use the notations $\langle D\rangle = (1 + D^2)^{1/2}$ and $F_D = D\langle D\rangle^{-1} = D(1 + D^2)^{-1/2}$ for a self-adjoint regular operator D on a Hilbert module.

Definition I.0.2. cf. [GM15, Definition A.1] Let D be a self-adjoint regular operator on a right Hilbert B-module E. For $0 \leq \alpha \leq 1$, let

$$\mathrm{Lip}^*_\alpha(D) \subseteq \mathrm{End}^*_B(E)$$

be the subspace consisting of elements $a \in \mathrm{End}^*_B(E)$ for which $a \operatorname{dom} D \subseteq \operatorname{dom} D$ and $[D, a]\langle D\rangle^{-\alpha}$ and $\langle D\rangle^{-\alpha}[D, a]$ extend to bounded adjointable operators. By [GM15, Proposition A.5], $\mathrm{Lip}^*_\alpha(D)$ is a $*$-algebra.

It is shown in §A.4.1 that $\mathrm{Lip}^*_\alpha(D)$ is a Banach $*$-algebra under an appropriate norm and is closed under the holomorphic functional calculus, but we do not use this here. We will also weaken our definition of unbounded cycles along the lines of [DM20, Definition 1.1] since morphisms between cycles may not naturally preserve a given smooth subalgebra.

Definition I.0.3. cf. [Wah07, Definition 2.4] [GM15, Definition A.2] [DM20, Definition 1.1] Let $0 \leq \alpha < 1$. An order-$\frac{1}{1-\alpha}$ A-B-cycle consists of an A-B-correspondence E and a regular operator D on E such that:

1. D is self-adjoint;
2. $(1 + D^2)^{-1}a$ is compact for all $a \in A$; and
3. A is contained in the operator norm closure of $\mathrm{Lip}^*_\alpha(D)$.

If E is a $\mathbb{Z}/2\mathbb{Z}$-graded A-B-correspondence (that is, with A acting by even operators), we require that D be an odd operator and call (A, E_B, D) an *even* cycle. If E is ungraded, (A, E_B, D) is *odd.*

If we have a dense subalgebra $\mathscr{A}$ of A which is contained in $\mathrm{Lip}^*_\alpha(D)$, we will call the cycle an order-$\frac{1}{1-\alpha}$ $\mathscr{A}$-B-cycle. If $\alpha = 0$ then we refer to order-1 cycles as unbounded Kasparov modules, and if $B = \mathbb{C}$, so that E is a Hilbert space, we call these cycles spectral triples.

Example I.0.4. [GM15, Remark A.0.3] Let X be a complete Riemannian manifold and V a vector bundle over X. If D is a self-adjoint elliptic pseudodifferential operator of order $m > 0$ acting on sections of V then $(C_0(X), L^2(X, V), D)$ is an order-m spectral triple.

The generalisation to 'higher order operators' does not interfere with the main topological result for unbounded Kasparov modules. The main tool in the proof is the integral formula†

$$(1 + D^2)^{-\alpha} = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda + 1 + D^2)^{-1} d\lambda, \tag{I.0.5}$$

†Referred to by some as the 'magic integral formula'.

norm-convergent for $0 < \mathfrak{R}(\alpha) < 1$. Its use in noncommutative geometry is due to Baaj and Julg [BJ83]; for more details we refer to [CP98, Lemma A.4]. We quote the following refinement of Baaj and Julg's bounded transform result which follows easily from the results of [Wah07, §2.1], [Gre12, §7], [GM15, Appendix A].

Theorem I.0.6. *Let D be a self-adjoint regular operator on a right Hilbert B-module E. Let S be an adjointable operator such that $S \operatorname{dom} D \subseteq \operatorname{dom} D$ and $[D,S]\langle D\rangle^{-\alpha}$ extends to a bounded operator for some $0 \leq \alpha < 1$. Then*

$$[F_D, S]\langle D\rangle^{\beta}$$

is bounded for $\beta < 1-\alpha$, with a bound $C_{\alpha+\beta}\left\|[D,S]\langle D\rangle^{-\alpha}\right\|$ where $C_{\alpha+\beta} > 0$ depends only on $\alpha+\beta$.

Corollary I.0.7. *cf. [Wah07, Definition 2.4] [GM15, Theorem A.6] [DM20, Proposition 1.7] Let (A, E_B, D) be an order $\frac{1}{1-\alpha}$ A-B-cycle. Then the bounded transform $D \mapsto F_D := D(1+D^2)^{-1/2}$ gives a bounded Kasparov module (A, E_B, F_D) of the same parity.*

We will also make occasional reference to the summability of Fredholm modules and spectral triples, confining ourselves to the case of unital C*-algebras. For this, we use the Schatten ideals

$$\mathcal{L}^p(H) := \{T \in K(H) \mid \operatorname{Tr}(|T|^p) < \infty\}$$

with exponent $p > 0$ for a Hilbert space H. For any $p > 0$, $\mathcal{L}^p(H)$ is a symmetrically quasinormed ideal in the bounded operators and, for $p \geq 1$, it is a symmetrically normed ideal.

Definition I.0.8. If A is unital, we say that a Fredholm module (A, H, F) is p-summable if

$$F^* - F, F^2 - 1 \in \mathcal{L}^{p/2}(H) \quad \text{and} \quad [F,a] \in \mathcal{L}^p(H)$$

for a in a dense $*$-subalgebra $\mathscr{A}$ of A.

If $\mathscr{A}$ is unital, we say that a higher order spectral triple $(\mathscr{A}, H, D)$ is *p-summable* if $(1+D^2)^{-1/2} \in \mathcal{L}^p(H)$.

Proposition I.0.9. *[FGM25, Theorem 2.2] cf. [SWW98, Proposition 1] With $\mathscr{A}$ unital, let $(\mathscr{A}, H, D)$ be a p-summable order m spectral triple. Writing A for the C*-algebra closure of $\mathscr{A}$, the Fredholm module (A, H, F_D) is mp-summable over $\mathscr{A}$.*

In the context of the above Proposition, q-summability of the Fredholm module for $q > mp$ follows immediately from Theorem I.0.6; to take $q = mp$ requires the careful use of an operator inequality.

Remark I.0.10. In this thesis, we have chosen to work only with ungraded C*-algebras. We therefore work with even and odd Kasparov modules, about which a small remark is in order. Let A and B be ungraded C*-algebras. By definition [Kas88, §2.22], $KK_1(A,B) = KK_0(A, B \otimes \mathscr{C}\ell_1)$, where $\mathscr{C}\ell_1$ is treated as a graded C*-algebra. The following is well known; variations can be found in [Con94, Proposition IV.A.13(b)] and [HR00, (8.1.10)]. A more sophisticated discussion could involve *multigradings* [HR00, Definition 8.1.11, §A.3].

If (A, E_B, F) is an odd bounded Kasparov module, we can build a bounded Kasparov A-$B \otimes \mathscr{C}\ell_1$-module

$$\left(A, (E \oplus E)_{B \otimes \mathscr{C}\ell_1}, \begin{pmatrix} & F \\ F & \end{pmatrix}\right)$$

where $B \otimes \mathscr{C}\ell_1$ acts on the right of $E \oplus E$ by

$$\begin{pmatrix} \xi & \eta \end{pmatrix} (b + c\gamma_1) = \begin{pmatrix} \xi & \eta \end{pmatrix} \begin{pmatrix} b & c \\ c & b \end{pmatrix} = \begin{pmatrix} \xi b + \eta c & \xi c + \eta b \end{pmatrix} \qquad (\xi, \eta \in E, b + c\gamma_1 \in B \otimes \mathscr{C}\ell_1), \quad \text{(I.0.11)}$$

and the grading on $E \oplus E$ is given by $\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$; cf. [HR00, (8.1.10)].

This process is completely reversible. Given a bounded Kasparov module $(A, E'_{B\otimes\mathscr{C}\ell_1}, F')$, the action of $\gamma_1 \in \mathscr{C}\ell_1$ on the right of E' identifies the even and odd parts of E'. Writing, therefore, $E' = E \oplus E$ for $E = E'^{\mathrm{ev}} = E'^{\mathrm{odd}}$, we write $(A, E'_{B\otimes\mathscr{C}\ell_1}, F')$ as

$$\left(A, (E\oplus E)_{B\otimes\mathscr{C}\ell_1}, \begin{pmatrix} & V \\ U & \end{pmatrix}\right),$$

where the action of $B \otimes \mathscr{C}\ell_1$ on the right of $E \oplus E$ is given by (I.0.11). The off-diagonal form of the operator is a consequence of the requirement that F' is odd. Since F' must also be linear in the action of $\mathscr{C}\ell_1$, we must have $U = V$. We thus obtain an odd bounded Kasparov A-B-module (A, E_B, U).

The same discussion applies equally to odd unbounded cycles.

I.1 Equivalence relations for KK-theory

Let A and B be C*-algebras. A *homotopy* between two bounded Kasparov A-B-modules is a Kasparov A-$C([0,1], B)$-module whose evaluations at $0, 1 \in [0,1]$ recover them [Kas81, Definition 4.2.2]. Homotopy is an equivalence relation and compatible with direct sums. The homotopy classes of bounded Kasparov A-B-modules, together with the direct sum, form a $\mathbb{Z}/2\mathbb{Z}$-graded abelian group $KK_*(A,B) = KK_0(A,B) \oplus KK_1(A,B)$, contravariant in A and covariant in B [Kas81, Theorem 4.1, Definition 4.4]. Another relation on bounded Kasparov modules is *operator homotopy* [Kas81, Definition 4.2.2]. If the C*-algebra A is separable, then operator homotopy, together with the addition of *degenerate* modules, is equivalent to homotopy. The details of homotopy for unbounded Kasparov modules have only recently been worked out [DM20, Kaa20]. It turns that out that, provided that A is separable, one can indeed obtain $KK_*(A,B)$ from homotopy classes of unbounded Kasparov modules.

On the other hand, the strongest reasonable equivalence relation in the bounded picture of KK-theory (apart from unitary equivalence) is locally compact perturbation. If (A, E_B, F) is a bounded Kasparov module and $T \in \mathrm{End}^*(E)$ is such that $Ta, aT \in \mathrm{End}^0(E)$ for all $a \in A$, then $(A, E_B, F+T)$ will still be a bounded Kasparov module. The only condition which is not immediate is that $((F+T)^2 - 1)a \in \mathrm{End}^0(E)$, demonstrated by the computation

$$((F+T)^2-1)a = (F^2-1)a + (F+T)Ta + TFa = (F^2-1)a + (F+T)Ta + T[F,a] + TaF.$$

It is perhaps unclear, in the unbounded picture of KK-theory, what should stand in for equivalence up to locally compact perturbation. The most immediate relation that suggests itself is equivalence up to bounded perturbation. If (A, E_B, D) is an unbounded Kasparov module and $T = T^* \in \mathrm{End}^*(E)$, then $(A, E_B, D+T)$ will still be an unbounded Kasparov module. The local compactness of the resolvent takes a little work, see e.g. [CP98, Lemma B.6]. One can similarly consider locally bounded perturbations, at least in the presence of an adequate approximate unit [Dun18, §4].

By applying Theorem I.0.6, we can study additive perturbations of higher order cycles in the following sense; cf. [CP98, Lemmas B.6–7].

Proposition I.1.1. *Let D_0 and D_1 be self-adjoint regular operators on right Hilbert B-modules E_0 and E_1. Suppose that there is an operator $a \in \mathrm{Hom}^*_B(E_0, E_1)$ such that $a \operatorname{dom} D_0 \subseteq \operatorname{dom} D_1$ and*

$$(D_1 a - a D_0)\langle D_0\rangle^{-\alpha}$$

extends to an adjointable operator for some $0 \le \alpha < 1$. Then, fixing $\beta < 1 - \alpha$,

$$(F_{D_1} a - a F_{D_0})\langle D_0\rangle^{\beta}$$

is bounded.

Proof. Consider the operators

$$D = \begin{pmatrix} D_0 & \\ & D_1 \end{pmatrix} \qquad S = \begin{pmatrix} & 0 \\ a & \end{pmatrix}$$

on $E_0 \oplus E_1$. Then

$$S \operatorname{dom} D = \begin{pmatrix} 0 \\ a \operatorname{dom} D_0 \end{pmatrix} \subseteq \begin{pmatrix} \operatorname{dom} D_0 \\ \operatorname{dom} D_1 \end{pmatrix} = \operatorname{dom} D$$

and

$$[D, S]\langle D\rangle^{-\alpha} = \begin{pmatrix} & 0 \\ (D_1 a - aD_0)\langle D_0\rangle^{-\alpha} & \end{pmatrix}.$$

By Theorem I.0.6,

$$[F_D, S]\langle D\rangle^{\beta} = \begin{pmatrix} & 0 \\ (F_{D_1} a - aF_{D_0})\langle D_0\rangle^{\beta} & \end{pmatrix}$$

is bounded for $\beta < 1 - \alpha$, as required. ☐

Corollary I.1.2. *Let D_0 and D_1 be self-adjoint regular operators on a right Hilbert B-module E with densely intersecting domains. Suppose that there is a bounded operator a such that $a \operatorname{dom} D_0 \subseteq \operatorname{dom} D_0 \cap \operatorname{dom} D_1$ and*

$$(D_1 - D_0)a\langle D_0\rangle^{-\alpha} \qquad [D_0, a]\langle D_0\rangle^{-\alpha}$$

extend to bounded operators for some $0 \leq \alpha < 1$. Then, fixing $\beta < 1 - \alpha$,

$$(F_{D_1} - F_{D_0})a\langle D_0\rangle^{\beta}$$

is bounded.

Proof. We have

$$(D_1 a - aD_0)\langle D_0\rangle^{-\alpha} = (D_1 - D_0)a\langle D_0\rangle^{-\alpha} + [D_0, a]\langle D_0\rangle^{-\alpha}$$

and

$$(F_{D_1} a - aF_{D_0})\langle D_0\rangle^{\beta} = (F_{D_1} - F_{D_0})a\langle D_0\rangle^{\beta} + [F_{D_0}, a]\langle D_0\rangle^{\beta}.$$

By Theorem I.0.6, $[F_{D_0}, a]\langle D_0\rangle^{\beta}$ is bounded, so $(F_{D_1} - F_{D_0})a\langle D_0\rangle^{\beta}$ is also, as required. ☐

In [CS86, §3], *cobordism* is introduced as another equivalence relation on bounded Kasparov modules; slightly weakening locally compact perturbation. (We remark that the similarly named equivalence relation of *bordism* of unbounded Kasparov modules [Hil10, DGM18] is unrelated and will not appear in this thesis.) First, we require a small Lemma.

Lemma I.1.3. *[CS86, §3] If (A, E_B, F) is a bounded Kasparov module and $p \in \operatorname{End}^*(E)$ is an even projection commuting with the representation of A such that $[F, p]a$ is compact for all $a \in A$, then (A, pE_B, pFp) is a Kasparov module.*

Definition I.1.4. [CS86, Definition 3.1] Two bounded Kasparov modules (A, E'_B, F_1) and (A, E''_B, F_2) of the same parity are *cobordant* if there exists a Kasparov module (A, E_B, F) of that parity and an partial isometry $v \in \operatorname{End}^*(E)$ (even if the parity is even), such that

- v commutes with (the representation of) A;
- $[F, v]a$ is compact for all $a \in A$;
- $(A, (1 - vv^*)E_B, (1 - vv^*)F(1 - vv^*))$ is unitarily equivalent to (A, E'_B, F_1); and
- $(A, (1 - v^*v)E_B, (1 - v^*v)F(1 - v^*v))$ is unitarily equivalent to (A, E''_B, F_2).

We call $(A, E_B, F; v)$ a cobordism.

It turns out that cobordism is an equivalence relation, and is compatible with direct sums [CS86, Lemma 3.3]. Even though apparently much stronger than homotopy, cobordism gives rise to the same KK-groups, provided A is separable [CS86, Theorem 3.7]. (Our definition differs slightly from that of [CS86, Definition 3.1], in that we deal only with trivially graded C*-algebras and work with odd as well as even Kasparov modules. By Remark I.0.10, it is straighforward to check that [CS86, Lemma 3.6, Theorem 3.7] are still valid.)

Example I.1.5. Suppose that two bounded Kasparov modules (A, E'_B, F_1) and (A, E''_B, F_2) of the same parity are unitarily equivalent, up to a locally compact perturbation, that is, there exists a unitary $U : E'_B \to E''_B$ (even if the parity is even), intertwining the representations of A, such that $(U^* F_2 U - F_1)a \in \mathrm{End}^0(E)$ for all $a \in A$. Then

$$\left(A, (E' \oplus E'')_B, \begin{pmatrix} F_1 & \\ & F_2 \end{pmatrix}\right) \qquad v = \begin{pmatrix} & 0 \\ U & \end{pmatrix}$$

constitute a cobordism between the two modules.

Lemma I.1.6. *If two bounded Kasparov modules $(A, E_{1,B}, F_1)$ and $(A, E_{2,B}, F_2)$ are cobordant, there exists a cobordism $(A, E_B, F; v)$ such that vv^*, v^*v, and F mutually commute.*

Proof. Let $(A, E'_B, F'; v')$ be any cobordism between $(A, E_{1,B}, F_1)$ and $(A, E_{2,B}, F_2)$. Let $w_1 : E_1 \to (1 - v'v'^*)E'$ and $w_2 : E_2 \to (1 - v'^*v')E'$ be the unitaries of the cobordism. Then

$$\left(A, E_1 \oplus E' \oplus E_2, F_1 \oplus F' \oplus F_2; w_1^* + v' + w_2\right)$$

is a cobordism between $(A, E_{1,B}, F_1)$ and $(A, E_{2,B}, F_2)$. We have

$$(w_1^* + v' + w_2)^*(w_1^* + v' + w_2) = 0 \oplus 1 \oplus 1 \qquad (w_1^* + v' + w_2)(w_1^* + v' + w_2)^* = 1 \oplus 1 \oplus 0.$$

We can check that

$$\begin{aligned}
[F_1 \oplus F' \oplus F_2, w_1^* + v' + w_2]a &= \left(F_1 w_1^* + F'(v' + w_2) - (w_1^* + v')F' - w_2 F_2\right) a \\
&= \left([F', v'] + F_1 w_1^* + F' w_2 - w_1^* F' - w_2 F_2\right) a \\
&= \left([F', v'] + w_1^* F'(1 - v'v'^*) + F' w_2 - w_1^* F' - (1 - v'^*v')F' w_2\right) a \\
&= \left([F', v'] - w_1^* F' v' v'^* + v'^* v' F' w_2\right) a \\
&= [F', v']a - w_1^*[F', v']av'^* - v'^*[F', v']aw_2
\end{aligned}$$

is compact for all $a \in A$, as required. □

I.1.1 Cobordism of higher order cycles and positive degeneracy

We shall make a natural generalisation of cobordism to unbounded cycles but, first, a Lemma.

Lemma I.1.7. *Let (A, E_B, D) be an order-$\frac{1}{1-\alpha}$ cycle and $p \in \mathrm{End}^*(E)$ a projection (even if the cycle is of even parity) such that p commutes with A and D. Then (A, pE_B, pDp) is an order-$\frac{1}{1-\alpha}$ cycle and, furthermore, $F_{pDp} = pF_D p$ on pE.*

A similar result to Lemma I.1.7 would follow from weaker assumptions than that p and D commute but we do without.

Definition I.1.8. Two order-$\frac{1}{1-\alpha}$ cycles (A, E'_B, D_1) and (A, E''_B, D_2) of the same parity are *cobordant* if there exist an order-$\frac{1}{1-\alpha}$ cycle (A, E_B, D) of that parity and a partial isometry $v \in \mathrm{End}^*(E)$ (even if the parity is even), such that

- v commutes with (the representation of) A, and vv^* and v^*v commute with D;
- $vA \subseteq \overline{\mathrm{Lip}^*_\alpha(D)}$;
- $(A, (1 - vv^*)E_B, (1 - vv^*)D(1 - vv^*))$ is unitarily equivalent to (A, E'_B, D_1); and
- $(A, (1 - v^*v)E_B, (1 - v^*v)D(1 - v^*v))$ is unitarily equivalent to (A, E''_B, D_2).

For a dense $*$-subalgebra $\mathscr{A} \subseteq A$, $(\mathscr{A}, E_B, D; v)$ is a cobordism between $(\mathscr{A}, E'_B, D_1)$ and $(\mathscr{A}, E''_B, D_2)$ if $v^*v\mathscr{A} \subseteq \mathscr{Q}$.

At the cost of further technicalities, we could proceed with weaker assumptions than that D commute with vv^* and v^*v. However, by a similar argument to Lemma I.1.6, this would not be worth the cost.

Proposition I.1.9. *cf. [CS86, Lemma 3.3] Cobordism of higher order cycles is an equivalence relation and is compatible with direct sums.*

Proof. For reflexivity, we take $v = 0 \in \mathrm{End}^*(E)$ to see that (A, E_B, D) is cobordant to itself.

For symmetry, note that $v^*A = (vA)^* \subseteq \overline{\mathrm{Lip}^*_\alpha(D)}$ so that making the substitution of v^* for v reverses the roles of (A, E'_B, D_1) and (A, E''_B, D_2).

For transitivity, suppose that $(A, E_B, D; v)$ is a cobordism between the cycles $(A, E_{1,B}, D_1)$ and $(A, E_{2,B}, D_2)$, and that $(A, E'_B, D'; v')$ is a cobordism between $(A, E_{2,B}, D_2)$ and $(A, E_{3,B}, D_3)$. Let $U : (1 - v^*v)E \to E_2$ and $U' : (1 - v'v'^*)E \to E_2$ be the unitary equivalences between the cycles

$$(A, (1 - v^*v)E_B, (1 - v^*v)D(1 - v^*v)) \qquad (A, (1 - v'v'^*)E'_B, (1 - v'v'^*)D'(1 - v'v'^*))$$

and the cycle $(A, E_{2,B}, D_2)$, respectively. Then

$$(A, (E \oplus E')_B, D \oplus D'; v + U'^*U + v')$$

is a cobordism between $(A, E_{1,B}, D_1)$ and $(A, E_{3,B}, D_3)$. We have

$$(v + U'^*U + v')(v + U'^*U + v')^* = vv^* \oplus 1 \qquad (v + U'^*U + v')^*(v + U'^*U + v') = 1 \oplus v'^*v'.$$

Furthermore,

$$\mathrm{Lip}^*_\alpha(D) \oplus \mathrm{Lip}^*_\alpha(D') \subseteq \mathrm{Lip}^*_\alpha(D \oplus D'),$$

so that $(v + v')A \subseteq \overline{\mathrm{Lip}^*_\alpha(D \oplus D')}$. Because D commutes with $(1 - v^*v)$ and D' commutes with $(1 - v'v'^*)$, $D'U'^*U = U'^*D_2U = U'^*UD$ on $E \oplus E'$. Hence

$$U'^* \mathrm{Lip}^*_\alpha(D_2)U \subseteq \mathrm{Lip}^*_\alpha(D \oplus D')$$

and so $U'^*UA = U'^*AU \subseteq U'^*\overline{\mathrm{Lip}^*_\alpha(D_2)}U \subseteq \overline{\mathrm{Lip}^*_\alpha(D \oplus D')}$ as required.

Finally, it is straightforward to check that direct sums of cobordisms are cobordisms of direct sums in an obvious way. ☐

Example I.1.10. Let (A, E'_B, D_1) and (A, E''_B, D_2) be two order-$\frac{1}{1-\alpha}$ cycles of the same parity. Suppose that there exists a unitary $U : E'_B \to E''_B$ (even if the parity is even), intertwining the representations of A, such that A is contained in the closure of the set of $a \in \mathrm{End}^*(E')$ for which $Ua \operatorname{dom} D_1 \subseteq \operatorname{dom} D_2$ and

$$(U^*D_2Ua - aD_1)\langle D_1\rangle^{-\alpha} \qquad U\langle D_2\rangle^{-\alpha}U(U^*D_2Ua - aD_1)$$

extend to adjointable operators on E'. Then

$$\left(A, (E' \oplus E'')_B, \begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix}\right) \qquad v = \begin{pmatrix} & 0 \\ U & \end{pmatrix}$$

constitute a cobordism between the two cycles.

Proposition I.1.11. *Given two cobordant order-$\frac{1}{1-\alpha}$ cycles (A, E'_B, D_1) and (A, E''_B, D_2), their bounded transforms (A, E'_B, F_{D_1}) and (A, E''_B, F_{D_2}) are cobordant and so they define the same element in $KK_*(A, B)$.*

Proof. Let $(A, E_B, D; v)$ be a cobordism between (A, E'_B, D_1) and (A, E''_B, D_2). By Lemma I.1.7, $(A, E_B, F_D; v)$ is a bounded cobordism between (A, E'_B, F_{D_1}) and (A, E''_B, F_{D_2}). □

A natural question to ask is whether one can identify unbounded cycles cobordant to the zero module. In [DM20, §3–4], several notions of degenerate module are surveyed and shown to be homotopic to zero. Instead of making a similar survey, we shall make the following definition, in the safety of the knowledge that it contains as special cases the *spectrally degenerate* cycles of [DM20, Definition 3.5], the *spectrally symmetric* cycles of [DM20, Definition 4.6] (which, in turn, include the *spectrally decomposable* cycles of [Kaa20, Definition 4.1]), the *Clifford symmetric* cycles of [DM20, Definition 4.13], and the *weakly degenerate* cycles of [DGM18, Definition 3.1].

Definition I.1.12. An order-$\frac{1}{1-\alpha}$ cycle (A, E_B, D) is *positively degenerate* if there exists a self-adjoint unitary $s \in \operatorname{End}^*(E)$ (odd if the cycle is of even parity), preserving the domain of D, such that

- As operators on $\operatorname{dom} D$, $Ds + sD \geq -c\langle D\rangle^\alpha$ for some constant $c \geq 0$ and
- $A \subseteq \overline{\mathscr{P}}$, where $\mathscr{P}$ is the set of $a \in \operatorname{Lip}^*_\alpha(D)$ such that $[s, a] = 0$.

Proposition I.1.13. *A positively degenerate order-$\frac{1}{1-\alpha}$ cycle (A, E_B, D) is cobordant to $(A, 0_B, 0)$.*

Proof. Let $s \in \operatorname{End}^*(E)$ be a symmetry implementing the degeneracy. Let N be the number operator and S the unilateral shift on $\ell^2(\mathbb{N}_{\geq 0})$. Then $(A, E_B \otimes \ell^2(\mathbb{N}_{\geq 0}), D \otimes 1 + s \otimes N)$ is an order-$\frac{1}{1-\alpha}$ cycle. The main point to check is the local compactness of the resolvent, for which we compute

$$(D \otimes 1 + s \otimes N)^2 = D^2 \otimes 1 + 1 \otimes N^2 + (Ds + sD) \otimes N \geq D^2 \otimes 1 + 1 \otimes N^2 - c\langle D\rangle^\alpha \otimes N.$$

Fix $\varepsilon \in (0, 1)$. The function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f : (x, y) \mapsto \varepsilon(x^2 + y^2) - c(1 + x^2)^{\alpha/2} y$$

has a global minimum. Hence, for large enough $\lambda > 0$,

$$\lambda + (D \otimes 1 + s \otimes N)^2 \geq (1 - \epsilon)(D^2 \otimes 1 + 1 \otimes N^2)$$

and so

$$a(\lambda + 1 + (D \otimes 1 + s \otimes \kappa N)^2)^{-1}$$

is compact. The constructed order-$\frac{1}{1-\alpha}$ cycle, together with the isometry $1 \otimes S$, implements the required cobordism. Using the relation $NS = S(N + 1)$, we check that

$$[D \otimes 1 + s \otimes N, (1 \otimes S)a] = s \otimes [N, S]a = s \otimes Sa$$

is bounded for $a \in \mathscr{P}$. □

We can now show that higher order cycles, subject to the equivalence relation of cobordism, form a group under direct sum.

Corollary I.1.14. *Given an order-$\frac{1}{1-\alpha}$ cycle (A, E_B, D),*

$$(A, E_B, D) \oplus (A, E_B^{(\mathrm{op})}, -D) = \left(A, (E \oplus E)^{(\mathrm{op})})_B, \begin{pmatrix} D & \\ & -D \end{pmatrix} \right),$$

where $E^{(\mathrm{op})}$ is E with the opposite grading if E is graded, is cobordant to $(A, 0_B, 0)$.

Proof. The symmetry $s = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ makes the direct sum cycle positively degenerate. □

Combining Propositions I.1.9, I.1.11 and Corollary I.1.14 proves

Theorem I.1.15. *Let $0 \leq \alpha < 1$. Cobordism classes of order-$\frac{1}{1-\alpha}$ A-B-cycles form a $\mathbb{Z}/2\mathbb{Z}$-graded abelian group which surjects onto $KK_*(A, B)$. Further, cobordism classes of higher order A-B-cycles (without a constraint on their order) form a $\mathbb{Z}/2\mathbb{Z}$-graded abelian group which surjects onto $KK_*(A, B)$.*

For the final statement, we note than any order-$\frac{1}{1-\alpha}$ cycle can be considered to be an order-$\frac{1}{1-\beta}$ cycle for $\alpha \leq \beta < 1$. It is presumably the case that cobordism of higher order cycles is strictly stronger than homotopy. It is possible that the addition of the functional dampening of [DM20] could make cobordism equivalent to homotopy. This remains a matter for future investigation.

I.2 Group-equivariant KK-theory

In this section we begin by recalling the definitions of equivariant KK-theory and the descent map, due to Kasparov [Kas88]. The first attempt to generalise equivariance to unbounded KK-theory is [JV87, §1], for the case of $KK^G(\mathbb{C}, \mathbb{C})$. The first detailed treatment is by Kucerovsky [Kuc94, §8], which we mildly generalise in §I.2.1 to apply to the higher order case and allow for local boundedness in the definition. We refer to §A.1.1 for some technical preliminaries about Hilbert C*-modules over locally compact Hausdorff spaces, mostly following [Kuc94, Appendix A].

The case of compact groups is much easier to handle in both the bounded and unbounded settings. This is because, given the action of a compact group on a Kasparov module, one can integrate using the Haar measure to produce a module for which the operator is actually invariant under the action of the group. This fact has led to the definition of unbounded equivariant KK-theory in the case of a compact group as unbounded Kasparov modules with group actions for which the operator is invariant under the action. Alas, this does not represent the full range of geometrical situations available under equivariant KK-theory.

The following definition introduces notation for tracking the action of operators implementing equivariance. Throughout this section, G is a locally compact group.

Definition I.2.1. Let E be a right Hilbert B-module and $\tau \in \operatorname{Aut} B$. We define $\operatorname{End}^{*;\tau}_B(E)$ to be the set of $\mathbb{C}$-linear maps $T : E \to E$ for which there exists a map $T^* : E \to E$ such that

$$\langle T(x) \mid y\rangle_B = \tau(\langle x \mid T^*(y)\rangle_B).$$

These maps are not B-linear; however they satisfy $T(xb) = T(x)\tau(b)$ since

$$\langle T(xb) \mid y\rangle_B = \tau(\langle xb \mid T^*(y)\rangle_B) = \tau(b^*)\tau(\langle x \mid T^*(y)\rangle_B) = \tau(b^*)\langle T(x) \mid y\rangle_B = \langle T(x)\tau(b) \mid y\rangle_B.$$

This gives an identification of $\operatorname{End}^{*;\tau}_B(E)$ with $\operatorname{Hom}^*_B(E, E \otimes_\tau B)$, where $E \otimes_\tau B$ is the internal tensor product of E with ${}_\tau B$. The adjoint $T^* \in \operatorname{End}^{*;\tau^{-1}}_B(E)$, since

$$\langle T^*(x) \mid y\rangle_B = \langle y \mid T^*(x)\rangle^*_B = \tau^{-1}(\langle T(y) \mid x\rangle^*_B) = \tau^{-1}(\langle x \mid T(y)\rangle_B).$$

The composition of $S \in \operatorname{End}^{*;\sigma}_B(E)$ and $T \in \operatorname{End}^{*;\tau}_B(E)$ is $ST \in \operatorname{End}^{*;\sigma\circ\tau}_B(E)$. In particular, if $\tau = \sigma^{-1}$ then ST is an adjointable operator.

Definition I.2.2. e.g. [Kas88, §1.2] Let $\beta : G \to \operatorname{Aut} B$ be an action of a group G on a C*-algebra B. A *G-equivariant* Hilbert B-module E is a Hilbert B-module equipped with a continuous $\mathbb{C}$-linear map $U : G \times E \to E$ such that

$$U_{gh} = U_g U_h \qquad U_g(xb) = U_g(x)\beta_g(b) \qquad \beta_g(\langle x \mid y\rangle_B) = \langle U_g(x) \mid U_g(y)\rangle_B$$

for $g, h \in G$, $x, y \in E$, and $b \in B$. We may equivalently say that $U_g \in \mathrm{End}_B^{*,\beta_g}(E)$ with the conditions

$$U_{gh} = U_g U_h \qquad U_{g^{-1}} = U_g^{-1} = U_g^*$$

for all $g, h \in G$.

Definition I.2.3. Let $\alpha : G \to \operatorname{Aut} A$ be an action of a group G on a C*-algebra A. An A-B-correspondence E is *G-equivariant* if it is a G-equivariant Hilbert B-module and

$$U_g(ax) = \alpha_g(a) U_g(x)$$

for all $g \in G$, $a \in A$ and $x \in E$.

Definition I.2.4. [Kas88, Definition 2.2] cf. [Kuc94, Definition 8.5, Remark] A bounded Kasparov A-B-module (A, E_B, F) is *G-equivariant* if E is a G-equivariant A-B-correspondence and, for all $a \in A$, the map $g \mapsto (U_g F U_g^* - F)a$ is norm-continuous from G into $\mathrm{End}^0(E)$.

Remark I.2.5. cf. [Kuc94, Definition 8.5, Remark] By Lemma A.1.8, the norm continuity of the map $g \mapsto (U_g F U_g^* - F)a$ into $\mathrm{End}^0(E)$ is equivalent to the condition that, when restricted to any compact subset $K \subseteq G$, the function $g \mapsto (U_g F U_g^* - F)a$ is in $\mathrm{End}^0(C(K, E))$.

We also record the following Definition.

Definition I.2.6. cf. [Pus11, Definition 2.7]. If A is a unital C*-algebra, we say that (A, H, F) is a p-summable G-equivariant Fredholm module if (A, H, F) is a Fredholm module, p-summable and G-equivariant, and $U_g F U_g^* - F \in \mathcal{L}^p(H)$ for all $g \in G$.

I.2.1 Uniform group equivariance

Again, throughout this section, G is a locally compact group. The following definition mildly generalises that of Kucerovsky, in that we allow for higher order Kasparov modules and for the equivariance to be checked locally in the algebra.

Definition I.2.7. cf. [Kuc94, Definition 8.7] Let (A, E_B, D) be an order-$\frac{1}{1-\alpha}$ A-B-cycle with E a G-equivariant A-B-correspondence. We say that (A, E_B, D) is *uniformly G-equivariant* if A is contained in the closure of $\mathscr{Q}$, the set of $a \in \mathrm{End}^*(E)$ such that $a \operatorname{dom} D \subseteq U_g \operatorname{dom} D$ for all $g \in G$ and the maps

$$g \mapsto \overline{(U_g D U_g^* a - aD)\langle D\rangle^{-\alpha}} \qquad g \mapsto \overline{U_g \langle D\rangle^{-\alpha} U_g^* (U_g D U_g^* a - aD)}$$

are $*$-strongly continuous as maps from G into $\mathrm{End}_B^*(E)$. If $U_g D U_g^* = D$ for all $g \in G$, we say that the cycle is *isometrically equivariant*. If $\mathscr{A}$ is a dense $*$-subalgebra of A contained in $\mathscr{Q}$, we say that $(\mathscr{A}, E_B, D)$ is a uniformly G-equivariant order-$\frac{1}{1-\alpha}$ $\mathscr{A}$-B-cycle.

An example where the extra freedom in our definition, as compared to [Kuc94, Definition 8.7], is needed is given in Proposition II.4.1.

Remarks I.2.8.

1. We remark that $\mathscr{Q} \subseteq \mathrm{Lip}_\alpha^*(D)$ by considering the conditions at $g = e$, the identity of the group. Indeed, $\mathscr{Q}$ is a right ideal of $\mathrm{Lip}_\alpha^*(D)$.
2. By Lemma A.1.12, the conditions on $a \in \mathscr{Q}$ are equivalent to the condition that $a \operatorname{dom} D \subseteq U_g \operatorname{dom} D$ and, when restricted to any compact subset $K \subseteq G$, the functions

$$g \mapsto \overline{(U_g D U_g^* a - aD)\langle D\rangle^{-\alpha}} \qquad g \mapsto \overline{U_g \langle D\rangle^{-\alpha} U_g^* (U_g D U_g^* a - aD)}$$

be in $\mathrm{End}^*(C(K, E))$.

3. When $\alpha = 0$, the conditions on $a \in \mathcal{Q}$ are equivalent to requiring that $[D, a]$ extend to an adjointable operator and
$$g \mapsto (U_g D U_g^* - D)a$$
be $*$-strongly continuous as a map from G into bounded operators. The higher order generalisation allows for higher order differential operators on manifolds, for example.

To prove that the bounded transform is well-defined, we use the results of §A.1.1, based on the approach of Kucerovsky [Kuc94, Chapter 8, Appendix A]; see also [AK23, Appendix A].

Theorem I.2.9. *[Kuc94, Proposition 8.11] Let (A, E_B, D) be a uniformly G-equivariant order-$\frac{1}{1-\alpha}$ cycle. Then (A, E_B, F_D) is a G-equivariant bounded Kasparov module.*

Proof. The only difference from the non-equivariant case is the need to show that, for every $a \in A$, $g \mapsto (F_D - U_g F_D U_g^*)a$ is norm-continuous as a map from G into $\mathrm{End}^0(E)$.

Fix $b \in \mathcal{Q}$, where $\mathcal{Q}$ is as in Definition I.2.7. By definition, the map $f : g \mapsto (U_g D U_g^* b - bD)\langle D\rangle^{-\alpha}$ is $*$-strongly continuous as a map from G into $\mathrm{End}^*(E)$. By Lemma A.1.12, this is equivalent to $f|_K$ residing in $\mathrm{End}^*(C(K, E))$ for every compact subset $K \subseteq G$.

Fix a compact subset $K \subseteq G$ and let $\tilde{E} = C(K, E)$. Define $\tilde{D}$ to be the self-adjoint regular operator on $\tilde{E}$ given by D at each point of K. Similarly, let $\tilde{b} \in \mathrm{End}^*(\tilde{E})$ be given by b at each point of K. Let U denote the $\mathbb{C}$-linear map from $\tilde{E}$ to itself given by $(U\xi)(g) = U_g \xi(g)$. Then
$$(U\tilde{D}U^*\tilde{b} - \tilde{b}\tilde{D})\langle\tilde{D}\rangle^{-\alpha}$$
is bounded. Applying Proposition I.1.1, the operator $(F_{U\tilde{D}U^*} - F_{\tilde{D}})\tilde{b}\langle\tilde{D}\rangle^{\beta}$ is bounded for all $\beta < 1 - \alpha$. By the functional calculus, $F_{U\tilde{D}U^*} = UF_{\tilde{D}}U^*$. Fixing an element $c \in A$, let $\tilde{c}$ denote the operator on $\tilde{E}$ given by $c \in \mathrm{End}^*(E)$ at every point of K. Since $\langle D\rangle^{-\beta}c \in \mathrm{End}^0(E)$,
$$\langle\tilde{D}\rangle^{-\beta}\tilde{c} \in C(K, \mathrm{End}^0(E)) = \mathrm{End}^0(\tilde{E}).$$
Hence
$$(UF_{\tilde{D}}U^* - F_{\tilde{D}})\tilde{b}\tilde{c} = (F_{U\tilde{D}U^*} - F_{\tilde{D}})\tilde{b}\langle\tilde{D}\rangle^{\beta}\langle\tilde{D}\rangle^{-\beta}\tilde{c}$$
is in $\mathrm{End}^0(\tilde{E}) = \mathrm{End}^0(C(K, E))$.

Define the map $f' : g \mapsto (F_D - U_g F_D U_g^*)bc$ from G into bounded operators on E. By Lemma A.1.8, the norm-continuity of f' is equivalent to the condition that $f'|_K$ be in $\mathrm{End}^0(C(K, E))$ for every compact subset $K \subseteq G$. By the inclusion of $A \subseteq \overline{\mathcal{Q}A}$, we are done. □

Remark I.2.10. Let A be a unital C*-algebra and $\mathscr{A}$ a dense unital $*$-subalgebra of A. Let $(\mathscr{A}, H, D)$ be a uniformly G-equivariant p-summable order m spectral triple. Because $(U_g F_D U_g^* - F_D)(1 + D^2)^{\beta/2}$ is bounded for $\beta < m^{-1}$, the G-equivariant Fredholm module (A, H, F_D) is q-summable over $\mathscr{A}$ for any $q > mp$ (see Definition I.2.6).

I.2.2 Descent and the dual Green–Julg map

An important feature of equivariant KK-theory is Kasparov's descent map
$$j_t^G : KK^G(A, B) \to KK(A \rtimes_t G, B \rtimes_t G)$$
for either topology $t \in \{u, r\}$, universal or reduced [Kas88, Theorem 3.11]. There can be other, exotic, topologies t for which there is a descent map [BEW15, §6] but we will not pursue this.

Definition I.2.11. [Kas88, Definition 3.8], [Bla98, Definition 20.6.1] Let E be a G-equivariant A-B-correspondence. The algebra $C_c(G,B)$ acts on the right of $C_c(G,E)$ by

$$(\xi f)(g) = \int_G \xi(h)\beta_h(f(h^{-1}g))d\mu(h) \qquad (\xi \in C_c(G,E), f \in C_c(G,B))$$

where β is the action of G on B. We define a right $C_c(G,B)$-valued inner product on $C_c(G,E)$ by

$$\langle \xi|\eta\rangle_{C_c(G,B)}(g) = \int_G \beta_{h^{-1}}(\langle \xi(h)|\eta(hg)\rangle_B)d\mu(h) \qquad (\xi,\eta \in C_c(G,E)).$$

The algebra $C_c(G,A)$ acts on the left of $C_c(G,E)$ by

$$(f\xi)(g) = \int_G f(h)U_h\xi(h^{-1}g)d\mu(h) \qquad (f \in C_c(G,A), \xi \in C_c(G,E))$$

where U is the representation of G on E. For $t \in \{u,r\}$, we denote by $E \rtimes_t G$ the $A \rtimes_t G$-$B \rtimes_t G$-correspondence obtained by completing $C_c(G,E)$ in the $C_c(G,B)$-valued inner product. We may also realise $E \rtimes_t G$ as the internal tensor product $E \otimes_B (B \rtimes_t G)$, but the left action of $A \rtimes_t G$ is difficult to see in this picture.

Proposition I.2.12. *[Kas88, Theorem 3.11] Let (A, E_B, F) be a G-equivariant bounded Kasparov module. Then, for $t \in \{u,r\}$, $(A \rtimes_t G, (E \rtimes_t G)_{B\rtimes_t G}, \tilde{F})$ is a bounded Kasparov module, where $\tilde{F}$ is the operator given on $\xi \in C_c(G,E) \subseteq E \rtimes_t G$ by $(\tilde{F}\xi)(g) = F(\xi(g))$.*

If G is a compact group and acts trivially on A there is the *Green–Julg* isomorphism

$$\Phi_G : KK^G(A,B) \to KK(A, B \rtimes G);$$

see [Bla98, 20.2.7(b)]. On the other hand, when G acts trivially on B, there is the *dual Green–Julg* map

$$\Psi^G : KK^G(A,B) \to KK(A \rtimes_u G, B)$$

which is an isomorphism when G is discrete [Bla98, 20.2.7(b)]. The existence of Ψ^G is proved in the next proposition, and then we present the isomorphism for discrete groups. The universal crossed product is needed because it is universal for covariant representations.

Proposition I.2.13. *Let (A, E_B, F) be a G-equivariant bounded Kasparov module, with G acting trivially on B. Then $(A \rtimes_u G, E_B, F)$ is a bounded Kasparov module, with the integrated representation of $A \rtimes_u G$.*

Proof. With α the action of G on A, π the representation of A on E, and U the representation of G on E, the pair (π, U) is a covariant representation of the C*-dynamical system (A, G, α). We obtain by [EKQR06, §A.2] the integrated representation $\pi \rtimes U$ of $A \rtimes_u G$ on E, and it is here that the universal crossed product is needed. We will consider the dense subalgebra $C_c(G,A) \subseteq A \rtimes_u G$. For an element $f \in C_c(G,A)$,

$$(F^* - F)(\pi \rtimes U)(f) = \int_G (F^* - F)\pi(f(g))U_g d\mu(g).$$

Because f is compactly supported and the integrand norm continuous, the integral converges. The integrand being valued in compact operators, the result is also compact. In the same way,

$$(F^2 - 1)(\pi \rtimes U)(f) = \int_G (F^2 - 1)\pi(f(g))U_g d\mu(g)$$

and

$$[F, (\pi \rtimes U)(f)] = \int_G [F, \pi(f(g))U_g]d\mu(g) = \int_G \left([F, \pi(f(g))]U_g + \pi(f(g))(F - U_gFU_g^*)U_g\right) d\mu(g)$$

are compact. By the density of $C_c(G,A) \subseteq A \rtimes_u G$ we are done. □

Proposition I.2.14. *Let $(A \rtimes_u G, E_B, F)$ be a bounded Kasparov module, with G a discrete group and $A \rtimes_u G$ represented nondegenerately on E. Then (A, E_B, F) is a G-equivariant bounded Kasparov module, with the group action given by $(U_g)_{g \in G} \subseteq C^*_u(G) \subseteq M(A \rtimes_u G)$, acting trivially on B.*

Proof. Because G is discrete, A is included in $A \rtimes_u G$. Hence,

$$(F^* - F)a \qquad (F^2 - 1)a \qquad [F, a]$$

are compact for all $a \in A$. Inside $M(A \rtimes_u G)$ are unitary elements $(U_g)_{g \in G}$ representing G, such that $aU_g \in A \rtimes_u G$ for all $a \in A$ and $g \in G$. Then

$$(F - U_g F U_g^*)a = [F, U_g]U_g^* a = [F, a] - [F, U_g^* a] = [F, a] + [F, aU_g]^*$$

is compact, as required. □

Before we define the descent map for uniformly equivariant cycles, let us introduce some notation.

Definition I.2.15. Let $\mathscr{A}$ be a dense $*$-subalgebra of a C*-algebra A. Let α be an action of a locally compact group G on A which preserves $\mathscr{A}$. If G is discrete, we write $\mathscr{A} \rtimes G$ for the algebraic crossed product, which is dense in $A \rtimes_t G$. For a non-discrete group, we will generalise this by defining $\mathscr{A} \rtimes G \subseteq A \rtimes_t G$ as the (dense) $*$-subalgebra generated by $\mathscr{A}$ and $C_c(G)$ under the canonical inclusions $\mathscr{A} \subseteq A \subseteq M(A \rtimes_t G)$ and $C_c(G) \subseteq C^*(G) \subseteq M(A \rtimes_t G)$.

Proposition I.2.16. *Let (A, E_B, D) be a uniformly G-equivariant order-$\frac{1}{1-\alpha}$ cycle. Then for either topology $t \in \{u, r\}$, $(A \rtimes_t G, (E \rtimes_t G)_{B \rtimes_t G}, \tilde{D})$ is an order-$\frac{1}{1-\alpha}$ cycle, where $\tilde{D}$ is the regular operator given on $\xi \in C_c(G, E) \subseteq E \rtimes_t G$ by $(\tilde{D}\xi)(g) = D(\xi(g))$.*

If, for a dense $*$-subalgebra $\mathscr{A} \subseteq A$, $(\mathscr{A}, E_B, D)$ is a uniformly G-equivariant order-$\frac{1}{1-\alpha}$ cycle, $(\mathscr{A} \rtimes G, (E \rtimes_t G)_{B \rtimes_t G}, \tilde{D})$ is an order-$\frac{1}{1-\alpha}$ cycle.

Proof. We have, for $f \in C_c(G, A)$ and $\xi \in C_c(G, E)$

$$((1 + \tilde{D}^2)^{-1} f\xi)(g) = \int_G (1 + D^2)^{-1} f(h) U_h \xi(h^{-1}g) d\mu(h).$$

As f is compactly supported and the integrand continuous, the integral converges. Observe that $(1 + \tilde{D}^2)^{-1} f$ is an element of $C_c(G, \mathrm{End}^0(E))$, given by $g \mapsto (1 + D^2)^{-1} f(g)$. By [Kas88, Proof of Theorem 3.11], $C_c(G, \mathrm{End}^0(E)) \subseteq \mathrm{End}^0(E \rtimes_t G)$, so $(1 + \tilde{D}^2)^{-1} f$ is compact.

Next, note that the closure of $\mathscr{Q} C_c(G)$ includes $A \rtimes_t G$. Let $a \in \mathscr{Q}$, $f \in C_c(G)$, and $\xi \in \operatorname{span}(C_c(G) \operatorname{dom} D) \subseteq C_c(G, \operatorname{dom} D)$. Then we find that

$$\begin{aligned}([\tilde{D}, af]\langle \tilde{D} \rangle^{-\alpha} \xi)(g) &= \int_G [D, aU_h] \langle \tilde{D} \rangle^{-\alpha} f(h) \xi(h^{-1}g) d\mu(h) \\ &= \int_G (Da - aU_h D U_h^*) U_h \langle \tilde{D} \rangle^{-\alpha} U_h^* U_h f(h) \xi(h^{-1}g) d\mu(h).\end{aligned}$$

As f is compactly supported and the integrand is continuous, the integral converges. Observe that the closure of $[\tilde{D}, af]\langle \tilde{D} \rangle^{-\alpha}$ is an element of $C_c(G, \mathrm{End}^*(E)_{*-s})$ given by

$$g \mapsto f(g) \overline{(Da - aU_g D U_g^*) U_g \langle D \rangle^{-\alpha} U_g^*}.$$

As $C_c(G, \mathrm{End}^*(E)_{*-s}) \subseteq \mathrm{End}^*(E \rtimes_t G)$ (see [Rae88, Lemma 7]), $[\tilde{D}, af]\langle \tilde{D} \rangle^{-\alpha}$ is bounded. Similarly, $\langle \tilde{D} \rangle^{-\alpha}[\tilde{D}, af]$ is bounded. Hence for $b \in \mathscr{A} \rtimes G$, $[\tilde{D}, b]\langle \tilde{D} \rangle^{-\alpha}$ and $\langle \tilde{D} \rangle^{-\alpha}[\tilde{D}, b]$ are bounded, proving the second statement. □

For uniformly equivariant cycles, we have a dual Green–Julg map for the universal crossed product.

Proposition I.2.17. *Let (A, E_B, D) be a uniformly G-equivariant order-$\frac{1}{1-\alpha}$ cycle, with G acting trivially on B. Then $(A \rtimes_u G, E_B, D)$ is an order-$\frac{1}{1-\alpha}$ cycle, with the integrated representation of $A \rtimes_u G$.*

If, for a dense $*$-subalgebra $\mathscr{A} \subseteq A$, $(\mathscr{A}, E_B, D)$ is a uniformly G-equivariant order-$\frac{1}{1-\alpha}$ cycle, with G acting trivially on B, $(\mathscr{A} \rtimes G, E_B, D)$ is an order-$\frac{1}{1-\alpha}$ cycle.

Proof. With α the action of G on A, π the representation of A on E, and U the representation of G on E, the pair (π, U) is a covariant representation of the C*-dynamical system (A, G, α) and we obtain the integrated representation $\pi \rtimes U$ of $A \rtimes_u G$ on E. For an element $f \in C_c(G, A)$,

$$(1 + D^2)^{-1}(\pi \rtimes U)(f) = \int_G (1 + D^2)^{-1}\pi(f(g))U_g d\mu(g).$$

As f is compactly supported and the integrand norm-continuous, the integral converges, and as the integrand is valued in compact operators, the integral is also compact. As in the proof of Proposition I.2.16, we note that the closure of $\mathcal{Q}C_c(G)$ includes $A \rtimes_u G$. Let $a \in \mathcal{Q}$, $f \in C_c(G)$, and $\xi \in \operatorname{dom} D$; then

$$\begin{aligned}
[D, (\pi \rtimes U)(af)]\langle D\rangle^{-\alpha}\xi &= \int_G f(g)[D, \pi(a)U_g]\langle D\rangle^{-\alpha}\xi d\mu(g) \\
&= \int_G f(g)(D\pi(a) - \pi(a)U_g DU_g^*)U_g\langle D\rangle^{-\alpha}\xi d\mu(g).
\end{aligned}$$

As f is compactly supported and the integrand is continuous, the integral converges. By Corollary A.1.10, $[D, (\pi \rtimes U)(af)]\langle D\rangle^{-\alpha}$ extends to an adjointable operator, as does $\langle D\rangle^{-\alpha}[D, (\pi \rtimes U)(af)]$. □

In order to display the inverse of the dual Green-Julg map for discrete groups, a dense subalgebra $\mathscr{A}$ of A is required.

Proposition I.2.18. *Let $(\mathscr{A} \rtimes G, E_B, D)$ be an order-$\frac{1}{1-\alpha}$ cycle, with G a discrete group and the representation of $\mathscr{A} \rtimes G$ on E nondegenerate. Then $(\mathscr{A}, E_B, D)$ is a uniformly G-equivariant order-$\frac{1}{1-\alpha}$ cycle, with group action given by $(U_g)_{g\in G} \subseteq C_u^*(G) \subseteq M(A \rtimes_u G)$, acting trivially on B.*

Proof. Because G is discrete, $\mathscr{A}$ is included in $\mathscr{A} \rtimes G$. Hence, $(1 + D^2)^{-1}a$ is compact and $[D, a]$ is bounded for all $a \in \mathscr{A}$. Inside $M(A \rtimes_u G)$ are unitary elements $(U_g)_{g\in G}$ representing G, such that $aU_g \in \mathscr{A} \rtimes G$ for all $a \in \mathscr{A}$ and $g \in G$. Then

$$U_g DU_g^* a - aD = U_g[D, U_g^* a]$$

so that $(U_g DU_g^* a - aD)\langle D\rangle^{-\alpha}$ and $\langle D\rangle^{-\alpha}U_g^*(U_g DU_g^* a - aD)$ are bounded, as required. □

Remark I.2.19. It is immediate that the bounded transform $(A \rtimes_t G, (E \rtimes_t G)_{B\rtimes_t G}, F_{\tilde{D}} = \tilde{F_D})$ of the descent $(A \rtimes_t G, (E \rtimes_t G)_{B\rtimes_t G}, \tilde{D})$ of a uniformly G-equivariant cycle (A, E_B, D) is exactly the descent of the bounded transform (A, E_B, F_D). The same is true for the dual Green–Julg map.

I.3 Quantum group–equivariant KK-theory

Quantum group–equivariant KK-theory, in the bounded picture, is due to Baaj and Skandalis [BS89]. A detailed account can be found in [Ver02]. We first recall the notions of a *C*-bialgebra* and a *locally compact quantum group*.

Definition I.3.1. e.g. [Tim08, Definitions 4.1.1,3] A *C*-bialgebra* is a C*-algebra S equipped with a *comultiplication* map, a coassociative, nondegenerate $*$-homomorphism $\Delta : S \to M(S \otimes S)$ such that $\Delta(S)(S \otimes 1)$ and $(1 \otimes S)\Delta(S)$ are contained in $S \otimes S$. A C*-bialgebra S is *simplifiable* if

$$\overline{\operatorname{span}}(\Delta(S)(S \otimes 1)) = S \otimes S = \overline{\operatorname{span}}((1 \otimes S)\Delta(S)).$$

A *von Neumann bialgebra* is a von Neumann algebra M with a *comultiplication* map, a coassociative, unital, normal $*$-homomorphism $\Delta : M \to M \otimes M$, the von Neumann tensor product.

Commutative C*-bialgebras are in duality with certain locally compact semigroups; see [Val85, §3] for precise statements.

Definition I.3.2. e.g. [Tim08, Chapter 8] A *locally compact quantum group* $\mathbb{G}$ is given by the equivalent data of either:

- A simplifiable C*-bialgebra $C_0^r(\mathbb{G})$ with left- and right-invariant, KMS, faithful weights; or
- A von Neumann bialgebra $L^\infty(\mathbb{G})$ with left- and right-invariant, normal, semifinite, faithful weights.

For the precise meaning of the adjectives on the weights, see e.g. [Tim08, §8.1.1-2], but we will not use these details. From such data, one obtains:

- The Hilbert space $L^2(\mathbb{G})$, on which $L^\infty(\mathbb{G})$ and $C_0^r(\mathbb{G})$ are represented, obtained by the GNS construction from the left Haar weight (of either algebra);
- The *universal* function algebra $C_0^u(\mathbb{G})$, which surjects onto $C_0^r(\mathbb{G})$;
- The *dual* locally compact quantum group $\hat{\mathbb{G}}$, for which $L^2(\hat{\mathbb{G}}) \cong L^2(\mathbb{G})$, and the C*-algebras $C_r^*(\mathbb{G}) := C_0^r(\hat{\mathbb{G}})$ and $C_u^*(\mathbb{G}) := C_0^u(\hat{\mathbb{G}})$;
- The *multiplicative unitary* $W \in M(C_0^r(\mathbb{G}) \otimes C_0^r(\hat{\mathbb{G}})) \subseteq B(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$ satisfying the equation $W_{12}W_{13}W_{23} = W_{23}W_{12}$ and, for $a \in C_0^r(\mathbb{G})$, $\Delta(a) = W^*(1 \otimes a)W$ on $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$; and
- A Banach algebra $L^1(\mathbb{G}) := L^\infty(\mathbb{G})_*$, the predual of $L^\infty(\mathbb{G})$.

We next recall the details of C*-bialgebra-coactions on C*-algebras and Hilbert modules.

Definition I.3.3. [EKQR06, Definitions 1.39, A.3] Let B and C be C*-algebras. The *C-multiplier algebra* of $B \otimes C$ is

$$M_C(B \otimes C) = \Big\{ m \in M(B \otimes C) \mid m(1 \otimes C) \cup (1 \otimes C)m \in B \otimes C \Big\}.$$

If E is a Hilbert B-module, the *C-multiplier module* of $E \otimes C_{B \otimes C}$ is the Hilbert $M_C(B \otimes C)$-module

$$M_C(E \otimes C) = \Big\{ m \in \operatorname{Hom}^*_{B \otimes C}(B \otimes C, E \otimes C) | \, m(1 \otimes C) \cup (1 \otimes C)m \in E \otimes C \Big\}.$$

Definition I.3.4. [BS89, §2], [Ver02, §3.1] A coaction of a C*-bialgebra S on a C*-algebra B is a coassociative nondegenerate $*$-homomorphism $\delta_B : B \to M_S(B \otimes S)$. A coaction of S on a Hilbert B-module E is a coassociative $\mathbb{C}$-linear map $\delta_E : E \to M_S(E \otimes S)$ such that

- $\delta_E(\xi)\delta_B(b) = \delta_E(\xi b)$ and $\langle \delta_E(\xi) \mid \delta_E(\eta) \rangle_{M_S(B \otimes S)} = \delta_B(\langle \xi \mid \eta \rangle_B)$ for all $\xi, \eta \in E$ and $b \in B$; and
- $\delta_E(E)(B \otimes S)$ is dense in $E \otimes S$.

Let $E \otimes_{\delta_B} (B \otimes S)$ be the internal tensor product of Hilbert modules where the left action of B on $B \otimes S$ is given by δ_B. For an element $\xi \in E$, denote by $T_\xi \in \operatorname{Hom}^*_{B \otimes S}(B \otimes S, E \otimes_{\delta_B} (B \otimes S))$ the map $b \otimes s \mapsto \xi \otimes_{\delta_B} (b \otimes s)$. A unitary $V_E \in \operatorname{Hom}^*_{B \otimes S}(E \otimes_{\delta_B} (B \otimes S), E \otimes S)$ is *admissible* if

- $V_E T_\xi \in M_S(E \otimes S)$ for all $\xi \in E$; and

- $(V_E \otimes_{\mathbb{C}} 1)(V_E \otimes_{\delta_B \otimes \mathrm{id}_S} 1) = (V_E \otimes_{\mathrm{id}_B \otimes \Delta_S} 1) \in \mathrm{Hom}^*_{B\otimes S\otimes S}(E \otimes_{\delta_B^2} (B \otimes S \otimes S), E \otimes S \otimes S)$, where $\delta_B^2 = (\delta_B \otimes \mathrm{id}_S)\delta_B = (\mathrm{id}_B \otimes \Delta_S)\delta_B$.

The data of a coaction on E is equivalent to the data of an admissible unitary V_E, by the identity $V_E T_\xi = \delta_E(\xi)$ for $\xi \in E$.

If A is a C*-algebra with an S-coaction δ_A, an A-B-correspondence E is *S-equivariant* if it possesses a Hilbert B-module coaction δ_E such that

$$\delta_A(a)\delta_E(\xi) = \delta_E(a\xi)$$

for all $a \in A$ and $\xi \in E$. In terms of the admissible unitary, this is equivalent to $V_E(a \otimes 1)V_E^* = \delta_A(a)$.

Definition I.3.5. cf. [Pod95, Definition 1.4(b)], [BSV03, §5.2] Let S be a C*-bialgebra. An S-coaction δ_B on a C*-algebra B satisfies the *Podleś condition* (sometimes called simply *continuity*) if $\overline{\mathrm{span}}(\delta_B(B)(1 \otimes S)) = B \otimes S$. An S-coaction δ_E on a Hilbert B-module E then automatically satisfies

$$\overline{\mathrm{span}}(\delta_E(E)(1 \otimes S)) = \overline{\mathrm{span}}(\delta_E(E)\delta_B(B)(1 \otimes S)) = \overline{\mathrm{span}}(\delta_E(E)(B \otimes S)) = E \otimes S$$

and $\overline{V_E(E \otimes_{\delta_B} (1 \otimes S))}$ is dense in $E \otimes S$.

Definition I.3.6. An *action* of a locally compact quantum group $\mathbb{G}$ on a C*-algebra B is a $C_0^r(\mathbb{G})$-coaction on B satisfying the Podleś condition. A $\mathbb{G}$-action on a Hilbert B-module E is a $C_0^r(\mathbb{G})$-coaction on E.

In the above Definition, the reduced C*-algebra is used following [Ver02] and [NV10, §4]. One could perhaps define the action of a quantum group $\mathbb{G}$ as a $C_0^u(\mathbb{G})$-coaction instead, as is done in [EKQR06], although it is unclear what the consequences of this would be, particularly for the descent map.

Definition I.3.7. [BS89, Définition 3.1] cf. [NV10, §4] Let A and B be C*-algebras equipped with coactions of a C*-bialgebra S. A bounded Kasparov A-B-module (A, E_B, F) is *S-equivariant* if E is an S-equivariant A-B-correspondence and for all $a \in A$ and $s \in S$

$$(V_E(F \otimes_{\delta_B} 1)V_E^* - F \otimes 1)a \otimes s$$

is compact. If A and B are C*-algebras with $\mathbb{G}$-actions, a bounded Kasparov module (A, E_B, F) is *$\mathbb{G}$-equivariant* if it is $C_0^r(\mathbb{G})$-equivariant.

I.3.1 Uniform quantum group equivariance

We make the following definition in the unbounded setting. To our knowledge, except in the case of the isometric coaction of a compact quantum group (see e.g. [GB16, Definition 2.3.1]), such a definiton has not appeared in the published literature (but see [Gof09, Definition 3.3.1]).

Definition I.3.8. Let A and B be C*-algebras equipped with coactions of a C*-bialgebra S. Fix $0 \le \alpha < 1$. For $a \in \mathrm{Lip}^*_\alpha(D)$ let $\mathscr{S}_a$ be the set of $s \in S$ such that $a \otimes s\,\mathrm{dom}(D \otimes 1) \subseteq V_E\,\mathrm{dom}(D \otimes_{\delta_B} 1)$ and

$$\Big(V_E(D \otimes_{\delta_B} 1)V_E^*(a \otimes s) - (a \otimes s)(D \otimes 1)\Big)\langle D \otimes 1\rangle^{-\alpha}$$
$$\text{and } V_E\langle D \otimes_{\delta_B} 1\rangle^{-\alpha}V_E^*\Big(V_E(D \otimes_{\delta_B} 1)V_E^*(a \otimes s) - (a \otimes s)(D \otimes 1)\Big)$$

extend to adjointable operators on $E \otimes S$. An order-$\frac{1}{1-\alpha}$ A-B-cycle (A, E_B, D) is *uniformly S-equivariant* if E is an S-equivariant A-B-correspondence and A is contained in the closure of

$$\mathscr{Q} = \left\{ a \in \mathrm{Lip}^*_\alpha(D) \,\middle|\, \overline{\mathscr{S}_a} = S \right\} .$$

If $V_E(D \otimes_{\delta_B} 1)V_E^* = D \otimes 1$, we say that the cycle is *isometrically equivariant.*

If A and B are C*-algebras with $\mathbb{G}$-actions, a cycle (A, E_B, D) is *uniformly* $\mathbb{G}$*-equivariant* if it is uniformly $C_0^r(\mathbb{G})$-equivariant.

If $\mathscr{A}$ is a dense $*$-subalgebra of A such that $\mathscr{A} \subseteq \mathscr{Q}$, we say that $(\mathscr{A}, E_B, D)$ is S-equivariant (or $\mathbb{G}$-equivariant, as the case may be).

Remark I.3.9. The dense subset $\mathscr{S}_a \subseteq S$ need not be the same for different $a \in \mathscr{Q}$. For many locally compact quantum groups, there may be a natural choice, fixed for all a. For a discrete quantum group $\mathbb{G}$, i.e. when $C_0(\mathbb{G})$ is isomorphic as an algebra to the C*-algebraic direct sum

$$\bigoplus_{\lambda \in \Lambda} M_{n_\lambda}(\mathbb{C})$$

of finite-dimensional matrix algebras, $\mathscr{S}_a$ would contain all elements of the algebraic direct sum. In this case, the admissible unitary would be labelled by the index set $\lambda \in \Lambda$, so that

$$V_E^\lambda \in \operatorname{Hom}_B^*(E \otimes_{\delta_B} (B \otimes \mathbb{C}^{n_\lambda}), E \otimes \mathbb{C}^{n_\lambda})$$

and the equivariance condition becomes that

$$\begin{aligned}&\big(V_E^\lambda(D \otimes_{\delta_B} 1)V_E^{\lambda *}(a \otimes 1_{n_\lambda}) - (a \otimes 1_{n_\lambda})(D \otimes 1_{n_\lambda})\big)\langle D \otimes 1_{n_\lambda}\rangle^{-\alpha} \\ &\qquad\text{and } V_E^\lambda \langle D \otimes_{\delta_B} 1\rangle^{-\alpha} V_E^{\lambda *}\big(V_E^\lambda(D \otimes_{\delta_B} 1)V_E^{\lambda *}(a \otimes 1_{n_\lambda}) - (a \otimes 1_{n_\lambda})(D \otimes 1_{n_\lambda})\big)\end{aligned}$$

be bounded for all $\lambda \in \Lambda$. (Note that there need not be any bound uniform in $\lambda \in \Lambda$.) For the dual $\hat{G}$ of a group G, we suspect it always makes sense to assume that $\mathscr{S}_a$ contains the right ideal $C_r^*(G)^\infty$ of smooth elements [WN92, §§2–3], as in Example I.3.11.

Theorem I.3.10. *A uniformly S-equivariant order-$\frac{1}{1-\alpha}$ cycle (A, E_B, D) gives rise to an S-equivariant bounded Kasparov module (A, E_B, F_D).*

Proof. The only difference from the non-equivariant case is the need to show that, for every $a \in A$ and $s \in S$, $(F_D \otimes 1 - V_E(F_D \otimes_{\delta_B} 1)V_E^*)a \otimes s$ is compact. Let $b \in \mathscr{Q}$ and $s \in \mathscr{S}_a$ so that

$$(V_E(D \otimes_{\delta_B} 1)V_E^* - D \otimes 1)(b \otimes s)\langle D \otimes 1\rangle^\beta$$

extends to an adjointable operator. By Corollary I.1.2,

$$(V_E(F_D \otimes_{\delta_B} 1)V_E^* - F_D \otimes 1)(b \otimes s)\langle D\rangle^\beta \otimes 1$$

is bounded for all $\beta < 1 - \alpha$. With $c \in A$,

$$(V_E(F_D \otimes_{\delta_B} 1)V_E^* - F_D \otimes 1)bc \otimes s = (V_E(F_D \otimes_{\delta_B} 1)V_E^* - F_D \otimes 1)(b \otimes s)(\langle D\rangle^\beta \otimes 1)\langle D\rangle^{-\beta}c \otimes 1$$

is compact and, by the density of $\mathscr{S}_a \subseteq S$ and the inclusion of $A \subseteq \overline{\mathscr{Q}A}$, we are done. □

Example I.3.11. Let G be a connected Lie group with a left-invariant Riemannian metric $\mathbf{g}$, such as the affine group $\mathbb{R} \rtimes \mathbb{R}_+^\times$ of the real line as the real hyperbolic plane. The left-invariant Riemannian metric on G is exactly determined by the inner product $\mathbf{g}_e$ on the tangent space $T_eG = \mathfrak{g}$ at the identity $e \in G$. The left-invariant differential operators and differential forms on G can be identified with $U(\mathfrak{g})$ and $\Lambda^*(\mathfrak{g})$, respectively. The Clifford algebra $\mathscr{C}\ell(\mathfrak{g})$ acts on the left of $\Lambda^*(\mathfrak{g})$. The Hodge–de Rham Dirac operator $d + \delta$ on $(G, \mathbf{g})$ can be written as

$$d + \delta = \sum_{i=1}^{\dim \mathfrak{g}} X_i \otimes \gamma_i,$$

where $X_i \in \mathfrak{g} \subseteq U(\mathfrak{g})$ and $\gamma_i \in \mathfrak{g} \subseteq \mathscr{C}\ell(\mathfrak{g})$. We have an isometrically G-equivariant spectral triple

$$(C_0(G), L^2(G, \Lambda^*(\mathfrak{g})), d + \delta).$$

Elements of $U(\mathfrak{g})$ act as affiliated operators on $C_r^*(G)$; see [WN92, §§2–3]. By abuse of notation, we also write $d + \delta \in U(\mathfrak{g}) \otimes \mathscr{C}\ell(\mathfrak{g})$ for the corresponding regular operator on $(C_r^*(G) \otimes \Lambda^*(\mathfrak{g}))_{C_r^*(G)}$. By Baaj–Skandalis duality [BS89, §6], it is reasonable to expect that

$$(\mathbb{C}, (C_r^*(G) \otimes \Lambda^*(\mathfrak{g}))_{C_r^*(G)}, d + \delta)$$

is a uniformly $\hat{G}$-equivariant $\mathbb{C}$-$C_r^*(G)$-unbounded Kasparov module. To see that it is, first consider the coaction on the module $(C_r^*(G) \otimes \Lambda^*(\mathfrak{g}))_{C_r^*(G)}$. The admissible unitary is a map from

$$(C_r^*(G) \otimes \Lambda^*(\mathfrak{g})) \otimes_{\delta_{C_r^*(G)}} (C_r^*(G) \otimes C_r^*(G)) = C_r^*(G) \otimes \Lambda^*(\mathfrak{g}) \otimes C_r^*(G)$$

to

$$(C_r^*(G) \otimes \Lambda^*(\mathfrak{g})) \otimes_{\mathbb{C}} C_r^*(G) = C_r^*(G) \otimes \Lambda^*(\mathfrak{g}) \otimes C_r^*(G).$$

Under these identifications,

$$T_{x \otimes \psi} : C_r^*(G) \otimes C_r^*(G) \to C_r^*(G) \otimes \Lambda^*(\mathfrak{g}) \otimes C_r^*(G) \qquad y \otimes z \mapsto x_{(1)} y \otimes \psi \otimes x_{(2)} z$$

$$x_{(1)} \otimes \psi \otimes x_{(2)} = \delta(x \otimes \psi) = V T_x = V(x_{(1)} \otimes \psi \otimes x_{(2)})$$

so V is just the identity in $\operatorname{End}^*_{C_r^*(G)}(C_r^*(G) \otimes \Lambda^*(\mathfrak{g}) \otimes C_r^*(G))$. Because $X_i \in \mathfrak{g}$, in the universal enveloping algebra $U(\mathfrak{g})$, $\Delta X_i = X_i \otimes 1 + 1 \otimes X_i$ and

$$(d + \delta) \otimes_{\delta_{C_r^*(G)}} 1 = \sum_i (X_i \otimes \gamma_i) \otimes_{\Delta_{U(\mathfrak{g})}} 1 = \sum_i (X_i \otimes \gamma_i \otimes 1 + 1 \otimes \gamma_i \otimes X_i).$$

Therefore,

$$V((d + \delta) \otimes_{\delta_{C_r^*(G)}} 1) V^* - (d + \delta) \otimes 1 = 1 \otimes \gamma_i \otimes X_i.$$

For $(\mathbb{C}, (C_r^*(G) \otimes \Lambda^*(\mathfrak{g}))_{C_r^*(G)}, d + \delta)$ to be $C_r^*(G)$-equivariant, we require a dense subalgebra of $C_r^*(G)$ in the common domain of the derivations $\mathfrak{g}$. There is in fact such a subalgebra, the right ideal $C_r^*(G)^\infty$ of smooth elements for the G-action on $C_r^*(G)$ by unitary multipliers [WN92, §§2–3].

I.3.2 Descent and the dual Green–Julg map

Crossed products are not defined in the generality of Hopf C*-algebra–coactions. One needs a well-defined notion of duality and, for that, we restrict to locally compact quantum groups. (It is possible to work in the greater generality of a weak Kac system [Ver02, §2.2], but we forgo this in the interests of readability.)

We use the symbol Σ for the flip map on a tensor product. Recall the multiplicative unitary $W \in M(C_0^r(\mathbb{G}) \otimes C_0^r(\hat{\mathbb{G}})) \subseteq B(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$ of a locally compact quantum group $\mathbb{G}$.

Definition I.3.12. [Tim08, Definition 7.3.1] cf. [BS93, Proposition 3.2, Définition 3.3] A locally compact quantum group $\mathbb{G}$ is *regular* if

$$\overline{\operatorname{span}}\{(\omega \otimes 1)(W\Sigma) \mid \omega \in B(L^2(\mathbb{G}))_*\} = K(L^2(\mathbb{G})).$$

Equivalently, $\mathbb{G}$ is regular if the reduced crossed product $C_0^r(\mathbb{G}) \rtimes_r \mathbb{G} \cong K(L^2(\mathbb{G}))$; see Definition I.3.14 below.

For example, every locally compact group G and its dual $\hat{G}$ are regular.

In the following proof, one might expect

$$(A \otimes 1)U(1 \otimes B) \subseteq A \otimes B$$

to hold automatically for a unitary $U \in M(A \otimes B)$ but this is not the case, as [LPRS87, Remark after Lemma 1.2] shows.

Lemma I.3.13. *Let E be a Hilbert B-module with a $\mathbb{G}$ action, $\mathbb{G}$ acting trivially on B. Then $C_u^*(\mathbb{G})$ is represented on E. Conversely, if $\mathbb{G}$ is a regular quantum group, a (nondegenerate) representation of $C_u^*(\mathbb{G})$ on a Hilbert B-module gives rise to a $\mathbb{G}$ action on E which is trivial on B.*

Proof. Let E be a Hilbert B-module with a $\mathbb{G}$ action, $\mathbb{G}$ acting trivially on B. The fundamental unitary V_E is then an element of $\operatorname{End}^*(E \otimes C_0^r(\mathbb{G}))$ and can be thought of as an element of $\operatorname{End}^*(E \otimes L^2(\mathbb{G}))$ by the left regular representation of $C_0^r(\mathbb{G})$. By [Kus01, Proposition 5.2], there is a nondegenerate representation of $C_u^*(\mathbb{G})$ on E.

On the other hand, suppose that $C_u^*(\mathbb{G})$ is represented nondegenerately by π on a Hilbert B-module E. Let $\hat{\mathcal{V}} \in M(C_0^r(\mathbb{G}) \otimes C_u^*(\mathbb{G}))$ be the unitary of [Kus01, Proposition 4.2]. By [Kus01, Corollary 4.3], we obtain an element $X = (\pi \otimes \mathrm{id})(\Sigma\hat{\mathcal{V}}\Sigma) \in \operatorname{End}^*(E \otimes S)$ such that $(1 \otimes \Delta)(X) = X_{12}X_{13}$. The only thing stopping X from being the admissible unitary of an action of $\mathbb{G}$ on E (with trivial action on B) is the possible failure of $(1 \otimes C_0^r(\mathbb{G}))X(E \otimes 1)$ to be contained in $E \otimes C_0^r(\mathbb{G})$. If we assume $\mathbb{G}$ to be regular, by [BS93, Proposition A.3(d)],

$$\overline{\operatorname{span}}(1 \otimes C_0^r(\mathbb{G}))X(\pi(C_u^*(\mathbb{G})) \otimes 1) = \pi(C_u^*(\mathbb{G})) \otimes C_0^r(\mathbb{G})$$

and therefore

$$(1 \otimes C_0^r(\mathbb{G}))X(E \otimes 1) = (1 \otimes C_0^r(\mathbb{G}))X(\pi(C_u^*(\mathbb{G}))E \otimes 1) \subseteq E \otimes C_0^r(\mathbb{G}),$$

as required. □

It is unclear if the converse statement of Lemma I.3.13 is true without the assumption of regularity.

Definition I.3.14. cf. [Ver02, Définitions 4.2, 5.1, Lemmes 4.1, 5.2] Let A be a C*-algebra with a $\mathbb{G}$-action. The *reduced* crossed product $A \rtimes_r \mathbb{G}$ is given by

$$\overline{\operatorname{span}}(\delta_A(A)(1 \otimes C_r^*(\mathbb{G}))) \subseteq M(A \otimes K(L^2(\mathbb{G}))).$$

There is also a *universal* crossed product $A \rtimes_u \mathbb{G}$; for a definition we refer to [Vae05, §2.3]. For our purposes, the following details will suffice. Let ${}_\pi E$ be a $\mathbb{G}$-equivariant A-B-correspondence, with $\mathbb{G}$ acting trivially on B. There is an *integrated* representation of $A \rtimes_u \mathbb{G}$ on E whose image is

$$\overline{\operatorname{span}}(\pi(A)C_u^*(\mathbb{G})) \subseteq \operatorname{End}^*(E).$$

If $\mathbb{G}$ is regular, the algebra $A \rtimes_u \mathbb{G}$ is universal for such integrated representations; if $\mathbb{G}$ is not regular $A \rtimes_u \mathbb{G}$ is universal for a slightly larger class of representations; see [Ver02, Définition 4.2] and [Vae05, §2.3]. There is a canonical surjection $A \rtimes_u \mathbb{G} \to A \rtimes_r \mathbb{G}$.

Let E be a right Hilbert B-module with an action of $\mathbb{G}$. For either topology $t \in \{u, r\}$, the crossed product Hilbert module $E \rtimes_t \mathbb{G}$ is given by the internal tensor product $E \otimes_B (B \rtimes_t \mathbb{G})$. By [Ver02, Lemme 5.2], $\operatorname{End}^0_B(E) \rtimes_t \mathbb{G}$ is naturally identified with $\operatorname{End}^0_{B \rtimes_t \mathbb{G}}(E \rtimes_t \mathbb{G})$.

In the locally compact quantum group setting, there is a descent map

$$j_t^{\mathbb{G}} : KK^{\mathbb{G}}(A, B) \to KK(A \rtimes_t \mathbb{G}, B \rtimes_t \mathbb{G})$$

for either topology $t \in \{u, r\}$, universal or reduced, generalising Kasparov's descent map for classical groups. If $\mathbb{G}$ is the dual of a classical group, descent is due to Baaj and Skandalis [BS89, Théorème 6.19], and in general due to Vergnioux [Ver02, Proposition 5.3]. In the locally compact quantum group setting, a refinement of the reduced descent is possible, to a map

$$J^{\mathbb{G}} : KK^{\mathbb{G}}(A, B) \to KK^{\hat{\mathbb{G}}}(A \rtimes_r \mathbb{G}, B \rtimes_r \mathbb{G})$$

whose composition with the forgetful functor $KK^{\hat{\mathbb{G}}} \to KK$ is $j_r^{\mathbb{G}}$. If $\mathbb{G}$ is regular, $C_0^r(\mathbb{G}) \rtimes_r \mathbb{G} \cong K(L^2(\mathbb{G})) \cong C_r^*(\mathbb{G}) \rtimes_r \hat{\mathbb{G}}$ and the maps $J^{\mathbb{G}}$ and $J^{\hat{\mathbb{G}}}$ are mutually inverse isomorphisms [BS93, Remarque 7.7(b)]. We refer to these isomorphisms as *Baaj–Skandalis duality.*

Proposition I.3.15. *[Ver02, Proposition 5.3] Let (A, E_B, F) be a $\mathbb{G}$-equivariant bounded Kasparov module. For $t \in \{u, r\}$, let ι be the inclusion $\operatorname{End}^0(E) \to M(\operatorname{End}^0(E) \rtimes_t \mathbb{G}) \cong \operatorname{End}^*_{B \rtimes_t \mathbb{G}}(E \rtimes_t \mathbb{G})$. Then $(A \rtimes_t \mathbb{G}, (E \rtimes_t \mathbb{G})_{B \rtimes_t \mathbb{G}}, \iota(F))$ is a bounded Kasparov module.*

If $\mathbb{G}$ is compact and acts trivially on A, we have the *Green–Julg* isomorphism

$$\Phi_{\mathbb{G}} : KK^{\mathbb{G}}(A, B) \to KK(A, B \rtimes \mathbb{G});$$

see [Ver02, Théorème 5.10]. On the other hand, when $\mathbb{G}$ acts trivially on B, there is a dual Green–Julg map for the universal crossed product

$$\Psi^{\mathbb{G}} : KK^{\mathbb{G}}(A, B) \to KK(A \rtimes_u \mathbb{G}, B)$$

which is an isomorphism when $\mathbb{G}$ is discrete [Ver02, Proposition 5.11].

Proposition I.3.16. *[Ver02, Proposition 5.11] Let (A, E_B, F) be a $\mathbb{G}$-equivariant bounded Kasparov module, with $\mathbb{G}$ acting trivially on B. Then $(A \rtimes_u \mathbb{G}, E_B, F)$ is a bounded Kasparov module, with the integrated representation of $A \rtimes_u \mathbb{G}$.*

Proposition I.3.17. *[Ver02, Proposition 5.11] Let $(A \rtimes_u \mathbb{G}, E_B, F)$ be a bounded Kasparov module, with $\mathbb{G}$ a discrete quantum group and $A \rtimes_u \mathbb{G}$ represented nondegenerately on E. Then (A, E_B, F) is a $\mathbb{G}$-equivariant bounded Kasparov module, with the coaction of $C_0^r(\mathbb{G})$ on E given by the action of $C_u^*(\mathbb{G}) \subseteq M(A \rtimes_u \mathbb{G})$ on E, acting trivially on B.*

In the unbounded setting, we have the following picture of descent.

Proposition I.3.18. *Let (A, E_B, D) be a uniformly $\mathbb{G}$-equivariant order-$\frac{1}{1-\alpha}$ cycle. For $t \in \{u, r\}$, let ι be the inclusion $\operatorname{End}^0(E) \to M(\operatorname{End}^0(E) \rtimes_t \mathbb{G}) \cong \operatorname{End}^*_{B \rtimes_t \mathbb{G}}(E \rtimes_t \mathbb{G})$. Then $(A \rtimes_t \mathbb{G}, (E \rtimes_t \mathbb{G})_{B \rtimes_t \mathbb{G}}, \iota(D))$ is an order-$\frac{1}{1-\alpha}$ cycle.*

If, for a dense $*$-subalgebra $\mathscr{A} \subseteq A$, $(\mathscr{A}, E_B, D)$ is a uniformly $\mathbb{G}$-equivariant order-$\frac{1}{1-\alpha}$ cycle, the data

$$\Big(\operatorname{span}\{(1 \otimes \omega)((\iota(a)^* \otimes s^*)X) \mid a \in \mathscr{A}, s \in \mathscr{S}_a, \omega \in L^1(\mathbb{G})\}, (E \rtimes_t \mathbb{G})_{B \rtimes_t \mathbb{G}}, \iota(D)\Big)$$

defines an order-$\frac{1}{1-\alpha}$ cycle, where X is a unitary on $(E \rtimes_t \mathbb{G}) \otimes C_0^r(\mathbb{G})$ described in the proof.

Proof. Note that the image of the representation of $A \rtimes_t \mathbb{G}$ is $\overline{\operatorname{span}}(\iota(A) C_t^*(\mathbb{G})) \subseteq \operatorname{End}^*(E \rtimes_t \mathbb{G})$. Using the identification $\operatorname{End}^0_B(E) \rtimes_t \mathbb{G} \cong \operatorname{End}^0_{B \rtimes_t \mathbb{G}}(E \rtimes_t \mathbb{G})$, we see that, for $a \in A$ and $f \in C_t^*(\mathbb{G})$,

$$(1 + \iota(D)^2)^{-1/2}(\iota(a) f) = \iota((1 + D^2)^{-1/2} a) f$$

is compact, cf. [Ver02, Démonstration du Proposition 5.3]. By the universality of the crossed product [Ver02, §4.1] [Vae05, §2.3], the morphism $\operatorname{End}^0(E) \rtimes_u \mathbb{G} \to \operatorname{End}^0(E) \rtimes_t \mathbb{G}$ gives rise to the morphism

$\iota : \mathrm{End}^0(E) \to M(\mathrm{End}^0(E) \rtimes_t \mathbb{G}) \cong \mathrm{End}^*(E \rtimes_t \mathbb{G})$ and a unitary $X \in M((\mathrm{End}^0(E) \rtimes_t \mathbb{G}) \otimes C_0^r(\mathbb{G})) \cong \mathrm{End}^*((\mathrm{End}^0(E) \rtimes_t \mathbb{G}) \otimes C_0^r(\mathbb{G}))$ such that

$$X(\iota(T) \otimes 1)X^* = (\iota \otimes \mathrm{id})(V_E(T \otimes_{\delta_B} 1)V_E^*)$$

for $T \in \mathrm{End}^0(E)$. Let $a \in \mathcal{Q}$ and $s \in \mathcal{S}_a$. For $X^*(\iota(a) \otimes s_1) \in \mathrm{End}^*((E \rtimes_t \mathbb{G}) \otimes C_0^r(\mathbb{G})$,

$$\begin{aligned}
&[\iota(D) \otimes 1, X^*(\iota(a) \otimes s)]\langle \iota(D)\rangle^{-\alpha} \\
&\quad = X^*\left(X(\iota(D) \otimes 1)X^*(\iota(a) \otimes s) - (\iota \otimes \mathrm{id})\left((a \otimes s)(D \otimes 1)\right)\right)\langle \iota(D)\rangle^{-\alpha} \\
&\quad = X^*(\iota \otimes \mathrm{id})\left(\left(V_E(D \otimes_{\delta_B} 1)V_E^*(a \otimes s) - (a \otimes s)(D \otimes 1)\right)(\langle D\otimes\rangle^{-\alpha}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\langle \iota(D)\rangle^{-\alpha}[\iota(D) \otimes 1, X^*(\iota(a) \otimes s)] \\
&\quad = X^*X\langle \iota(D)\rangle^{-\alpha}X^*\left(X(\iota(D) \otimes 1)X^*(\iota(a) \otimes s) - (\iota \otimes \mathrm{id})\left((a \otimes s)(D \otimes 1)\right)\langle \iota(D)\rangle^{-\alpha}\right) \\
&\quad = X^*(\iota \otimes \mathrm{id})\left(V_E\langle D \otimes_{\delta_B} 1\rangle^{-\alpha}V_E^*\left(V_E(D \otimes_{\delta_B} 1)V_E^*(a \otimes s) - (a \otimes s)(D \otimes 1)\right)\right)
\end{aligned}$$

are adjointable. The representation of $A \rtimes_t \mathbb{G}$ on $E \rtimes_t \mathbb{G}$ consists of

$$\begin{aligned}
\overline{\mathrm{span}}(\iota(A)C_t^*(\mathbb{G})) &= \overline{\mathrm{span}}\left\{\iota(a)(1 \otimes \omega)(X) \,\middle|\, a \in A, \omega \in L^1(\mathbb{G})\right\} \\
&= \overline{\mathrm{span}}\left\{\iota(a)(1 \otimes \eta_1^*)X(1 \otimes \eta_2^*) \,\middle|\, a \in A, \eta_1, \eta_2 \in L^2(\mathbb{G})\right\} \\
&= \overline{\mathrm{span}}\left\{(1 \otimes \eta_1^*)(\iota(a)^* \otimes s^*)X(1 \otimes \eta_2^*) \,\middle|\, a \in A, s \in C_0^r(\mathbb{G}), \eta_1, \eta_2 \in L^2(\mathbb{G})\right\} \\
&\subseteq \overline{\mathrm{span}}\left\{(1 \otimes \eta_1^*)(\iota(a)^* \otimes s^*)X(1 \otimes \eta_2^*) \,\middle|\, a \in \mathcal{Q}, s \in \mathcal{S}_a, \eta_1, \eta_2 \in L^2(\mathbb{G})\right\}
\end{aligned}$$

by the density of $\mathcal{S}_a^* \subseteq C_0^r(\mathbb{G})$ and the inclusion $A \subseteq \overline{\mathcal{Q}}$. □

We also have a realisation of the dual Green–Julg map on uniformly equivariant unbounded Kasparov modules.

Proposition I.3.19. *Let (A, E_B, D) be a uniformly $\mathbb{G}$-equivariant order-$\frac{1}{1-\alpha}$ cycle, with $\mathbb{G}$ acting trivially on B. Then $(A \rtimes_u \mathbb{G}, E_B, D)$ is an order-$\frac{1}{1-\alpha}$ cycle, with the integrated representation of $A \rtimes_u \mathbb{G}$.*

If, for a dense $*$-subalgebra $\mathcal{A} \subseteq A$, $(\mathcal{A}, E_B, D)$ is a uniformly $\mathbb{G}$-equivariant unbounded Kasparov module, with G acting trivially on B, then

$$\left(\mathrm{span}\{(1 \otimes \omega)((a^* \otimes s^*)V_E) \,|\, a \in \mathcal{A}, s \in \mathcal{S}_a, \omega \in L^1(\mathbb{G})\}, E_B, D\right)$$

is an order-$\frac{1}{1-\alpha}$ cycle.

Proof. The only point which is not immediate is the boundedness of commutators with D. Let $a \in \mathcal{Q}$ and $s \in \mathcal{S}_a$ and let $\omega \in L^1(\mathbb{G})$, so that

$$(1 \otimes \omega)((a^* \otimes s^*)V_E)$$

is in the integrated representation of $A \rtimes_u \mathbb{G}$ on E. By the uniform equivariance condition,

$$[D, (1 \otimes \omega)((a^* \otimes s^*)V_E)] = (1 \otimes \omega)\left(\left(V_E(D \otimes 1)V_E^*(a \otimes s) - (a \otimes s)(D \otimes 1)\right)^* V_E\right)$$

and so $(1 \otimes \omega)((a^* \otimes s^*)V_E) \in \mathrm{Lip}_\alpha^*(D)$. The representation of $A \rtimes_t \mathbb{G}$ on $E \rtimes_t \mathbb{G}$ consists of

$$\begin{aligned}
\overline{\mathrm{span}}(AC_u^*(\mathbb{G})) &= \overline{\mathrm{span}}\left\{a(1 \otimes \omega)(V_E) \,\middle|\, a \in A, \omega \in L^1(\mathbb{G})\right\} \\
&= \overline{\mathrm{span}}\left\{a(1 \otimes \eta_1^*)V_E(1 \otimes \eta_2^*) \,\middle|\, a \in A, \eta_1, \eta_2 \in L^2(\mathbb{G})\right\} \\
&= \overline{\mathrm{span}}\left\{(1 \otimes \eta_1^*)(a^* \otimes s^*)V_E(1 \otimes \eta_2^*) \,\middle|\, a \in A, s \in C_0^r(\mathbb{G}), \eta_1, \eta_2 \in L^2(\mathbb{G})\right\} \\
&\subseteq \overline{\mathrm{span}}\left\{(1 \otimes \eta_1^*)(a^* \otimes s^*)V_E(1 \otimes \eta_2^*) \,\middle|\, a \in \mathcal{Q}, s \in \mathcal{S}_a, \eta_1, \eta_2 \in L^2(\mathbb{G})\right\}
\end{aligned}$$

by the density of $\mathcal{S}_a L^2(\mathbb{G}) \subseteq C_0^r(\mathbb{G})L^2(\mathbb{G}) \subseteq L^2(\mathbb{G})$ and the inclusion $A \subseteq \overline{\mathcal{Q}}$. □

For the inverse map, more structure is required, including the presence of a dense subalgebra $\mathscr{A}$ of A. A discrete quantum group $\mathbb{G}$ has a compact dual, whose polynomial algebra we denote by $\mathcal{O}(\hat{\mathbb{G}})$. We write $\mathscr{A} \rtimes \mathbb{G}$ for the subalgebra of $A \rtimes_u \mathbb{G}$ generated by $\mathscr{A}$ and $\mathcal{O}(\hat{\mathbb{G}})$.

Proposition I.3.20. *Let $(\mathscr{A} \rtimes \mathbb{G}, E_B, D)$ be an order-$\frac{1}{1-\alpha}$ cycle, with $\mathbb{G}$ a discrete quantum group and the representation of $\mathscr{A} \rtimes \mathbb{G}$ on E nondegenerate. Then $(\mathscr{A}, E_B, D)$ is a uniformly $\mathbb{G}$-equivariant order-$\frac{1}{1-\alpha}$ cycle, with the $\mathbb{G}$-action on E given by Lemma I.3.13 and trivial on B.*

Proof. Because $\mathbb{G}$ is discrete, $\mathscr{A}$ is included in $\mathscr{A} \rtimes \mathbb{G}$. Hence $(1 + D^2)^{-1}a$ is compact and $[D, a]$ is bounded for all $a \in \mathscr{A}$. The inclusion $C^*_u(\mathbb{G}) \subseteq M(A \rtimes_u \mathbb{G})$ gives a (nondegenerate) representation π of $C^*_u(\mathbb{G})$ on E. Because $\mathbb{G}$ is discrete, it is regular. Applying Lemma I.3.13, we obtain an action of $\mathbb{G}$ on E, acting trivially on B. Let V_E be the admissible unitary. Discreteness means that $C_0(\mathbb{G})$ is isomorphic as an algebra to the C*-algebraic direct sum

$$\bigoplus_{\lambda \in \Lambda} M_{n_\lambda}(\mathbb{C})$$

of finite-dimensional matrix algebras. The admissible unitary is the direct sum over the index set $\lambda \in \Lambda$ of

$$V_E^\lambda \in \pi(\mathcal{O}(\hat{\mathbb{G}})) \otimes M_{n_\lambda}(\mathbb{C}) \subseteq \pi(C^*_u(\mathbb{G})) \otimes M_{n_\lambda}(\mathbb{C}) \subseteq \operatorname{End}^*_B(E \otimes \mathbb{C}^{n_\lambda}),$$

cf. [VY20, §4.2.3] for the inclusion in the polynomial subalgebra. Then, for $a \in \mathscr{A}$,

$$\Big(V_E^\lambda (D \otimes 1) V_E^{\lambda *}(a \otimes 1) - (a \otimes 1)(D \otimes 1)\Big)\langle D \otimes 1\rangle^{-\alpha} = V_E^\lambda \left[D \otimes 1, V_E^{\lambda *}(a \otimes 1)\right] \langle D \otimes 1\rangle^{-\alpha}$$

and

$$V_E^\lambda \langle D \otimes 1\rangle^{-\alpha} V_E^{\lambda *}\Big(V_E^\lambda (D \otimes 1) V_E^{\lambda *}(a \otimes 1) - (a \otimes 1)(D \otimes 1)\Big) = V_E^\lambda \langle D \otimes 1\rangle^{-\alpha}\left[D \otimes 1, V_E^{\lambda *}(a \otimes 1)\right]$$

are bounded for all $\lambda \in \Lambda$, because $V_E^{\lambda *}(a \otimes 1) \in \pi(\mathcal{O}(\hat{\mathbb{G}}))\mathscr{A} \otimes M_{n_\lambda}(\mathbb{C}) \subseteq (\mathscr{A} \rtimes \mathbb{G}) \otimes M_{n_\lambda}(\mathbb{C})$. ☐

Remark I.3.21. It is clear that the bounded transform $(A \rtimes_t \mathbb{G}, (E \rtimes_t \mathbb{G})_{B \rtimes_t \mathbb{G}}, F_{\iota(D)} = \iota(F_D))$ of the descent $(A \rtimes_t \mathbb{G}, (E \rtimes_t \mathbb{G})_{B \rtimes_t \mathbb{G}}, \iota(D))$ of a uniformly $\mathbb{G}$-equivariant cycle (A, E_B, D) is exactly the descent of the bounded transform (A, E_B, F_D). The same is true for the dual Green–Julg map.

I.4 The Kasparov product

Let A, B, and C be (complex) C*-algebras with A separable. The *internal Kasparov product* is a $\mathbb{Z}$-bilinear pairing

$$KK_i(A, B) \times KK_j(B, C) \to KK_{i+j}(A, C) \qquad (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \otimes_B \mathbf{y},$$

for $i, j \in \mathbb{Z}/2\mathbb{Z}$. It has a number of nice properties: it is contravariant in A, covariant in C, and associative, in the sense that, if B is separable and D is a C*-algebra,

$$(\mathbf{x} \otimes_B \mathbf{y}) \otimes_C \mathbf{z} = \mathbf{x} \otimes_B (\mathbf{y} \otimes_C \mathbf{z}) \in KK_{i+j+k}(A, D)$$

for all $\mathbf{z} \in KK_k(C, D)$. The pairing also exists, and has these same properties, for G-equivariant KK-theory, for G a σ-compact locally compact group [Kas88, Theorems 2.11,14], and for S-equivariant KK-theory, for S a σ-unital C*-bialgebra [BS89, Théorèmes 5.3,5].

Example I.4.1. [Kas88, Definition 2.15] [BS89, Remarque 5.11(1)] Let G be a σ-compact locally compact group. The *Kasparov representation ring* $R(G)$ is defined to be the graded abelian group $KK^G_*(\mathbb{C}, \mathbb{C})$ with multiplication given by the internal product. Similarly, for S a σ-unital C*-bialgebra and $\mathbb{G}$ a σ-compact locally compact quantum group, the Kasparov representation rings $R(S)$ and

$R(\mathbb{G})$ are defined to be the graded abelian groups $KK_*^S(\mathbb{C},\mathbb{C})$ and $KK_*^{\mathbb{G}}(\mathbb{C},\mathbb{C})$, respectively, with multiplication again given by the internal product in each case. Each of $R(G)$, $R(S)$, and $R(\mathbb{G})$ is a $\mathbb{Z}/2\mathbb{Z}$-graded unital ring; $R(G)$ is commutative, while $R(S)$ is commutative only if S is commutative, and $R(\mathbb{G})$ is commutative only if $\mathbb{G}$ is a (classical) locally compact group. If G is a compact group, $R(G)$ agrees with the usual representation ring. If G is discrete, $R(\hat{G})$ is isomorphic to the group ring $\mathbb{Z}[G]$.

Many subsequent results in this thesis will involve taking the Kasparov product of (usually unbounded) Kasparov modules. In the interests of economy, we will not give separate statements for different combinations of parities. Instead, we will abuse the symbol $\tilde{\otimes}$ to represent a flexible tensor product of possibly $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert modules and operators thereon. Let E_1 be a Hilbert B-module and E_2 a B-C-correspondence. Let T be a regular (or, as a special case, adjointable) operator on E_1.

- In the case when E_1 and E_2 are both graded, $\tilde{\otimes}_B$ will simply mean the graded tensor product.
- In the case when E_1 is graded and E_2 ungraded, $E_1 \tilde{\otimes}_B E_2$ will refer to the plain tensor product, giving an ungraded Hilbert C-module, and we will write $T \tilde{\otimes} 1 := T \otimes 1$.
- In the case when E_1 is ungraded and E_2 graded, $E_1 \tilde{\otimes}_B E_2$ will again refer to the plain tensor product, but this time we will write $T\tilde{\otimes}1 = T\otimes 1$ if $T\tilde{\otimes}1$ is understood to be even or $T\tilde{\otimes}1 := T\otimes\gamma_2$ if $T \tilde{\otimes} 1$ is understood to be odd, where γ_2 is the grading on H_2.
- If both E_1 and E_2 are ungraded, we will let $E_1 \tilde{\otimes}_B E_2 = E_1 \otimes_B E_2 \otimes_{\mathbb{C}} \mathbb{C}^2$, with a grading given by $1 \otimes 1 \otimes \sigma_3$, and write $T\tilde{\otimes} 1 = D \otimes 1 \otimes 1$ if $T\tilde{\otimes} 1$ is understood to be even or $T\tilde{\otimes} 1 = D \otimes 1 \otimes \sigma_1$ if $T \tilde{\otimes} 1$ is understood to be odd, where σ_1, σ_2, and σ_3 are the Pauli matrices.

Let E_1 be a Hilbert B-module and E_2 a Hilbert C-module. We will write $E_1 \tilde{\otimes}_{\mathbb{C}} E_2$ for the external tensor product, with the same parity adjustments as above. For S a regular (or, as a special case, adjointable) operator on E_2,

- In the case when E_1 is graded and E_2 ungraded, we will write $1 \tilde{\otimes} S = 1 \otimes S$ if $1 \tilde{\otimes} S$ is understood to be even or $1 \tilde{\otimes} S = \gamma_1 \otimes S$ if $1 \tilde{\otimes} S$ is understood to be odd, where γ_1 is the grading on H_1.
- In the case when E_1 is ungraded and E_2 graded, we will write $1 \tilde{\otimes} S = 1 \otimes S$.
- If both E_1 and E_2 are ungraded, we will write $1 \tilde{\otimes} S = 1 \otimes S \otimes 1$ if $1 \tilde{\otimes} S$ is understood to be even or $1 \tilde{\otimes} S = 1 \otimes S \otimes \sigma_2$ if $1 \tilde{\otimes} S$ is understood to be odd, where σ_1, σ_2, and σ_3 are the Pauli matrices.

As all our C*-algebras are assumed to be ungraded, this abuse of notation should hopefully cause no confusion. As an alternative, one could use the machinery of *multigradings* [HR00, §A.3]. That Theorem I.4.2 (and so also Theorem I.4.2) is valid for each combination of parities is well known; see [HR00, Proposition 9.2.5, Exercises 9.8.1–2].

Theorem I.4.2. *[CS84, Definition A.1, Theorem A.5] [Kas88, Definition 2.10, Theorem 2.11] [BS89, Définition 5.2, Théorème 5.3] Let A, B, and C be C*-algebras with A separable. Let $(A, E_{1,B}, F_1)$, $(B, E_{2,C}, F_2)$, and $(A, E_1 \tilde{\otimes}_B E_{2,C}, F)$ be bounded Kasparov modules. If*

1. *For all $\xi \in E_1$, with $T_\xi \in \mathrm{Hom}_C^*(E_2, E_1 \tilde{\otimes}_B E_2)$ given by $\eta \to \xi \tilde{\otimes} \eta$,*

$$\left[\begin{pmatrix} F & \\ & F_2 \end{pmatrix}, \begin{pmatrix} 0 & T_\xi \\ T_\xi^* & \end{pmatrix} \right]$$

   is a compact operator on $(E_1 \tilde{\otimes}_B E_2) \oplus E_2$; and

2. *There exists* $0 \leq \kappa < 2$ *such that*

$$a\Big(F(F_1 \mathbin{\tilde{\otimes}} 1) + (F_1 \mathbin{\tilde{\otimes}} 1)F\Big)a^* \geq -\kappa a a^*$$

modulo $\operatorname{End}^0(E_1 \mathbin{\tilde{\otimes}}_B E_2)$

then $(A, E_1 \mathbin{\tilde{\otimes}}_B E_{2,C}, F)$ *represents the Kasparov product*

$$[(A, E_{1,B}, F_1)] \otimes_B [(B, E_{2,C}, F_2)].$$

If $(A, E_{1,B}, F_1)$, $(B, E_{2,C}, F_2)$, *and* $(A, E_1 \mathbin{\tilde{\otimes}}_B E_{2,C}, F)$ *are* G*-equivariant for* G *a* σ*-compact locally compact group, the latter represents the* G*-equivariant Kasparov product of the other two. Similarly, if the three bounded Kasparov modules are* S*-equivariant for* S *a* σ*-unital C*-bialgebra,* $(A, E_1 \mathbin{\tilde{\otimes}}_B E_{2,C}, F)$ *represents the* S*-equivariant product of the other two. Moreover, whether in equivariant KK-theory or not, a module* $(A, E_1 \mathbin{\tilde{\otimes}}_B E_{2,C}, F)$ *satisfying the above conditions can always be found and is unique up to homotopy.*

The following is essentially the state of the art for conditions for the unbounded Kasparov product, without assuming the existence of an approximate unit as in [Dun22]. The statements about the product in equivariant KK-theory follow immediately from the above; cf. [Kuc94, Theorem 8.12].

Theorem I.4.3. *cf. [Kuc94, Theorem 8.12], [Kuc97, Theorem 13], [GM15, Theorem A.7], [Dun22, Definition 3.2, Theorem 3.3] Let* A*,* B*, and* C *be C*-algebras with* A *separable. Let* $(A, E_{1,B}, D_1)$*,* $(B, E_{2,C}, D_2)$*, and* $(A, E_1 \mathbin{\tilde{\otimes}}_B E_{2,C}, D)$ *be order-*$\frac{1}{1-\alpha}$ *cycles. If*

1. *For all* ξ *in a dense subspace of* E_1*, with* $T_\xi \in \operatorname{Hom}^*_C(E_2, E_1 \mathbin{\tilde{\otimes}}_B E_2)$ *given by* $\eta \to \xi \mathbin{\tilde{\otimes}} \eta$*,*

$$\begin{pmatrix} 0 & T_\xi \\ T_\xi^* & \end{pmatrix} \begin{pmatrix} \operatorname{dom} D \\ \operatorname{dom} D_2 \end{pmatrix} \subseteq \begin{pmatrix} \operatorname{dom} D \\ \operatorname{dom} D_2 \end{pmatrix}$$

and

$$\left[\begin{pmatrix} D & \\ & D_2 \end{pmatrix}, \begin{pmatrix} 0 & T_\xi \\ T_\xi^* & \end{pmatrix} \right] \begin{pmatrix} \langle D \rangle^{-\alpha} & \\ & \langle D_2 \rangle^{-\alpha} \end{pmatrix}$$

extends to an adjointable operator on $(E_1 \mathbin{\tilde{\otimes}}_B E_2) \oplus E_2$*; and*

2. *We have* $\operatorname{dom} D \subseteq \operatorname{dom}(D_1 \mathbin{\tilde{\otimes}} 1)$ *and there exists* $\lambda \geq 0$ *such that*

$$\langle (D_1 \mathbin{\tilde{\otimes}} 1)\psi \mid D\psi \rangle + \langle D\psi \mid (D_1 \mathbin{\tilde{\otimes}} 1)\psi \rangle \geq -\lambda \langle \psi \mid \langle D \rangle \psi \rangle$$

then $(A, E_1 \mathbin{\tilde{\otimes}}_B E_{2,C}, D)$ *represents the Kasparov product*

$$[(A, E_{1,B}, D_1)] \otimes_B [(B, E_{2,C}, D_2)].$$

If $(A, E_{1,B}, D_1)$, $(B, E_{2,C}, D_2)$, *and* $(A, E_1 \mathbin{\tilde{\otimes}}_B E_{2,C}, D)$ *are uniformly* G*-equivariant for* G *a* σ*-compact locally compact group, the latter represents the* G*-equivariant Kasparov product of the other two. Similarly, if the three bounded Kasparov modules are uniformly* S*-equivariant for* S *a* σ*-unital C*-bialgebra,* $(A, E_1 \mathbin{\tilde{\otimes}}_B E_{2,C}, D)$ *represents the* S*-equivariant product of the other two.*

Conditions 1. and 2. in both Theorems I.4.2 and I.4.3 are referred to as the *connection* and *positivity* conditions, respectively. Both Theorems are sometimes referred to as ‘guess and check’ methods. The positivity condition of Theorem I.4.3 is, in some sense, not well adapted to higher order cycles. By considering classical differential operators it may be desirable and, indeed, by inspecting [Dun22, Proof of Lemma 3.9], may be possible to adapt the form bound to include a factor of $\langle S \rangle^\alpha$ on the right-hand side.

For any C*-algebra C, there is a homomorphism $\sigma_C : KK_*(A,B) \to KK_*(A \otimes C, B \otimes C)$ given by taking an bounded Kasparov module (A, E_B, F) to $(A \otimes C, (E \otimes_{\mathbb{C}} C)_{B\otimes C}, F \otimes 1)$. In the unbounded picture, σ_C takes an unbounded cycle (A, E_B, D) to $(A \otimes C, (E \otimes_{\mathbb{C}} C)_{B\otimes C}, D \otimes 1)$. If C is a G-C*-algebra for some locally compact group G, $\sigma_C : KK^G_*(A,B) \to KK^G_*(A \otimes C, B \otimes C)$ is defined in the same way, where $A \otimes C$ and $B \otimes C$ have the diagonal actions of G. If C is a S-C*-algebra for some commutative C*-bialgebra, $\sigma_C : KK^S_*(A,B) \to KK^S_*(A \otimes C, B \otimes C)$ is again defined in the same way [Ver02, §3.3]. A C*-algebra A permits a nondegenerate $*$-homomorphism $\phi : A \otimes A \to M(A)$ such that $\phi(a \otimes 1) = a = \phi(1 \otimes a)$ if and only if A is commutative [Wor80, §2]. The commutativity of S ensures that there is a nondegenerate $*$-homomorphism $S \otimes S \to M(S)$, Gelfand dual to the diagonal embedding, which allows one to define a coaction of S on $A \otimes C$ and $B \otimes C$. It is perhaps easier to think of this in terms of the Gelfand dual of S, which is a locally compact semigroup [Val85] [BS89, Exemple 1.4(3)].

Let A_1, A_2, B_1, and B_2 be (complex) C*-algebras with A_1 and A_2 separable. The *external Kasparov product* is a $\mathbb{Z}$-bilinear pairing

$$KK_i(A_1, B_1) \times KK_j(A_2, B_2) \to KK_{i+j}(A_1 \otimes A_2, B_1 \otimes B_2) \qquad (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \otimes_{\mathbb{C}} \mathbf{y},$$

for $i, j \in \mathbb{Z}/2\mathbb{Z}$. It is defined in terms of the internal product, by

$$\mathbf{x} \otimes_{\mathbb{C}} \mathbf{y} = \sigma_{A_2}(\mathbf{x}) \otimes_{B_1 \otimes A_2} \sigma_{B_1}(\mathbf{y}).$$

As a consequence, it too has a number of desirable properties: it is contravariant in A_1 and A_2, covariant in B_1 and B_2, and associative. Furthermore, it is commutative, that is, $\mathbf{x} \otimes_{\mathbb{C}} \mathbf{y} = \mathbf{y} \otimes_{\mathbb{C}} \mathbf{x}$. The pairing also exists, and has these same properties, for G-equivariant KK-theory, for G a σ-compact locally compact group [Kas88, Theorems 2.11,14], and for S-equivariant KK-theory, for S a σ-unital commutative C*-bialgebra [Ver02, §3.3]. We should also mention here that the external product has been generalised to the setting of locally compact quantum groups by [NV10], using the machinery of the Drinfeld double and braided tensor products.

A key feature of unbounded KK-theory is the fact that the external product becomes completely constructive. Extending [BJ83, §3] to our setting, the external product of order-$\frac{1}{1-\alpha}$ cycles (A_1, E_{1,B_1}, D_1) and (A_2, E_{2,B_2}, D_2) is the order-$\frac{1}{1-\alpha}$ cycle

$$(A_1 \otimes A_2, (E_1 \tilde{\otimes}_{\mathbb{C}} E_2)_{B_1 \otimes B_2}, D_1 \tilde{\otimes} 1 + 1 \tilde{\otimes} D_2).$$

For G a locally compact group or S a commutative C*-bialgebra, the external product of G-equivariant or S-equivariant higher order cycles is defined in the same way.

In many examples, the internal product of order-$\frac{1}{1-\alpha}$ cycles $(A, E_{1,B}, D_1)$ and $(B, E_{2,C}, D_2)$ can be represented by an order-$\frac{1}{1-\alpha}$ cycle of the form

$$(A, (E_1 \otimes_B E_2)_C, D_1 \tilde{\otimes} 1 + 1 \tilde{\otimes}_\nabla D_2).$$

Here, one has to make sense of the operator $1 \tilde{\otimes}_\nabla D_2$, using the data of a *connection*; see [Mes12]. Further, the combined operator $D_1 \tilde{\otimes} 1 + 1 \tilde{\otimes}_\nabla D_2$ has to be shown to be self-adjoint, have locally compact resolvent, and satisfy the positivity condition; the current state of the art for guaranteeing these is by checking that $D_1 \tilde{\otimes} 1$ and $1 \tilde{\otimes}_\nabla D_2$ *weakly anticommute*; see [LM19]. Finally, one has to check that $1 \tilde{\otimes}_\nabla D_2$ has (relatively) bounded commutators with the algebra, which is the most fragile part of the procedure, being most likely to fail. Together, these techniques are known as the *constructive* unbounded Kasparov product, the eventual hope being to make it truly constructive under reasonable assumptions. This project has been pursued by a number of authors, most notably Mesland; see [Mes12, KL13, MR16, LM19, Dun20] and the survey [Mes24].

Chapter II

Noncommutative-geometric group theory

In this Chapter we employ tools from geometric group theory to study the geometry of group C*-algebras and Fell bundles. We construct spectral triples for group C*-algebras from matrix-valued weights. Our innovation, on the one hand, consists in extending the theory to locally compact groups and in working naturally with Fell bundles; on the other hand, it lies in exhibiting the nontriviality of the resulting KK-classes using carefully chosen "directed length" functions from CAT(0) spaces. We also study the geometry of group extensions using the unbounded Kasparov product.

II.1 KK-theory of group algebras

In this section, we make a study of the KK-groups

$$KK^G_*(A, C_0(G, B)) \qquad KK^{\hat{G}}_*(A \rtimes_r G, B) \qquad KK_*(A \rtimes_r G, B)$$

for G a locally compact group and G-C*-algebras A and B. The commuting diagram

$$\begin{tikzcd} KK_*^G(A, C_0(G,B)) \arrow[r, "J^G", "\sim"'] \arrow[dr, "j_r^G"'] & KK_*^{\hat{G}}(A \rtimes_r G, C_0(G,B) \rtimes_r G) \arrow[r, "\otimes[L^2(G,B)]", "\sim"'] \arrow[d, "r^{\hat{G},1}"] & KK_*^{\hat{G}}(A \rtimes_r G, B) \arrow[d, "r^{\hat{G},1}"] \\ & KK_*(A \rtimes_r G, C_0(G,B) \rtimes_r G) \arrow[r, "\otimes[L^2(G,B)]", "\sim"'] & KK_*(A \rtimes_r G, B) \end{tikzcd}$$

summarises their relationship. By the external Kasparov product, the KK-group $KK^G(A, C_0(G,A))$ is a left module over the unital ring $R(G) := KK^G(\mathbb{C},\mathbb{C})$ and a right module over the unital ring $KK^G(C_0(G), C_0(G))$. By Baaj–Skandalis duality, $KK^G(C_0(G), C_0(G))$ is isomorphic to the Kasparov representation ring $R(\hat{G}) := KK^{\hat{G}}(\mathbb{C},\mathbb{C})$. Recall that, if G is discrete, $R(\hat{G})$ is isomorphic to the group ring $\mathbb{Z}[G]$. This right module structure naturally accounts for the action of G on $KK^G(A, C_0(G,B))$ by the automorphism of right translation on $C_0(G,B)$. As a consequence, the restriction map

$$r_{\hat{G},1} : KK_*^{\hat{G}}(A \rtimes_r G, B) \to KK_*(A \rtimes_r G, B)$$

factors through $KK_*^{\hat{G}}(A \rtimes_r G, B) \otimes_{R(\hat{G})} \mathbb{Z}$.

We refer to §A.2 for details of proper actions, our conventions for crossed products, and the definition of a cut-off function, among other things. Throughout this Chapter, we take all groups to be σ-compact unless otherwise mentioned.

II.1.1 The Dirac, dual Dirac, and γ-elements

Let X be a Riemannian manifold, with perhaps infinitely many connected components. We will make the simplifying assumption that all the components of X are of the same dimension n. By $C_\tau(X)$ we denote the algebra of sections, vanishing at infinity, of the (complex) Clifford bundle $\mathscr{C\ell}(T^*X)$ of the cotangent space of X [Kas88, Definition 4.1]. Beware that $C_\tau(X)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded C*-algebra. An isometric action on X by a group G pulls back to an action on the bundle $\mathscr{C\ell}(T^*X)$. With the action of $\mathscr{C\ell}(T^*X)$ on differential forms by Clifford multiplication, we obtain an isometrically G-equivariant even spectral triple

$$(C_\tau(X), L^2(\Omega^*X), d + d^*) \tag{II.1.1}$$

for the Hodge–de Rham Dirac operator $d + d^*$, representing an element of $KK_0^G(C_\tau(X), \mathbb{C})$. For our purposes, it will be preferable to replace the Clifford bundle algebra $C_\tau(X)$ with $C_0(X)$. Doing this is contingent on the existence and choice of a spin^c structure on X, preserved by the action of G. One standard reference on spin^c structures is [LM89, Appendix D]. We will give a brief summary.

The group $\mathrm{Spin}^c(n)$ is a central extension of $SO(n)$ by $\mathbb{T}$. The Riemannian structure implies a reduction of the structure group of the frame bundle of X along $O(n) \hookrightarrow GL(n)$. A spin^c structure on X is a further reduction of the structure group of the frame bundle of X along $\mathrm{Spin}^c(n) \to O(n)$. Since the image of $\mathrm{Spin}^c(n) \to O(n)$ is $SO(n)$, a spin^c structure includes a choice of orientation. We will think of a spin^c structure as a more elaborate kind of 'orientation'; indeed, it is sometimes called a K-orientation. Another way of expressing this, more in tune with the machinery of KK-theory, can be found in [Ply86, §2]. A spin^c structure can be described as the data of an orientation ε on X together with a Morita equivalence bimodule $\mathscr{E}$ [Ply86, Definition 2.2]. If n is even we require that $\mathscr{E}$ be a $\mathbb{Z}/2\mathbb{Z}$-graded Morita equivalence bimodule between $C_\tau(X)$ and $C_0(X)$. If n is odd we require that $\mathscr{E}$ be a Morita equivalence bimodule between $C_\tau(X)^{\mathrm{ev}}$ and $C_0(X)$. If n is odd, $\mathscr{E}$ can be given the structure of a left $C_\tau(X)$-module by using the unit pseudoscalar [Ply86, §2.6]. The *fundamental spinor bundle* $\not{S}$ is the complex Hermitian vector bundle determined by $\Gamma_0(X, \not{S}) \cong \mathscr{E}$. The fibres of $\not{S}$ are irreducible representations of $\mathscr{C\ell}_n$ and it is $\mathbb{Z}/2\mathbb{Z}$-graded if and only if n is even. When we compose the cycle of (II.1.1) with the dual of $\Gamma_0(X, \not{S})$, we obtain an isometrically G-equivariant spectral triple

$$(C_0(X), L^2(X, \not{S}), \not{D}),$$

where $\not{S}$ is the Atiyah–Singer Dirac operator, representing a class $\alpha \in KK_n^G(C_0(X), \mathbb{C})$ (where, as usual, n is taken modulo 2).

We will later want to allow for group actions which are not orientation preserving, for instance, a group of reflections of $\mathbb{R}^n$. Just as $\mathrm{Spin}^c(n)$ is a central extension of $SO(n)$ by $\mathbb{T}$, there is a central extension of $O(n)$ by $\mathbb{T}$ called $\mathrm{Pin}^c(n)$. A pin^c structure is exactly what is left over when one removes the data of an orientation from a spin^c structure. Following [Ply86, §2.7], given a spin^c structure $(\varepsilon, \mathscr{E})$, the reversed spin^c structure is $(-\varepsilon, \mathscr{E}^{(\mathrm{op})})$, where $-\varepsilon$ is the reversed orientation and $\mathscr{E}^{(\mathrm{op})}$ is $\mathscr{E}$ with, in the case of n even, the opposite grading. We will say that a group G acts on X by pin^c isometries if it acts by spin^c-structure preserving and spin^c-structure reversing isometries. Because the composition of two spin^c-structure reversing isometries is spin^c-structure preserving, the subgroup of G acting by spin^c-structure preserving isometries is of index 2.

In [Kas88, Definition 5.1], Kasparov uses the term *special* manifold, repurposed from [Ree83], for a Riemannian manifold with a suitable *dual Dirac* element of KK-theory. A spin^c Riemannian n-manifold X, on which a locally compact group G acts isometrically and by spin^c-automorphisms, is special if there exists an element $\beta \in KK_n^G(\mathbb{C}, C_0(X))$ satisfying

$$\alpha \otimes_{\mathbb{C}} \beta = 1 \in KK_0^G(C_0(X), C_0(X))$$

where $\alpha \in KK_0^G(C_0(X), \mathbb{C})$ is the Atiyah–Singer Dirac element, as above. The element $\gamma := \beta \otimes_{C_0(X)} \alpha \in KK^G(\mathbb{C}, \mathbb{C})$ is automatically an idempotent; we shall shortly return to its significance.

The motivating example of a special manifold is a simply connected manifold of non-positive sectional curvature [Kas88, §5.3]. Let X be a simply connected spin^c Riemannian n-manifold of non-positive sectional curvature on which a locally compact group G acts by spin^c isometries. Fix $x_0 \in X$ and let $\rho : X \to [0, \infty)$ be given by $\rho(x) = d(x_0, x)$. The dual Dirac element β is represented by the uniformly G-equivariant unbounded Kasparov module

$$(\mathbb{C}, \Gamma_0(X, \not{S})_{C_0(X)}, \rho d\rho)$$

where $\rho d\rho \in \Omega^1 X$ acts on $\Gamma_c(X, \not{S})$ by Clifford multiplication [Kas95, Definition 5.4] [Kuc94, Chapter 8]. An early study of the properties of $\rho d\rho$ as an unbounded operator is in [Luk77].

In fact, for any almost connected group G, the quotient $X = G/K$ by the maximal compact subgroup K is a special manifold [Kas88, Theorem 5.7] (modulo a spin^c caveat). Kasparov shows this by an inductive construction, using the derived series of the Lie group G/N, where N is a compact normal subgroup of G. However, although the ingredients can be assembled as unbounded Kasparov modules, the final construction involves a troublesome Kasparov product. We will return to the issue of Kasparov products for group extensions in §II.4. In the case of an almost connected group G, Kasparov shows that the element $\gamma \in KK^G(\mathbb{C}, \mathbb{C})$ is independent of the choice of special manifold, and is so denoted $\gamma_{(G)}$.

Let A and B be G-C*-algebras, with A separable. By [Kas88, Corollary 5.7], if G is almost connected with maximal compact subgroup K, the restriction map

$$KK_*^G(A, B) \to KK_*^K(A, B)$$

is surjective with kernel $(1 - \gamma_{(G)}) KK_*^G(A, B)$. That is,

$$\gamma_{(G)} KK_*^G(A, B) \cong KK_*^K(A, B),$$

where $\gamma_{(G)} \in KK^G(\mathbb{C}, \mathbb{C})$ acts on $KK_*^G(A, B)$ by the external Kasparov product. In other words, when γ acts on a KK-group, it has the effect of reducing the equivariance to the maximal compact subgroup. It is therefore of interest to understand for which almost connected groups G one has $\gamma_{(G)} = 1$, and so $KK_*^G(A, B) \cong KK_*^K(A, B)$. This question was answered conclusively by Julg and Kasparov [JK95, Theorem 8.2]: $\gamma_{(G)} = 1$ if and only if the quotient of G by its radical, that is, its maximal solvable connected normal closed subgroup, is locally isomorphic to a product of a compact group and a finite number of real and complex Lorentz groups. The fact that the real Lorentz groups $SO_0(n,1)$ and the complex Lorentz groups $SU(n,1)$ have $\gamma_{(G)} = 1$ was proved in [Kas84] and [JK95], respectively, in each case using a representative for the γ-element based on the action of the group on a sphere. We shall outline these constructions and lift them to unbounded KK-theory in §III.2.1, using the technology of conformal equivariance developed in Chapter III.

Example II.1.2. Again, let G be an almost connected group and let A and B be G-C*-algebras, with A separable. By [Kas88, Theorem 5.8], $\gamma_{(G)}$ always acts as the identity on $KK^G_*(A, C_0(G,B))$. Hence

$$KK^G_*(A, C_0(G,B)) = \gamma_{(G)} KK^G_*(A, C_0(G,B)) \cong KK^K_*(A, C_0(G,B)).$$

Futher, using also the Green–Julg isomorphism, we record that

$$\begin{aligned} KK^G_*(\mathbb{C}, C_0(G)) &\cong KK^K_*(\mathbb{C}, C_0(G)) \\ &\cong KK_*(\mathbb{C}, C_0(G) \rtimes K) \\ &\cong KK_*(\mathbb{C}, C_0(G/K)) \\ &\cong \begin{cases} \mathbb{Z} \oplus 0 & \dim G/K \equiv 0 \\ 0 \oplus \mathbb{Z} & \dim G/K \equiv 1 \end{cases} \pmod 2 \end{aligned}$$

since G/K is homeomorphic to $\mathbb{R}^{\dim G/K}$; see e.g. [CH16, Theorem 2.E.16]. As a ring, using also Baaj–Skandalis duality,

$$\begin{aligned} R(\hat{G}) &\cong KK^G_*(C_0(G), C_0(G)) \\ &= \gamma_{(G)} KK^G_*(C_0(G), C_0(G)) \\ &\cong KK^K_*(C_0(G), C_0(G)) \\ &\cong KK^K_*(C_0(K) \otimes C_0(\mathbb{R}^{\dim G/K}), C_0(K) \otimes C_0(\mathbb{R}^{\dim G/K})) \\ &\cong KK^K_*(C_0(K), C_0(K)) \\ &\cong R(\hat{K}). \end{aligned}$$

The idea of the γ-element can be applied to other than almost connected groups; a survey can be found in [AJV19, §§3.5, 4.4]. A γ-element for locally compact groups acting on trees was constructed by Julg and Valette [JV84]. These same authors made a generalisation to the case of reductive Lie group over a nonarchimedean local field acting on its Bruhat–Tits building in [JV87]; we return to discuss buildings in §II.3.3. In [KS91], Kasparov and Skandalis placed these constructions in context: Dirac and dual Dirac elements α and β are constructed for locally compact groups acting on Euclidean buildings with product $\beta \otimes \alpha$ the γ-element; see also [Jul89].

The method of Kasparov and Skandalis was formalised by Tu, who simultaneously generalised it to the setting of groupoid equivariant KK-theory [Tu00]. Recently, Nishikawa extended the idea of the γ-element to allow its identification without needing classes α and β [Nis19]. Such a generalised γ-element in $KK^G(\mathbb{C}, \mathbb{C})$ is called an element with *property (γ)*, and is unique when it exists, so includes the γ-elements already discussed. In [BGHN20], an element with property (γ) is constructed for groups acting properly on CAT(0) cube complexes; we will have more to say about these latter in §II.3.3. We record, in particular, that an element in $KK^G(\mathbb{C}, \mathbb{C})$ with property (γ) restricts to $1 \in KK^K(\mathbb{C}, \mathbb{C})$ for every compact subgroup K of G.

II.1.2 The Pimsner exact sequences for groups acting on trees

In [Pim86], six-term exact sequences are given for the K-theory and K-homology of crossed product C*-algebras by groups acting on trees. We shall give a brief outline. To set notation we make

Definition II.1.3. [Pim86, §1] cf. [Ser80, Definition 1] An *oriented graph* X is given by the data of

- sets X^0 and X^1 of vertices and edges, respectively;
- *origin* and *terminus* maps $o, t : X^1 \to X^0$.

An *oriented tree* is a connected oriented graph X with no cycles. We shall from now on refer to oriented trees simply as trees.

Fix a locally compact group G acting on a tree X and a separable G-C*-algebra A. Let Σ be the quotient graph of X by G. To the set of edges X^1, we adjoint an 'edge at infinity', which we denote ∞. For any vertex $P \in X^0$, we denote by X^1_P the set of edges whose terminus is closer than their origin to P, in other words, edges 'pointing toward' P. We denote by χ_P the characteristic function of $X^1_P \sqcup \{\infty\}$ on $X^1 \sqcup \{\infty\}$. We construct the C*-subalgebra $C_+(X^1)$ of $C_0(X^1 \sqcup \{\infty\})$ generated by $C_0(X^1)$ and χ_P for all $P \in X^0$. The C*-algebra $C_+(X^1, A) := C_+(X^1) \otimes A$ fits into the G-equivariant exact sequence

$$0 \longrightarrow C_0(X^1, A) \longrightarrow C_+(X^1, A) \longrightarrow A \longrightarrow 0 \ . \tag{II.1.4}$$

Fixing $P \in X^0$, $\rho : a \to \chi_P a \chi_P$ gives a completely positive cross-section. (Beware that ρ is not equivariant. By [Tho01, Theorem 1.1], the six term exact sequences in equivariant KK-theory of [BS89, Théorème 7.2] can nevertheless be constructed.) The extension class associated with the sequence (II.1.4) is

$$(A, C_0(X^1, A)_{C_0(X^1,A)}, 2\chi_P - 1) \tag{II.1.5}$$

as a G-equivariant bounded Kasparov module.

By [Pim86, Proposition 10] and [BS89, Remarque 7.5(3)], one can construct two elements $\alpha \in KK^G(C_0(X^0), C_+(X^1))$ and $\beta \in KK^G(C_+(X^1), C_0(X^0))$ satisfying

$$\alpha \otimes_{C_+(X^1)} \beta = 1 \in KK^G(C_0(X^0), C_0(X^0)) \qquad \beta \otimes_{C_0(X^0)} \alpha = 1 \in KK^G(C_+(X^1), C_+(X^1)).$$

The existence of α and β make $C_+(X^1)$ and $C_0(X^0)$ KK-equivalent as G-C*-algebras. By [Bla98, Examples 19.1.2(c)], $C_+(X^1, A)$ and $C_0(X^0, A)$ are also KK-equivalent as G-C*-algebras.

Putting all of this together, we obtain two six-term exact sequences by [BS89, Théorème 7.2] and [Tho01, Theorem 1.1]. For any separable G-C*-algebra B, the sequence

$$\begin{tikzcd}
KK_0^G(B, C_0(X^1, A)) \arrow[r] & KK_0^G(B, C_0(X^0, A)) \arrow[r] & KK_0^G(B, A) \arrow[d] \\
KK_1^G(B, A) \arrow[u] & KK_1^G(B, C_0(X^0, A)) \arrow[l] & KK_1^G(B, C_0(X^1, A)) \arrow[l]
\end{tikzcd}$$

is exact and for any G-C*-algebra B, the sequence

$$\begin{tikzcd}
KK_0^G(C_0(X^1, A), B) \arrow[d] & KK_0^G(C_0(X^0, A), B) \arrow[l] & KK_0^G(A, B) \arrow[l] \\
KK_1^G(A, B) \arrow[r] & KK_1^G(C_0(X^0, A), B) \arrow[r] & KK_1^G(C_0(X^1, A), B) \arrow[u]
\end{tikzcd}$$

is exact.

We can present these sequences in terms of the Baaj–Skandalis duals. By [Pim86, Lemma 4], there is a $\hat{G}$-equivariant exact sequence

$$0 \longrightarrow C_0(X^1, A) \rtimes_r G \longrightarrow C_+(X^1, A) \rtimes_r G \longrightarrow A \rtimes_r G \longrightarrow 0. \tag{II.1.6}$$

Fixing $P \in X^0$, $\rho : a \to \chi_P a \chi_P$ gives a $\hat{G}$-equivariant completely positive cross-section. The extension class associated with the sequence (II.1.6) is

$$(A \rtimes_r G, C_0(X^1, A) \rtimes_r G_{C_0(X^1,A)\rtimes_r G}, 2\chi_P - 1) \tag{II.1.7}$$

as a $\hat{G}$-equivariant bounded Kasparov module, which is Baaj–Skandalis dual to the extension class (II.1.5).

By Baaj–Skandalis duality, $C_+(X^1, A) \rtimes_r G$ and $C_0(X^0, A) \rtimes_r G$ are KK-equivalent as $\hat{G}$-C*-algebras. Further, it is useful to incorporate the $\hat{G}$-equivariant Morita equivalences between $C_0(X^0, A) \rtimes_r G$ and $\bigoplus_{P\in\Sigma^0} A \rtimes_r G_P$ and between $C_0(X^1, A) \rtimes_r G$ and $\bigoplus_{y\in\Sigma^1} A \rtimes_r G_y$. (Recall that Σ is the quotient of X by G.) For vertices $P \in \Sigma^0$, the inclusions $\tau_P : G_P \to G$ have open image and give rise to homomorphisms $\tau_P : C^*_r(G_P) \to C^*_r(G)$. Let

$$\tau_* = \sum_{P\in\Sigma^0} \tau_{P*} \qquad \tau^* = \sum_{P\in\Sigma^0} \tau^*_P.$$

For edges $y \in \Sigma^1$, the injections $\sigma_{\overline{y}} : G_y \to G_{o(y)}$ and $\sigma_y : G_y \to G_{t(y)}$ similarly have open image and give rise to homomorphisms $\sigma_{\overline{y}} : C^*_r(G_y) \to C^*_r(G_{o(y)})$ and $\sigma_y : C^*_r(G_y) \to C^*_r(G_{t(y)})$. Let

$$\sigma_* = \sum_{y\in\Sigma^1} (\sigma_{y*} - \sigma_{\overline{y}*}) \qquad \sigma^* = \sum_{y\in\Sigma^1} (\sigma^*_y - \sigma^*_{\overline{y}}).$$

We obtain two six-term exact sequences by [BS89, Théorème 7.2]. For any separable $\hat{G}$-C*-algebra B, provided that Σ is finite, the sequence

$$\begin{CD}
\bigoplus_{y\in\Sigma^1} KK_0^{\hat{G}}(B, A\rtimes_r G_y) @>{\sigma_*}>> \bigoplus_{P\in\Sigma^0} KK_0^{\hat{G}}(B, A\rtimes_r G_P) @>{\tau_*}>> KK_0^{\hat{G}}(B, A\rtimes_r G)\\
@A{\partial}AA @. @VV{\partial}V\\
KK_1^{\hat{G}}(B, A\rtimes_r G) @<<{\tau_*}< \bigoplus_{P\in\Sigma^0} KK_1^{\hat{G}}(B, A\rtimes_r G_P) @<<{\sigma_*}< \bigoplus_{y\in\Sigma^1} KK_1^{\hat{G}}(B, A\rtimes_r G_y)
\end{CD}$$

is exact. For any $\hat{G}$-C*-algebra B (and with no restriction on Σ), the sequence

$$\begin{CD}
\bigoplus_{y\in\Sigma^1} KK_0^{\hat{G}}(A\rtimes_r G_y, B) @<<{\sigma^*}< \bigoplus_{P\in\Sigma^0} KK_0^{\hat{G}}(A\rtimes_r G_P, B) @<<{\tau^*}< KK_0^{\hat{G}}(A\rtimes_r G, B)\\
@V{\partial}VV @. @AA{\partial}A\\
KK_1^{\hat{G}}(A\rtimes_r G, B) @>{\tau^*}>> \bigoplus_{P\in\Sigma^0} KK_1^{\hat{G}}(A\rtimes_r G_P, B) @>{\sigma^*}>> \bigoplus_{y\in\Sigma^1} KK_1^{\hat{G}}(A\rtimes_r G_y, B)
\end{CD} \tag{II.1.8}$$

is exact. These sequences remain exact if one forgets the $\hat{G}$-equivariance everywhere, which is the form in which they are presented in [Pim86, Theorem 17]. One can then also remove the need for Σ to be finite if B is *KK-compact*; see §II.1.4 and [Pim86, Theorem 18].

II.1.3 Induction from cocompact subgroups

The following Proposition is well known but we have been unable to locate a suitable reference, so we sketch a proof; but cf. [Bla06, Theorem 20.5.5] and [MN06, §3.2]. An analogous result for quantum group equivariant KK-theory can be found in [NV10, Proposition 4.7]. The proof of the following Proposition uses the induction homomorphism $i^{H,G} : KK^H_*(A, B) \to KK^G_*(C_0(G, A)^H, C_0(G, B)^H)$ of [Kas88, Theorem 3.5, §3.6]. We have chosen not to give a fuller account of this map, partly as there are technical issues arising in the unbounded picture which would necessitate the use of symmetric operators and so half-closed chains.

Proposition II.1.9. *Let H be a cocompact closed subgroup of a locally compact group G. Let A be a G-C^*-algebra and B an H-C^*-algebra. There is an isomorphism $KK^H_*(A,B) \cong KK^G_*(A, C_0(G,B)^H)$.*

The isomorphism can be interpreted as a KK-theoretic analogue of Frobenius reciprocity.

The closed subgroup H of G acts on G by right translation. This means that, for a C*-algebra B with an action β of H, $C_0(G,B)$ has a left action α of G by left translation and a right action β' of H given by the combination of right translation and the action on H. That is,

$$\alpha_g(f)(s) = f(g^{-1}s) \qquad \beta'_h(f)(s) = \beta_h(f(sh)).$$

The group G then acts on $C_0(G,A)^H$ by $\alpha'_g(f)(h) = f(g^{-1}h)$.

Proof. Let $\mathbf{x} \in KK^H(A,B)$ and $\mathbf{y} \in KK^G(A, C_0(G,B)^H)$. By [Kas88, Theorem 3.5, §3.6], there is a homomorphism $i^{H,G} : KK^H_*(A,B) \to KK^G_*(C_0(G,A)^H, C_0(G,B)^H)$. By [Kas88, Proof of Corollary 3.15], $i^{H,G}$ is related to descent by

$$i^{H,G}(\mathbf{x}) = [{}^H C_0(G,A)] \otimes_{C_0(G,A)\rtimes_r H} j^H_r(\sigma_{C_0(G)}(\mathbf{x})) \otimes_{C_0(G,B)\rtimes_r H} [({}^H C_0(G,B))^*].$$

Because G acts on A and on $C_0(G,B)^H$, by [Kas88, Lemma 3.6],

$$C_0(G,A)^H \cong C(G/H, A)$$

and

$$C_0(G, C_0(G,B)^H)^H \cong C(G/H, C_0(G,B)^H);$$

see also [RW98, Hooptedoodle 6.15]. Let $[\lambda] \in KK^H(A, C_0(G,A)^H)$ be given by the homomorphism $\lambda : A \to C(G/H, A)$ taking an element of A to a constant function. The product

$$[\lambda] \otimes_{C_0(G,A)^H} i^{H,G}(\mathbf{x})$$

is an element of $KK^G(A, C_0(G,B)^H)$. Let $[\psi] \in KK^H(C_0(G,B)^H, B)$ be given by the homomorphism $C_0(G,B)^H \to B$ of evaluation at the identity in G. The product

$$r^{G,H}(\mathbf{y}) \otimes_{C_0(G,B)^H} [\psi]$$

is an element of $KK^H(A,B)$. Note that $i^{H,G}([\psi]) \in KK^G(C(G/H, C_0(G,B)^H), C_0(G,B)^H)$ is equal to $\sigma_{C_0(G,B)^H}([\psi'])$ where $[\psi'] \in KK(C(G/H), \mathbb{C})$ is given by evaluation at the identity coset in G/H. Note also that $[\lambda] \otimes_{C(G/H,A)} \sigma_A([\psi']) \in KK^G(A,A)$ is the identity. By [Kas88, Theorem 3.6], $i^{H,G} \circ r^{G,H} = \sigma_{C(G/H)}$. By careful use of the relationship between the exterior and interior Kasparov products,

$$\begin{aligned}
&[\lambda] \otimes_{C(G/H,A)} i^{H,G}\Big(r^{G,H}(\mathbf{y}) \otimes_{C_0(G,B)^H} [\psi]\Big) \\
&\quad = [\lambda] \otimes_{C(G/H,A)} i^{H,G}(r^{G,H}(\mathbf{y})) \otimes_{C(G/H, C_0(G,B)^H)} i^{H,G}([\psi]) \\
&\quad = [\lambda] \otimes_{C(G/H,A)} \sigma_{C(G/H)}(\mathbf{y}) \otimes_{C(G/H, C_0(G,B)^H)} \sigma_{C_0(G,B)^H}([\psi']) \\
&\quad = [\lambda] \otimes_{C(G/H,A)} \Big(\mathbf{y} \otimes_{\mathbb{C}} [\psi']\Big) \\
&\quad = [\lambda] \otimes_{C(G/H,A)} \Big([\psi'] \otimes_{\mathbb{C}} \mathbf{y}\Big) \\
&\quad = [\lambda] \otimes_{C(G/H,A)} \sigma_A([\psi']) \otimes_A \mathbf{y} \\
&\quad = \mathbf{y}.
\end{aligned}$$

On the other hand, note that $r^{G,H}([({}^H C_0(G,B))^*]) \otimes_{C_0(G,B)^H} [\psi] \in KK^H(C_0(G,B) \rtimes_r H, B)$ is equal to $j_r^H(\sigma_B([\eta])) \otimes_{C_0(H) \rtimes_r H} [L^2(H,B)]$ where $\sigma : C_0(G) \to C_0(H)$ is given by the H-orbit of the identity in G. By careful use of the relationship between the exterior and interior Kasparov products,

$$\begin{aligned}
&j_r^H(\sigma_{C_0(G)}(\mathbf{x})) \otimes_{C_0(G,B)\rtimes_r H} [({}^H C_0(G,B))^*] \otimes_{C_0(G,B)^H} [\psi] \\
&\quad = j_r^H(\sigma_{C_0(G)}(\mathbf{x})) \otimes_{C_0(G,B)\rtimes_r H} j_r^H(\sigma_B([\eta])) \otimes_{C_0(H)\rtimes_r H} [L^2(H,B)] \\
&\quad = j_r^H\Big(\sigma_{C_0(G)}(\mathbf{x}) \otimes_{C_0(G,B)} \sigma_B([\eta])\Big) \otimes_{C_0(H)\rtimes_r H} [L^2(H,B)] \\
&\quad = j_r^H\Big(\mathbf{x} \otimes_{\mathbb{C}} [\eta]\Big) \otimes_{C_0(H)\rtimes_r H} [L^2(H,B)] \\
&\quad = j_r^H\Big([\eta] \otimes_{\mathbb{C}} \mathbf{x}\Big) \otimes_{C_0(H)\rtimes_r H} [L^2(H,B)] \\
&\quad = j_r^H\Big(\sigma_A([\eta]) \otimes_{\mathbb{C}} \sigma_{C_0(H)}(\mathbf{x})\Big) \otimes_{C_0(H)\rtimes_r H} [L^2(H,B)].
\end{aligned}$$

Finally, we have

$$\begin{aligned}
&r^{G,H}\Big([\lambda] \otimes_{C_0(G,A)^H} i^{H,G}(\mathbf{x})\Big) \otimes_{C_0(G,B)^H} [\psi] \\
&= r^{G,H}\Big([\lambda] \otimes_{C_0(G,A)^H} [{}^H C_0(G,A)] \\
&\qquad\qquad \otimes_{C_0(G,A)\rtimes_r H} j_r^H(\sigma_{C_0(G)}(\mathbf{x})) \otimes_{C_0(G,B)\rtimes_r H} [({}^H C_0(G,B))^*]\Big) \otimes_{C_0(G,B)^H} [\psi] \\
&= r^{G,H}\Big([\lambda] \otimes_{C_0(G,A)^H} [{}^H C_0(G,A)]\Big) \\
&\qquad\qquad\qquad \otimes_{C_0(G,A)\rtimes_r H} j_r^H\Big(\sigma_A([\eta]) \otimes_{\mathbb{C}} \sigma_{C_0(H)}(\mathbf{x})\Big) \otimes_{C_0(H)\rtimes_r H} [L^2(H,B)] \\
&= r^{G,H}([\lambda]) \otimes_{C_0(G,A)^H} [{}^H C_0(H,A)] \otimes_{C_0(H,A)\rtimes_r H} j_r^H(\sigma_{C_0(H)}(\mathbf{x})) \otimes_{C_0(H)\rtimes_r H} [L^2(H,B)] \\
&= [L^2(H,A)^*] \otimes_{C_0(H,A)\rtimes_r H} j_r^H(\sigma_{C_0(H)}(\mathbf{x})) \otimes_{C_0(H)\rtimes_r H} [L^2(H,B)] \\
&= \mathbf{x}.
\end{aligned}$$

We hence obtain the required isomorphism of KK-groups. □

Remark II.1.10. [EKQR06, Example A.12] Let A be a G-C*-algebra and denote by $\underline{A}$ the same C*-algebra with the trivial G-action. The C*-algebras $C_0(G,A)$ and $C_0(G,\underline{A})$ are G-equivariantly isomorphic.

Corollary II.1.11. *Let H be a closed subgroup of a locally compact group G and let A and B be G-C*-algebras. Calling the inclusion $\varphi : H \hookrightarrow G$, there is a homomorphism*

$$\varphi^* r^\varphi : KK_*^G(A, C_0(G,B)) \to KK_*^H(A, C_0(H,B))$$

which is an isomorphism if H is cocompact in G.

Proof. Let $\underline{B}$ be the C*-algebra B equipped with the trivial G-action. Remark that $C_0(G, C_0(H,B))^H$ is G-equivariantly isomorphic to

$$C_0(G, C_0(H,\underline{B}))^H \cong C_0(G \times H)^H \otimes \underline{B} \cong C_0(G) \otimes \underline{B} \cong C_0(G,B).$$

The conclusion follows from applying Proposition II.1.9. □

Example II.1.12. If K is a compact group and A and B are K-C*-algebras,

$$KK_*^K(A, C(K,B)) \cong KK_*^{\{e\}}(A, C(\{e\},B)) = KK_*(A,B).$$

The C*-algebra $C^*(K)$ of a compact group K is isomorphic to a direct sum of matrix algebras $\bigoplus_{\lambda\in\Lambda} M_{\dim\lambda}(\mathbb{C})$ over the set Λ of the equivalence classes of irreducible unitary representations of K. By [Kas88, Theorem 2.9],

$$KK_*(C^*(K),\mathbb{C}) = KK_*\left(\bigoplus_{\lambda\in\Lambda} M_{\dim\lambda}(\mathbb{C}), \mathbb{C}\right) \cong \prod_{\lambda\in\Lambda} KK_*(M_{\dim\lambda}(\mathbb{C}),\mathbb{C}) = \prod_{\lambda\in\Lambda} \mathbb{Z} \oplus 0.$$

The descent map from $KK_*^K(\mathbb{C}, C(K))$ to $KK_*(C^*(K), \mathbb{C})$ is injective, taking $1 \in KK_*(\mathbb{C}, \mathbb{C}) \cong KK_*^K(\mathbb{C}, C(K))$ to the left regular representation in $KK_*(C^*(K), \mathbb{C})$. Remark also that, by the dual Green–Julg isomorphism,

$$R(\hat{K}) = KK^{\hat{K}}(\mathbb{C}, \mathbb{C}) \cong KK(C_0(K), \mathbb{C})$$

as abelian groups. (The right-hand side is not a ring.)

Example II.1.13. If G is abelian, $C^*(G)$ is the commutative C*-algebra of functions on the Pontryagin dual group $\hat{G}$. Within locally compact abelian groups, it is natural for our purposes to restrict to those which are compactly generated. In some sense, this corresponds to the finite-dimensionality of the geometry of $C^*(G)$. A compactly generated locally compact abelian group can always be decomposed as

$$G \cong \mathbb{R}^m \times \mathbb{Z}^n \times K$$

for integers m, n and a compact group K, see e.g. [CH16, Example 5.A.3]. Because $\mathbb{R}^m \times \mathbb{Z}^n$ is a cocompact subgroup of $\mathbb{R}^m \times \mathbb{Z}^n \times K$ and a cocompact subgroup of $\mathbb{R}^{m+n}$, we have an isomorphism

$$\begin{aligned} KK_i^{\mathbb{R}^m \times \mathbb{Z}^n \times K}(A, C_0(\mathbb{R}^m \times \mathbb{Z}^n \times K, B)) &\cong KK_i^{\mathbb{R}^m \times \mathbb{Z}^n}(A, C_0(\mathbb{R}^m \times \mathbb{Z}^n, B)) \\ &\cong KK_i^{\mathbb{R}^{m+n}}(A, C_0(\mathbb{R}^{m+n}, B)) \\ &\cong KK_{i+m+n}(A, B). \end{aligned}$$

In contrast,

$$KK_*(C^*(G), \mathbb{C}) = \prod_{\lambda \in \Lambda_K} KK_{*+m}(C(\mathbb{T}^n), \mathbb{C}) \cong \prod_{\lambda \in \Lambda_K} (\mathbb{Z}^{2^{n-1}} \oplus \mathbb{Z}^{2^{n-1}}).$$

As $KK_*^G(\mathbb{C}, C_0(G))$ is singly generated, it is straightforward to check that the descent map to $KK_*(C^*(G), \mathbb{C})$ is injective. By Proposition II.1.9, Baaj–Skandalis duality, and the Green–Julg isomorphism, as an abelian group,

$$\begin{aligned} R(\hat{G}) &= KK_*^G(C_0(G), C_0(G)) \\ &\cong KK_*^{\mathbb{R}^m \times \mathbb{Z}^n}(C_0(\mathbb{R}^m \times \mathbb{Z}^n \times K), C_0(\mathbb{R}^m \times \mathbb{Z}^n)) \\ &\cong KK_*^{\mathbb{R}^m \times \mathbb{T}^n}(C_0(\mathbb{R}^m \times \mathbb{Z}^n \times K) \rtimes (\mathbb{R}^m \times \mathbb{Z}^n), C_0(\mathbb{R}^m \times \mathbb{Z}^n) \rtimes (\mathbb{R}^m \times \mathbb{Z}^n)) \\ &\cong KK_*^{\mathbb{R}^m \times \mathbb{T}^n}(C_0(K), \mathbb{C}) \\ &\cong \gamma_{(\mathbb{R}^m \times \mathbb{T}^n)} KK_*^{\mathbb{T}^n}(C_0(K), \mathbb{C}) \\ &= KK_*^{\mathbb{T}^n}(C_0(K), \mathbb{C}) \\ &\cong KK_*(C_0(K), C^*(\mathbb{T}^n)) \\ &\cong \bigoplus_{\mathbb{Z}^n} KK_*(C_0(K), \mathbb{C}). \end{aligned}$$

It is not difficult to check that $R(\hat{G})$ is isomorphic as a ring to $R(\mathbb{T}^n) \otimes_{\mathbb{Z}} R(\hat{K}) \cong R(\hat{K})[X_1, \dots, X_n]$ by considering the range of the two injections $R(\mathbb{T}^n) \hookrightarrow R(\hat{G})$ and $R(\hat{K}) \hookrightarrow R(\hat{G})$.

II.1.4 Restriction to compact subgroups

We say that a C*-algebra A is *KK-compact* if $KK_*(A, \cdot)$ is continuous, i.e. commutes with direct limits [Uuy11, Definition 2.9]. To set notation, if $(B_i, \phi_{ij})_{i,j \in \Lambda}$ is an inductive system of C*-algebras over a directed set Λ (see e.g. [Bla98, §3.3]), if A is KK-compact then

$$KK_*(A, \varinjlim B_i) \cong \varinjlim KK_*(A, B_i).$$

The limit on the right is the algebraic direct limit of abelian groups. In particular, if $(C_i)_{i \in I}$ is a collection of C*-algebras,

$$KK_*\Big(A, \bigoplus_{i \in I} C_i\Big) \cong \bigoplus_{i \in I} KK_*(A, C_i).$$

The most important example of a KK-compact C*-algebra is $\mathbb{C}$; a sufficient condition for A to be KK-compact is that it satisfy the Universal Coefficient Theorem and have finitely-generated K-theory [Uuy11, Definition 2.10.1].

Proposition II.1.14. *Let G be a noncompact locally compact group for which the connected component of the identity G_0 is compact. Let A and B be G-C*-algebras with A a KK-compact C*-algebra. If K is a compact subgroup of G,*

$$r^{G,K} : KK^G_*(A, C_0(G,B)) \to KK^K_*(A, C_0(G,B))$$

is zero.

Proof. By [CH16, Corollary 2.E.7(2)], we can find a compact open subgroup H of G containing K. It will suffice to show that

$$r^{G,H} : KK^G_*(A, C_0(G,B)) \to KK^H_*(A, C_0(G,B))$$

is zero. Because H is open, the quotient space $X := G/H$ is discrete and G is H-equivariantly homeomorphic to $H \times X$. Choose $\sigma : X \to G$ such that $H\sigma(Hg) = Hg$ for $Hg \in X$. By Corollary II.1.11,

$$\sigma^* r^{H,1} : KK^H_*(A, C_0(G,B)) \to KK_*(A, C_0(X,B))$$

is an isomorphism. It therefore suffices to show that

$$r^{G,1} = r^{H,1} r^{G,H} : KK^G_*(A, C_0(G,B)) \to KK_*(A, C_0(G,B))$$

is zero.

Since $e \in G$ has a compact open neighbourhood H, so does every element of G. Let Σ be the set of compact open subsets of G, partially ordered by inclusion. For $U, V \in \Sigma$ such that $U \subseteq V$, denote by $\phi_{U,V}$ the inclusion $C(U,B) \hookrightarrow C(V,B)$. We thus form the inductive system $(C(U,B), \phi_{U,V})_{U,V \in \Sigma}$ of C*-algebras. For every $U \in \Sigma$, there is an inclusion $\phi_U : C(U,B) \hookrightarrow C_0(G,B)$, making $C_0(G,B)$ isomorphic to the direct limit $\varinjlim C(U,B)$. Suppose that A is KK-compact, so that

$$KK_*(A, C_0(G,B)) \cong \varinjlim KK_*(A, C(U,B)).$$

Let $\mathbf{x} \in KK^G_*(A, C_0(G,B))$, so that $r^{G,1}(\mathbf{x}) \in KK_*(A, C_0(G,B))$. By the definition of the algebraic direct limit, there must exist some $U \in \Sigma$ and $\mathbf{y} \in KK_*(A, C(U,B))$ such that $\mathbf{y} \otimes_{C(U,B)} [\phi_U] = r^{G,1}(\mathbf{x})$. Let $V \in \Sigma$ such that $U \subseteq V$. Now, with $p_V \in C_0(G,B)$ the projection onto $C(V,B) \subset C_0(G,B)$ and $[p_V] \in KK_0(C_0(G,B), C_0(G,B))$,

$$r^{G,1}(\mathbf{x}) \otimes_{C_0(G,B)} [p_V] = \mathbf{y} \otimes_{C(U,B)} [\phi_U] \otimes_{C_0(G,B)} [p_V] = \mathbf{y} \otimes_{C(U,B)} [\phi_U] = r^{G,1}(\mathbf{x}).$$

In particular,

$$r^{G,1}(\mathbf{x}) \otimes_{C_0(G,B)} [p_V] = r^{G,1}(\mathbf{x}) \otimes_{C_0(G,B)} [p_U].$$

By the noncompactness of G and the properness of its action on itself by left translation, choose $g \in G$ such that $gU \cap U = \emptyset$. Taking $V = gU \cup U$, we see that $r^{G,1}(\mathbf{x}) \otimes_{C_0(G,B)} [p_{gU}] = 0$. Let α and β be the actions of G on A and $C_0(G,B)$ respectively. Because $\mathbf{x} \in KK^G_*(A, C_0(G,B))$,

$$[\alpha_{g^{-1}}] \otimes_A r^{G,1}(\mathbf{x}) \otimes_{C_0(G,B)} [\beta_g] = r^{G,1}(\mathbf{x}).$$

Hence

$$\begin{aligned}[\alpha_{g^{-1}}] \otimes_A r^{G,1}(\mathbf{x}) \otimes_{C_0(G,B)} [p_{gU}] \otimes_{C_0(G,B)} [\beta_g] &= [\alpha_{g^{-1}}] \otimes_A r^{G,1}(\mathbf{x}) \otimes_{C_0(G,B)} [\beta_g] \otimes_{C_0(G,B)} [p_U] \\ &= r^{G,1}(\mathbf{x}) \otimes_{C_0(G,B)} [p_U]\end{aligned}$$

and so $r^{G,1}(\mathbf{x}) = 0$, as required. □

As a consequence, we have the following counterpoint to Corollary II.1.11.

Proposition II.1.15. *Let G be a locally compact group and A and B be G-C*-algebras. Let K be a compact subgroup of G. Call the inclusion $\iota_{K,G} : K \hookrightarrow G$. The homomorphism*

$$\iota^*_{K,G} r^{G,K} : KK^G_*(A, C_0(G, B)) \to KK^K_*(A, C(K, B))$$

is an isomorphism if G is compact and zero if G is noncompact and either

1. *The connected component G_0 of the identity is noncompact or*
2. *A is KK-compact.*

Proof. Let K be a compact subgroup of a locally compact group G. By Corollary II.1.11,

$$KK^K_*(A, C(K, B)) \cong KK^{\{e\}}_*(A, C(\{e\}, B)) = KK_*(A, B).$$

With the inclusion $\iota_{\{e\},G} : \{e\} \hookrightarrow G$, the homomorphism

$$\iota^*_{\{e\},G} r^{G,1} : KK^G_*(A, C_0(G, B)) \to KK^{\{e\}}_*(A, C(\{e\}, B)) = KK_*(A, B)$$

factors through

$$\iota^*_{K,G} r^{G,K} : KK^G_*(A, C_0(G, B)) \to KK^K_*(A, C(K, B))$$

so one is zero if and only if the other is zero.

If G_0 is compact but G is noncompact and A is KK-compact, the result follows directly from Proposition II.1.14. On the other hand, suppose that the connected component G_0 of the identity is noncompact. Let K_0 be the maximal compact subgroup of G_0. With $\iota_{G_0,G} : G_0 \hookrightarrow G$ and $\iota_{K_0,G_0} : K_0 \hookrightarrow G_0$ and $\iota_{\{e\},K_0} : \{e\} \hookrightarrow K_0$ the inclusion homomorphisms,

$$\iota^*_{\{e\},G} \circ r^{G,1} = \iota^*_{\{e\},K_0} \circ \iota^*_{K_0,G_0} \circ \iota^*_{G_0,G} \circ r^{G,1}.$$

So it will suffice for 1. to show that

$$\iota^*_{K_0,G_0} : KK_*(A, C_0(G_0, B)) \to KK_*(A, C_0(K_0, B))$$

is zero. By [CH16, Theorem 2.E.16], G_0 is homeomorphic to $\mathbb{R}^n \times K_0$ for some $n \geq 1$. The class $[\iota_{K_0,G_0}] \in KK_0(C_0(G_0), C_0(K_0))$, which implements $\iota^*_{K_0,G_0}$ by the external product on the right, is equal to $\sigma_{C_0(K_0)}(\mathbf{w})$ where $\mathbf{w} \in KK_0(\mathbb{C}, C_0(\mathbb{R}^n))$ is the class given by point evaluation (unique because $\mathbb{R}^n$ is path connected). But, since $\mathbb{R}^n$ is path-connected and noncompact, $\mathbf{w}$ is homotopy equivalent to zero, and we are done. □

Remark II.1.16. Let G be a locally compact group and let A and B be G-C*-algebras. If H is an open subgroup of G, there are inclusions

$$\lambda : A \rtimes_r H \hookrightarrow A \rtimes_r G \qquad \mu : B \rtimes_r H \hookrightarrow B \rtimes_r G.$$

It is routine to check, using in particular the nondegeneracy of λ and μ, that the diagram

$$\begin{tikzcd}
KK^G_*(A, B) \arrow[r, "j^G_r"] \arrow[dd, "r^{G,H}"] & KK_*(A \rtimes_r G, B \rtimes_r G) \arrow[dr, "\lambda^*"] & \\
& & KK_*(A \rtimes_r H, B \rtimes_r G) \\
KK^H_*(A, B) \arrow[r, "j^H_r"] & KK_*(A \rtimes_r H, B \rtimes_r H) \arrow[ur, "\mu_*"] &
\end{tikzcd}$$

commutes. As a special case, with the inclusion $\mu' : C_0(G,B) \rtimes_r H \hookrightarrow C_0(G,B) \rtimes_r G$, we have the commuting diagram

$$\begin{tikzcd}
KK_*^G(A, C_0(G,B)) \arrow[r, "j_r^G"] \arrow[dd, "r^{G,H}"] & KK_*(A \rtimes_r G, C_0(G,B) \rtimes_r G) \cong KK_*(A \rtimes_r G, B) \arrow[d, "\lambda^*"] \\
 & KK_*(A \rtimes_r H, C_0(G,B) \rtimes_r G) \cong KK_*(A \rtimes_r H, B) \\
KK_*^H(A, C_0(G,B)) \arrow[r, "j_r^H"] & KK_*(A \rtimes_r H, C_0(G,B) \rtimes_r H) \arrow[u, "\mu'_*"]
\end{tikzcd} \cdot$$

In particular, if G is noncompact and H is a compact open subgroup, Proposition II.1.14 implies that

$$\lambda^* : KK_*(A \rtimes_r G, B) \to KK_*(A \rtimes H, B)$$

is zero on the image of $KK_*^G(A, C_0(G,B))$ under the descent map. Taking into account Baaj–Skandalis duality, this implies also that

$$\lambda^* \circ r^{\hat{G},1} : KK_*^{\hat{G}}(A \rtimes_r G, B) \to KK_*(A \rtimes H, B)$$

is zero.

Remark II.1.17. Let G be an locally compact group, A a G-C*-algebra, and B a C*-algebra with the trivial action of G. Let H be an open subgroup of G and denote by λ_r and λ_u the inclusions $A \rtimes_r H \hookrightarrow A \rtimes_r G$ and $A \rtimes_u H \hookrightarrow A \rtimes_u G$. Let τ_G and τ_H be the quotient maps $A \rtimes_u G \to A \rtimes_r G$ and $A \rtimes_u H \to A \rtimes_r H$, respectively. With Ψ^G and Ψ^H the dual Green–Julg maps, the diagram

$$\begin{tikzcd}
KK_*^{\hat{G}}(A \rtimes_r G, B) \arrow[r, "r^{\hat{G},1}"'] & KK_*(A \rtimes_r G, B) \arrow[r, "\tau_G^*"] \arrow[d, "\lambda_r^*"] & KK_*(A \rtimes_u G, B) \arrow[d, "\lambda_u^*"] & KK_*^G(A,B) \arrow[l, "\Psi^G"'] \arrow[d, "r^{G,H}"] \\
 & KK_*(A \rtimes_r H, B) \arrow[r, "\tau_H^*"] & KK_*(A \rtimes_u H, B) & KK_*^H(A,B) \arrow[l, "\Psi^H"]
\end{tikzcd}$$

commutes. As a consequence, if G is infinite and discrete and H is a finite subgroup, because the dual Green–Julg maps Ψ^G and Ψ^H are isomorphisms,

$$r^{G,H} \circ (\Psi^G)^{-1} \circ \tau_G^* \circ r^{\hat{G},1} : KK_*^{\hat{G}}(A \rtimes_r, B) \to KK_*^K(A,B)$$

is zero. In particular, if $A = B = \mathbb{C}$,

$$r^{G,H} \circ (\Psi^G)^{-1} \circ \tau_G^* \circ r^{\hat{G},1} : KK_*^{\hat{G}}(C_r^*(G), \mathbb{C}) \to KK_*^K(\mathbb{C}, \mathbb{C})$$

is zero, which means that, for $\mathbf{x} \in KK_*^{\hat{G}}(A \rtimes_r G, B)$, the element $(\Psi^G)^{-1} \circ \tau_G^* \circ r^{\hat{G},1}$ of $KK^G(\mathbb{C}, \mathbb{C})$ cannot be 1 nor any of the generalised γ-elements discussed at the end of §II.1.1.

Remark II.1.18. Let G be a (countably) infinite discrete group. For any finite subgroup K of G, denote by $\lambda_{K,G}$ the inclusion $C^*(K) \hookrightarrow C_r^*(G)$. Remark II.1.16 says that

$$\lambda^* \circ r^{\hat{G},1} : KK_*^{\hat{G}}(C_r^*(G), \mathbb{C}) \to KK_*(C^*(K), \mathbb{C})$$

is zero. Let $\mathbf{x} \in KK_*^{\hat{G}}(C_r^*(G), \mathbb{C})$. If $p \in C_r^*(G)$ is a projection supported on K, that is, inside the subalgebra $C^*(K)$, representing a class in $K_0(C_r^*(G))$, then its pairing with $r^{\hat{G},1}(\mathbf{x})$ is

$$[p] \otimes_{C_r^*(G)} r^{\hat{G},1}(\mathbf{x}) = [p] \otimes_{C^*(K)} [\lambda] \otimes_{C_r^*(G)} r^{\hat{G},1}(\mathbf{x}) = 0. \tag{II.1.19}$$

Suppose that

1. $C_r^*(G)$ satisfies the Universal Coefficient Theorem;
2. $K_0(C_r^*(G))$ is free and represented by projections in $C_r^*(G)$ which are supported on finite subgroups of G; and
3. $K_1(C_r^*(G))$ is zero.

For example, G could be

- The free product of a finite number of finite groups [Cun83, §3]; or
- The (countable) direct limit of finite groups, including any torsion abelian (discrete) group, such as the Prüfer p-group $\mathbb{Z}(p^\infty)$ cf. [Bla98, Definition 22.3.4(N2)].

(We remark that, by [Tu99, Théorème 9.3, Proposition 10.7], a sufficient condition for $C_r^*(G)$ to satisfy the Universal Coefficient Theorem is that G have the *Haagerup property*; see [CCJJV01].) By the Universal Coefficient Theorem [Bla98, Theorem 23.1.1] and the freeness of $K_*(C_r^*(G))$, there are isomorphisms

$$KK_0(C_r^*(G), \mathbb{C}) \cong \mathrm{Hom}(K_0(C_r^*(G)), \mathbb{Z}) \qquad KK_1(C_r^*(G), \mathbb{C}) = \mathrm{Hom}(K_1(C_r^*(G)), \mathbb{Z}) = 0.$$

In particular, if an element $\mathbf{y} \in KK_0(C_r^*(G), \mathbb{C})$ pairs trivially with every element of $K_0(C_r^*(G))$, then $\mathbf{y} = 0$. By assumption, all the elements of $K_0(C_r^*(G))$ are supported on finite subgroups. So, if $\mathbf{x} \in KK_*^{\hat{G}}(C_r^*(G), \mathbb{C})$, (II.1.19) implies that $r^{\hat{G},1}(\mathbf{x}) = 0$. In other words, the forgetful map

$$r^{\hat{G},1} : KK_*^{\hat{G}}(C_r^*(G), \mathbb{C}) \to KK_*(C_r^*(G), \mathbb{C})$$

is zero. By Baaj–Skandalis duality, this implies also that the descent map

$$j_r^G : KK_*^G(\mathbb{C}, C_0(G)) \to KK_*(C_r^*(G), \mathbb{C})$$

is zero.

We shall show in Example II.3.18 that, for a free product $H_1 * H_2$ of finite groups H_1 and H_2, the KK-group

$$KK_*^{\hat{G}}(C_r^*(G), \mathbb{C})$$

is nonzero, so the forgetful map $r^{\hat{G},1}$ is really losing substantial information.

II.2 Unbounded Kasparov modules from weights on groups

II.2.1 Length functions and weights on groups

The building of spectral triples for group C*-algebras has its origin in Connes's 1989 paper [Con89]. There is actually more than one such construction present in the article, but the most influential has been the first. A *length function* on a discrete group G is a map $\ell : G \to \mathbb{R}_+$ such that

1. $\ell(gh) \leq \ell(g) + \ell(h)$,
2. $\ell(g^{-1}) = \ell(g)$, and
3. $\ell(e) = 0$

for $g, h \in G$ and e the identity. If, in addition, $(1 + \ell^2)^{-1} \in C_0(G)$, then

$$(C_r^*(G), \ell^2(G), M_\ell) \tag{II.2.1}$$

is a spectral triple, where the operator M_ℓ is multiplication by ℓ [Con89, Lemma 5].

Any isometric action of G on a metric space (X, d) and choice of a point $x_0 \in X$ gives rise to a length function $\ell : g \mapsto d(g \cdot x_0, x_0)$. By Proposition A.2.2, the condition that $(1+\ell^2)^{-1} \in C_0(G)$ is equivalent to the properness of the action. Conversely, given a length function ℓ on G, the expression $d : (g, h) \mapsto \ell(gh^{-1})$ defines a pseudometric on G and gives rise to a metric on the quotient space of G by the equivalence relation $g \sim h \iff d(g, h) = 0$.

The isometry groups of spectral triples of the form (II.2.1) were studied by Park [Par95a, Par95b]. More recently, other authors have considered the *quantum isometry groups* [BS10] and e.g. [GB16, Chapter 8]. Rieffel addressed whether the construction gives rise to a *compact quantum metric space*, first for $G = \mathbb{Z}^n$ [Rie02] and later, with other authors, for hyperbolic [OR05] and nilpotent groups [CR17]. In [BCL06], spectral triples for group C*-algebras were considered for their categorical properties. The length functions were here allowed to be $\mathbb{R}$-valued to include the number operator on $\mathbb{Z}$. Following the construction of spectral triples for crossed products by $\mathbb{Z}$ in [CMRV08, BMR10] spectral triples for crossed products by discrete groups in general were constructed in [HSWZ13, Pat14].

In spite of the pervasiveness of the use of length functions to build spectral triples of the form (II.2.1), the construction suffers from a serious drawback. Since the operator M_ℓ is positive, the spectral triple must represent the zero class in the K-homology of $C_r^*(G)$. Even where the length function is allowed to be $\mathbb{R}$-valued, no serious attempt has been made to produce examples other than the aforementioned number operator on $\mathbb{Z}$. In [Rub22, AGIR22], where spectral triples for twisted crossed products are considered, the length function is permitted to be matrix-valued, cf. [HSWZ13, Remark 2.15]. Still, beyond $\mathbb{Z}^n$, no new examples are given. Further, this loosening to matrix-valued weights comes at the cost of the geometrical interpretation of ℓ as a (pseudo)metric on G. A further generalisation can be found in [GRU19, §2.2.4], in which *semifinite spectral triples* are built from weights valued in bounded operators on an infinite dimensional Hilbert space.

We will consider how the situation can be remedied, using both ingredients dating back a half-century and new ideas. We also work in the generality of locally compact groups although, for nondiscrete G, $C_r^*(G)$ is non-unital and so we enter the realm of non-compact noncommutative geometry. We make the following boilerplate definition.

Definition II.2.2. Given a locally compact group G and a finite-dimensional complex vector space V, a *weight* is a continuous function

$$\ell : G \to \operatorname{End} V.$$

If V is $\mathbb{Z}/2\mathbb{Z}$-graded, we require that ℓ be odd. We say that ℓ is

- self-adjoint if $\ell^* = \ell$;
- *proper* if $(1+\ell^*\ell)^{-1} \in C_0(G, \operatorname{End} V) = C_0(G) \otimes \operatorname{End} V$; and
- *translation-bounded* if, for all $g \in G$, $\sup_{h\in G} \|\ell(gh) - \ell(h)\| < \infty$ and there exists a neighbourhood U of the identity in G such that $\sup_{g\in U, h\in G} \|\ell(gh) - \ell(h)\| < \infty$.

When G is discrete, our definition coincides with [Rub22, Definition 6.1].

For the following, we use the notion of a k-space; see e.g. [Wil70, Definition 43.8]. (Further details can be found in Appendix A.1.1, particularly in Definition A.1.6.) Note that any locally compact space is a k-space.

Lemma II.2.3. *Let X be a k-space, Y a locally compact Hausdorff space, and E a locally convex complete topological vector space. Let $f : X \times Y \to E$ be a continuous function. The following are equivalent:*

1. *The function $\Lambda(f) : X \to C(Y, E)$ given by $\Lambda(f)(x)(y) = f(x, y)$ is an element of $C(X, C_b(Y, E)_\beta)$, where β is the strict topology; and*
2. *For every compact subset $K \subseteq X$, $\sup_{x\in K, y\in Y} \|f(x, y)\| < \infty$.*

Proof. To show 1. $\Rightarrow$ 2., suppose that $\zeta \in C(X, C_b(Y,E)_\beta)$. Let K be a compact subset of X. Since K is compact, its image under ζ is also a compact set $\zeta(K) \subseteq C_b(Y,E)_\beta$. A compact subset of a topological vector space is closed and bounded so, in particular, $\zeta(K)$ is bounded in κ. By [Buc58, Theorem 1(iii)], $\zeta(K)$ is uniformly bounded, i.e. $\sup_{x\in K, y\in Y}\|f(x,y)\| < \infty$.

We now show 2. $\Rightarrow$ 1.. By e.g. [Eng89, Theorem 3.4.3] (cf. [Eng89, §2.6]), a function $f : X \times Y \to E$ is continuous if and only if $\Lambda(f) \in C(X, C(Y,E)_\kappa)$, where κ is the compact-open topology on $C(Y,E)$. Assume 2. holds. By taking $K = \{x\}$ for each $x \in X$, we obtain that $\zeta \in C(X, C_b(Y,E)_\kappa)$. Now let K be any compact subset of X. By [Buc58, Theorem 1(iii)], $\zeta(K)$ is bounded in β. By [Buc58, Theorem 1(iv)], on any β-bounded set, β and κ coincide. Hence the restriction $\zeta|_K$ is an element of $C(K, C_b(Y, \operatorname{End} V)_\beta)$. By e.g. [Wil70, Lemma 43.10], the fact that $\zeta|_K \in C(K, C_b(Y,E)_\beta)$ for all compact subsets $K \subseteq X$ is equivalent to $\zeta \in C(X, C_b(Y,E)_\beta)$. ☐

Lemma II.2.4. *Let G be a locally compact group, V a finite-dimensional complex vector space, and $\ell : G \to \operatorname{End} V$ a weight. The following are equivalent:*

1. *For all $g \in G$,*
$$\sup_{h\in G}\|\ell(gh) - \ell(h)\| < \infty$$
and there exists a neighbourhood U of the identity in G such that
$$\sup_{g\in U, h\in G}\|\ell(gh) - \ell(h)\| < \infty.$$

2. *For every compact subset $K \subseteq G$,*
$$\sup_{g\in K, h\in G}\|\ell(gh) - \ell(h)\| < \infty.$$

3. *The function $\zeta : G \to C(G, \operatorname{End} V)$ given by*
$$\zeta(g)(h) = \ell(gh) - \ell(h)$$
is an element of $C(G, C_b(G, \operatorname{End} V)_\beta)$, where β is the strict topology.

Proof. Suppose that 1. holds and let K be a compact subset of G. The open sets $(Ug)_{g\in K}$ cover K. Let $Ug_1, \dots, Ug_k$ be a finite subcover. We have

$$\begin{aligned}
\sup_{g\in K, h\in G}\|\ell(gh) - \ell(h)\| &\le \max_{1\le i\le k}\sup_{g\in Ug_i, h\in G}\|\ell(gh) - \ell(h)\| \\
&= \max_{1\le i\le k}\sup_{g\in U, h\in G}\|\ell(gg_i^{-1}h) - \ell(h)\| \\
&\le \max_{1\le i\le k}\sup_{g\in U, h\in G}\left(\|\ell(gg_i^{-1}h) - \ell(g_i^{-1}h)\| + \|\ell(g_i^{-1}h) - \ell(h)\|\right) \\
&\le \sup_{g\in U, h\in G}\|\ell(gh) - \ell(h)\| + \max_{1\le i\le k}\sup_{h\in G}\|\ell(g_i^{-1}h) - \ell(h)\| \\
&< \infty,
\end{aligned}$$

that is, 2. is satisfied.

Suppose that 2. holds and, by the local compactness of G, take an open neighbourhood U of the identity in G contained in a compact set K. Then

$$\sup_{g\in U, h\in G}\|\ell(gh) - \ell(h)\| \le \sup_{g\in K, h\in G}\|\ell(gh) - \ell(h)\| < \infty,$$

so 1. is satisfied.

That 3. $\Leftrightarrow$ 2., is a consequence of Lemma II.2.3 with $f : G \times G \to \operatorname{End} V$ given by $f(g,h) = \ell(gh) - \ell(h)$. By the continuity of ℓ and of group multiplication, f is continuous and, furthermore, $\zeta = \Lambda(f)$. ☐

In our construction of weights from directed length functions in §II.3, we will have a bound

$$\sup_{h \in G} \|\ell(gh) - \ell(h)\| = \|\ell(g)\|,$$

which implies the translation-boundedness of ℓ.

Remark II.2.5. Let X be a locally compact Hausdorff space and V a finite dimensional complex vector space. The $*$-strong topology on $\operatorname{End}^*_{C_0(X)} C_0(X,V) \cong C_b(X,V)$ coincides with the strict topology β on $C_b(X,V)$. This follows from the definition of each topology and the finite-dimensionality of V; for further details see [Gab21].

Example II.2.6. cf. [Rub22, Examples 6.2–3] Let G be a compactly generated locally compact abelian group. There is an isomorphism

$$G \cong \mathbb{R}^m \times \mathbb{Z}^n \times K$$

for integers m, n, and a compact group K, see e.g. [CH16, Example 5.A.3]. The group G acts properly on $\mathbb{R}^{m+n}$ by translation, with K the stabiliser at every point. Let $(v_i)_{i=1}^{m+n}$ be a basis of $\mathbb{R}^{m+n}$. To simplify notation, we will also write $(v_i)_{i=1}^{m+n}$ for their images in $\mathscr{C}\ell_{m+n}$. Let V be a Clifford module for $\mathscr{C}\ell_{m+n}$ and define a weight $\ell : G \to \operatorname{End} V$ on the group by

$$\ell(g) = \sum_{j=1}^{m+n} g_j v_j$$

where g_j is a real or integer component of g. The self-adjointness of $v_j \in \mathscr{C}\ell_{m+n}$ means that ℓ is self-adjoint. The weight ℓ is a homomorphism from G to the additive group of $\operatorname{End} V$, so

$$\|\ell(gh) - \ell(h)\| = \|\ell(g)\|$$

for all $g, h \in G$. We have

$$\ell(g)^2 = \sum_{i,j=1}^{m+n} g_i g_j \langle v_i \mid v_j \rangle,$$

which is a positive-definite quadratic form in g, and so ℓ is proper.

Example II.2.7. Let K be a compact group. Let V be equal to the $\mathbb{Z}/2\mathbb{Z}$-graded vector space $\mathbb{C} \oplus 0$ and let $\ell : k \mapsto 0$. Because K is compact, ℓ is proper; it would not be otherwise.

Example II.2.8. Let k be a local field with absolute value $|\cdot|$, such as $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$, or $\mathbb{F}_p((t))$. Define a weight $\ell : k^\times \to \mathbb{C}$ for the multiplication group of k by $\ell(a) = \log|a|$. The weight is clearly self-adjoint, and being, in fact, a homomorphism, is translation bounded. The weight ℓ is proper since the subgroup $|k^\times|$ of $\mathbb{R}^\times$ is either all of $\mathbb{R}^\times$, if k is archimedean, or equal to $c^{\mathbb{Z}}$ for some $c > 1$, if k is nonarchimedean.

On the other hand, let k be a locally compact field with absolute value $|\cdot|$ which is not local. Then $|\cdot|$ must be the trivial absolute value, giving k the discrete topology. Then $|k^\times| = \{1\}$ and, if we define $\ell : a \to \log|a|$, ℓ is zero. Only if k is a finite field is ℓ a proper weight.

If we had a third hand, we could consider a field k with absolute value $|\cdot|$ which is not locally compact. As $k^\times$ is not necessarily locally compact, let us equip it with the discrete topology. Suppose further that $|k^\times|$ is a dense subset of $\mathbb{R}^\times$. For instance, consider $\mathbb{Q}$ with the usual archimedean absolute value, in which case $|\mathbb{Q}^\times| = \mathbb{Q}^\times_+$, or consider the p-adic complex numbers $\mathbb{C}_p$, for which $|\mathbb{C}_p^\times| = c^{\mathbb{Q}}$, for some $c > 1$. Let $\ell : a \to \log|a|$. Although ℓ is self-adjoint and translation bounded, it is not proper.

II.2.2 Fell bundles

One of the goals of this Chapter is to study KK-theory for dynamical systems and to provide tools for the future study of such. Partial dynamical systems are naturally captured by the ideal of a *Fell bundle*. A standard reference from this point of view, in the discrete group case, is [Exe17].

Definition II.2.9. [FD88b, Definitions VIII.2.2,3.1,16.2] cf. [MW08, Definition 1.1] A *Fell bundle* $\mathscr{B}$ over a locally compact group G is a Banach bundle over G with a continuous bilinear associative multiplication map $\cdot : \mathscr{B} \times \mathscr{B} \to \mathscr{B}$ and a continuous conjugate-linear antiautomorphic involution $^* : \mathscr{B} \to \mathscr{B}$, such that

1. $B_g \cdot B_h \subseteq B_{gh}$ and $(B_g)^* \subseteq B_{g^{-1}}$;
2. The fibre B_e at the identity $e \in G$ is a C*-algebra with respect to the multiplication, involution, and norm on $\mathscr{B}$; and
3. For each $g \in G$, the fibre B_g is a partial imprimitivity B_e-B_e-bimodule with the module actions determined by the multiplication on $\mathscr{B}$ and inner products given by

$$
{}_{B_e}\langle a \mid b \rangle = ab^* \qquad \langle a \mid b \rangle_{B_e} = a^* b
$$

   for $a, b \in B_g$.

These axioms imply that $B_{g^{-1}}$ is isomorphic to the dual bimodule of B_g by the involution and that multiplication induces an isomorphism of $B_g \otimes_{B_e} B_h$ with a partial imprimitivity B_e-B_e-subbimodule of B_{gh} [MW08, Lemma 1.2].

Definition II.2.10. [EN02, Definitions 2.2,7, Proposition 2.10] Let $\mathscr{B}$ be a Fell bundle over a locally compact group G. A *section* of $\mathscr{B}$ is a continuous function y from G to $\mathscr{B}$ such that $y(g) \in B_g$. The space of compactly supported sections is denoted $C_c(\mathscr{B})$. The convolution product of sections $y, z \in C_c(\mathscr{B})$ is given by

$$
(yz)(g) = \int_G y(h) z(h^{-1} g) d\mu(h).
$$

The adjoint of a section is given by $y^*(g) = \Delta(g^{-1}) y(g^{-1})^*$. With these operations, $C_c(\mathscr{B})$ is a $*$-algebra. The Hilbert B_e-module $L^2(\mathscr{B})$ is defined as the completion of $C_c(\mathscr{B})$ under the B_e-valued right inner product given by

$$
\langle \xi \mid \eta \rangle_{B_e} = (\xi^* \eta)(e) = \int_G \xi(g)^* \eta(g) d\mu(g) \qquad (\xi, \eta \in C_c(\mathscr{B})).
$$

The representation of $C_c(\mathscr{B})$ given by left multiplication on $L^2(\mathscr{B})$ when completed gives the reduced C*-algebra $C_r^*(\mathscr{B})$. The elements of $\mathscr{B}$ itself act naturally as multipliers on $C_r^*(\mathscr{B})$.

The Fell bundle $\mathscr{B}$ gives its C*-algebra $C_r^*(\mathscr{B})$ a canonical $\hat{G}$-action δ [EN02, Proposition 2.10] cf. [KMQW10, Proposition 3.1, Remark 3.2]. The coaction $\delta : C_r^*(\mathscr{B}) \to M(C_r^*(\mathscr{B}) \otimes C_r^*(G))$ is given by

$$
\delta(y) = \int_G y(g) \otimes u_g d\mu(g) \qquad (y \in C_c(\mathscr{B})), \tag{II.2.11}
$$

where $u_g \in M(C_r^*(G))$ are the unitaries corresponding to elements of G. Indeed, [Qui96, Corollary 3.9] says that a C*-algebra A is isomorphic to $C_r^*(\mathscr{B})$ for some Fell bundle $\mathscr{B}$ over a discrete group if and only if A has a coaction of $\hat{G}$ and the conditional expectation to its fixed point algebra is faithful. This is very far from being true for a non-discrete group; see [LPRS87, Example 2.3(6)]. The Hilbert B_e-module $L^2(\mathscr{B})$ also carries a $\hat{G}$-action (with $\hat{G}$ acting trivially on B_e), given on $C_c(\mathscr{B}) \subseteq L^2(\mathscr{B})$ by the same formula (II.2.11), for which the representation of $C_r^*(\mathscr{B})$ is $\hat{G}$-equivariant.

Examples II.2.12.

1. [FD88b, Example VIII.2.7] The *group bundle* $\mathscr{B}$ is defined as $\mathbb{C} \times G$ as a Banach bundle over G. We define multiplication on $\mathscr{B}$ by

$$(z,g)(w,h) = (zw, gh)$$

and an involution by

$$(z,g)^* = (\overline{z}, g^{-1}).$$

There is an isomorphism between $C_r^*(\mathscr{B})$ and $C_r^*(G)$.

2. [FD88b, §VIII.4.2] Let A be a C*-algebra with an action α of a locally compact group G. We define the *semidirect product* Fell bundle $\mathscr{B}$ as $A \times G$ as a Banach bundle over G. We define multiplication on $\mathscr{B}$ by

$$(a,g)(b,h) = (a\alpha_g(b), gh)$$

and an involution by

$$(a,g)^* = (\alpha_{g^{-1}}(a)^*, g^{-1}).$$

There is an isomorphism between $C_r^*(\mathscr{B})$ and $A \rtimes_{\alpha,r} G$.

3. [Exe97, EL97] cf. [FD88b, §VIII.4.7] Let A be a C*-algebra and G a locally compact group. A *twisted action* of G on A is a continuous map $\alpha : G \to \mathrm{Aut}(A)$ and a strictly continuous map $\sigma : G \times G \to UM(A)$ to the group of unitary multipliers of A, satisfying

$$\alpha_e = \mathrm{id}_A \qquad \alpha_g \circ \alpha_h = \mathrm{Ad}(\sigma(g,h)) \circ \alpha_{gh}$$
$$\sigma(g,e) = \sigma(e,g) = 1 \qquad \sigma(g,h)\sigma(gh,k) = \alpha_g(\sigma(h,k))\sigma(g,hk).$$

The *twisted semidirect* Fell bundle $\mathscr{B}$ is defined as $A \times G$ as a Banach bundle over G. We define multiplication on $\mathscr{B}$ by

$$(a,g)(b,h) = (a\alpha_g(b)\sigma(g,h), gh)$$

and an involution by

$$(a,g)^* = (\alpha_{g^{-1}}(a)^*\sigma(g^{-1},g)^*, g^{-1}).$$

There is an isomorphism between $C_r^*(\mathscr{B})$ and the reduced twisted crossed product C*-algebra $A \rtimes_{\alpha,r}^{\sigma} G$.

4. [FD88b, §VIII.4.8] Consider the special case of 3. when $A = \mathbb{C}$ (and so $\alpha = \mathrm{id}$). Then $\sigma : G \times G \to \mathbb{T}$ is just a 2-cocycle. The *twisted group bundle* $\mathscr{B}$ is $\mathbb{C} \times G$ as a Banach bundle over G. Multiplication on $\mathscr{B}$ is given by

$$(z,g)(w,h) = (zw\sigma(g,h), gh)$$

and involution by

$$(z,g)^* = (\overline{z}\sigma(g^{-1},g)^*, g^{-1}).$$

There is an isomorphism between $C_r^*(\mathscr{B})$ and the reduced twisted group C*-algebra $C_r^*(G,\sigma)$.

II.2.2.1 Saturated and fissured bundles

In [CNNR11, §2.1], an unbounded Kasparov module is constructed from a C*-algebra A with a circle action to its fixed-point algebra, under a certain assumption. This condition, the spectral subspace assumption, is a weakening of the condition of saturation. We will give the following generalisation, which reduces to [CNNR11, Definition 2.2] when $G = \mathbb{Z}$. In the case when G is discrete, the condition is simpler to state and Theorem II.2.14 can be proved along the lines of [CNNR11, Lemmas 2.4,8].

Definition II.2.13. cf. [CNNR11, Definition 2.2] Let $\mathscr{B}$ be a Fell bundle over a locally compact group G. Let $\hat{\mathscr{B}}$ be the continuous field of C*-algebras over G with fibre

$$\hat{B}_g = B_g \otimes_{B_e} B_{g^{-1}} = \mathrm{End}^0_{B_e}(B_g) \trianglelefteq B_e$$

at $g \in G$. The C*-algebra of its continuous sections $\Gamma_0(\hat{\mathscr{B}})$ is an ideal of $C_0(G, B_e)$. We say that $\mathscr{B}$ is *fissured* if $\Gamma_0(\hat{\mathscr{B}})$ is a complemented ideal of $C_0(G, B_e)$, that is, when there exists an ideal $J \lhd C_0(G, B_e)$ such that $C_0(G, B_e) = \Gamma_0(\hat{\mathscr{B}}) \oplus J$. If $\Gamma_0(\hat{\mathscr{B}}) = C_0(G, B_e)$, i.e. $\hat{B}_g = B_e$ for all $g \in G$, $\mathscr{B}$ is *saturated* [FD88b, Definition VIII.2.8].

For instance, all of Examples II.2.12 are saturated [FD88b, §4.3,7]. Fissuration is not to be confused with semi-saturation [Exe17, Definition 16.10(b)]. An example of a Fell bundle which is fissured but not saturated is that associated to the partial Bernoulli action of a discrete group G; see [Exe17, Definition 5.12] and also [Exe17, Proposition 5.7].

Theorem II.2.14. *Let $\mathscr{B}$ be a Fell bundle over a locally compact group G. Let*

$$\Omega : C^*_r(\mathscr{B}) \rtimes \hat{G} \to \mathrm{End}^*(L^2(\mathscr{B}))$$

be the integrated representation. Then the image of Ω is $\mathrm{End}^0(L^2(\mathscr{B}))$ if and only if $\mathscr{B}$ is fissured. Further, Ω is an isomorphism from $C^*_r(\mathscr{B}) \rtimes \hat{G}$ to $\mathrm{End}^0(L^2(\mathscr{B}))$ if and only if $\mathscr{B}$ is saturated.

To prove this, we will use a result of [Aba18]. (Note that the arXiv version [Aba18] corrects an error in the published version [Aba03] of the article.) By [Aba18, Proposition 9.1], there is a presentation of $C^*_r(\mathscr{B}) \rtimes \hat{G}$ as a C*-algebra of kernels. A compactly supported continuous function $k : G \times G \to \mathscr{B}$ such that $k(g, h) \in B_{gh^{-1}}$ is a *compactly supported kernel* of $\mathscr{B}$. By [Aba18, Proposition 6.1], the set of such kernels forms a $*$-algebra $\mathbf{k}_c(\mathscr{B})$ with the convolution product

$$(k_1 k_2)(g, h) = \int_G k_1(g, s) k_2(s, h) d\mu(s)$$

and involution $k^*(g, h) = k(h, g)^*$. There is an *integrated* representation Ω of $\mathbf{k}_c(\mathscr{B})$ on $L^2(\mathscr{B})$ given by $(\Omega(k)\xi)(g) = \int_G k(g, h)\xi(h) d\mu(h)$. There is in addition a faithful representation of $\mathbf{k}_c(\mathscr{B})$ on $L^2(\mathscr{B}) \otimes L^2(G)$ under which the norm completion of $\mathbf{k}_c(\mathscr{B})$ is isomorphic to $C^*_r(\mathscr{B}) \rtimes \hat{G}$.

Proposition II.2.15. *[Aba18, Proposition 6.9] Let $\mathscr{B}$ be a Fell bundle over a locally compact group G. Let*

$$\Omega : C^*_r(\mathscr{B}) \rtimes \hat{G} \to \mathrm{End}^*(L^2(\mathscr{B}))$$

be the integrated representation on the Hilbert B_e-module $L^2(\mathscr{B})$. There is an ideal $I \subseteq C^*_r(\mathscr{B}) \rtimes_r \hat{G}$ which is represented faithfully by Ω as $\Omega(I) = \mathrm{End}^0(L^2(\mathscr{B}))$. The ideal

$$I_c = \{k \in \mathbf{k}_c(\mathscr{B}) \mid k(g, h) = \xi(g)\eta(h)^*, \xi, \eta \in C_c(\mathscr{B})\}$$

of $\mathbf{k}_c(\mathscr{B})$ is dense in I. The kernel of Ω is equal to the annihilator ideal

$$J = \left\{ a \in C^*_r(\mathscr{B}) \rtimes \hat{G} \middle| \forall x \in I, ax = 0 \right\}.$$

In particular, the image of Ω is $\mathrm{End}^0(L^2(\mathscr{B}))$ if and only if I is a complemented ideal and Ω is an isomorphism from $C^*_r(\mathscr{B}) \rtimes \hat{G}$ to $\mathrm{End}^0(L^2(\mathscr{B}))$ if and only if $I = C^*_r(\mathscr{B}) \rtimes \hat{G}$.

In particular, [Aba18, Proposition 6.9] already implies the saturated case of Theorem II.2.14.

Proof of Theorem II.2.14. First, let

$$I_c' = \{k \in \mathbf{k}_c(\mathscr{B}) \mid k(g,h) \in B_g \otimes_{B_e} B_{h^{-1}}\}$$

An application of [FD88a, Theorem II.14.6] gives us that I_c is dense in I_c' in the inductive limit topology and so that the norm closure of I_c' is I.

Second, the algebra $C_c(G, B_e)$ acts on $\mathbf{k}_c(\mathscr{B})$ by

$$(fk)(g,h) = f(g)k(g,h) \qquad (kf)(g,h) = k(g,h)f(h) \qquad (f \in C_c(G,B_e), k \in \mathbf{k}_c(\mathscr{B})).$$

These extend to a nondegenerate injective $*$-homomorphism $\varphi : C_0(G, B_e) \to M(C_r^*(\mathscr{B}) \rtimes \hat{G})$.

Suppose that $C_r^*(\mathscr{B}) \rtimes \hat{G} = I \oplus J$ for an ideal J of $C_r^*(\mathscr{B}) \rtimes \hat{G}$. Then there exists a projection $P \in M(C_r^*(\mathscr{B}) \rtimes \hat{G})$ such that

$$P(C_r^*(\mathscr{B}) \rtimes \hat{G}) = (C_r^*(\mathscr{B}) \rtimes \hat{G})P = I.$$

At $g \in G$, P is given by the projection $P(g) \in \mathrm{End}^0_{B_e}(B_g) = \hat{B}_g$ for which

$$P(g)B_{gh^{-1}} = B_g \otimes_{B_e} B_{h^{-1}} \qquad B_{hg^{-1}}P(g) = B_h \otimes_{B_e} B_{g^{-1}}.$$

Since $P(g)B_e = B_g \otimes_{B_e} B_{g^{-1}} = \hat{B}_g$, we have

$$P\varphi(C_c(G,B_e)), \varphi(C_c(G,B_e))P \subseteq \varphi(\Gamma_c(\hat{\mathscr{B}})).$$

For $a \in \hat{B}_g$, choose $b \in B_e$ such that $P(g)b = a$. Then, choosing $f_0 \in C_c(G, B_e)$ such that $f_0(g) = b$, we obtain $f_1 \in P\varphi(C_c(G, B_e))$ such that $f_1(g) = a$ by taking

$$f_1(h) = (P\varphi(f_0))(h).$$

Similarly, we can find $f_2 \in \varphi(C_c(G, B_e))P$ such that $f_2(g) = a$. Applying [FD88a, Corollary II.14.7], we obtain that

$$P\varphi(C_0(G,B_e)) = \varphi(C_0(G,B_e))P = \varphi(\Gamma_0(\hat{\mathscr{B}})).$$

This means that $C_0(G, B_e) = \Gamma_0(\hat{\mathscr{B}}) \oplus J'$ where

$$\varphi(J') = (1-P)\varphi(C_0(G,B_e)) = \varphi(C_0(G,B_e))(1-P).$$

On the other hand, suppose that $\mathscr{B}$ is fissured. Because $\Gamma_0(\hat{\mathscr{B}})$ is a complemented ideal of $C_0(G, B_e)$, there is a projection $p \in M(C_0(G, B_e)) \cong C_b(G, M(B_e)_\beta)$ such that

$$pC_0(G,B_e) = C_0(G,B_e)p = \Gamma_0(\hat{\mathscr{B}}).$$

At $g \in G$, p is given by the projection $p(g) \in M(B_e)$ for which

$$p(g)B_e = B_e p(g) = \hat{B}_g.$$

Since

$$p(g)B_{gh^{-1}} = B_g \otimes_{B_e} B_{g^{-1}} \otimes_{B_e} B_{gh^{-1}} \subseteq B_g \otimes_{B_e} B_{h^{-1}}$$

and

$$B_g \otimes_{B_e} B_{h^{-1}} = B_g \otimes_{B_e} B_{g^{-1}} \otimes_{B_e} B_g \otimes_{B_e} B_{h^{-1}} \subseteq B_g \otimes_{B_e} B_{g^{-1}} \otimes_{B_e} B_{gh^{-1}},$$

we have $p(g)B_{gh^{-1}} = B_g \otimes_{B_e} B_{h^{-1}}$ and, similarly, $B_{hg^{-1}}p(g) = B_h \otimes_{B_e} B_{g^{-1}}$. This means that

$$\varphi(p)\mathbf{k}_c(\mathscr{B}), \mathbf{k}_c(\mathscr{B})\varphi(p) \subseteq I_c'.$$

For $a \in B_g \otimes_{B_e} B_{h^{-1}}$, choose $b \in B_{gh^{-1}}$ such that $p(g)b = a$. Then, choosing $f_0 \in \mathbf{k}_c(\mathscr{B})$ such that $f_0(g,h) = b$, we obtain $f_1 \in \varphi(p)\mathbf{k}_c(\mathscr{B})$ such that $f_1(g,h) = a$ by taking

$$f_1(h) = (\varphi(p)f_0)(h).$$

Similarly, we can find $f_2 \in \mathbf{k}_c(\mathscr{B})\varphi(p)$ such that $f_2(g,h) = a$. Applying [FD88a, Theorem II.14.6], we obtain that

$$\varphi(p)(C_r^*(\mathscr{B}) \rtimes \hat{G}) = (C_r^*(\mathscr{B}) \rtimes \hat{G})\varphi(p) = I.$$

This means that $C_r^*(\mathscr{B}) \rtimes \hat{G} = I \oplus J$ where

$$J = (1-\varphi(p))(C_r^*(\mathscr{B}) \rtimes \hat{G}) = (C_r^*(\mathscr{B}) \rtimes \hat{G})(1-\varphi(p)).$$

By Proposition II.2.15, $\Omega(C_r^*(\mathscr{B}) \rtimes \hat{G}) = \mathrm{End}^0(L^2(\mathscr{B}))$ if and only if I is a complemented ideal, in other words

$$I \oplus J = C_r^*(\mathscr{B}) \rtimes \hat{G}.$$

Hence, $I \trianglelefteq (C_r^*(\mathscr{B}) \rtimes \hat{G})$ is complemented if and only if $\Gamma_0(\hat{\mathscr{B}}) \trianglelefteq C_0(G, B_e)$ is complemented. □

Remark II.2.16. Let $A = C_r^*(\mathscr{B}) \rtimes_r \hat{G}$ as a G-C*-algebra and denote by $\underline{A}$ the same C*-algebra with the trivial G-action. Applying Baaj–Skandalis duality and the Morita equivalence between B_e and $\mathrm{End}^0(L^2(\mathscr{B}))$,

$$\begin{aligned} KK_*^{\hat{G}}(C_r^*(\mathscr{B}), B_e) &\cong KK_*^G(A, C_0(G, B_e)) \\ &\cong KK_*^G(A, C_0(G, \mathrm{End}^0(L^2(\mathscr{B})))) \\ &\cong KK_*^{\hat{G}}(A \rtimes_r G, \mathrm{End}^0(L^2(\mathscr{B}))), \end{aligned}$$

where B_e and $\mathrm{End}^0(L^2(\mathscr{B}))$ both carry the trivial $\hat{G}$- and G-actions. By forgetting the $\hat{G}$-equivariance, we also obtain an isomorphism

$$KK_*(C_r^*(\mathscr{B}), B_e) \cong KK_*(A \rtimes_r G, \mathrm{End}^0(L^2(\mathscr{B}))).$$

If $\mathscr{B}$ is saturated, A is isomorphic to $\mathrm{End}^0(L^2(\mathscr{B}))$, so

$$KK_*^{\hat{G}}(C_r^*(\mathscr{B}), B_e) \cong KK_*^G(A, C_0(G, \underline{A})) \cong KK_*^{\hat{G}}(A \rtimes_r G, A)$$

and $KK_*(C_r^*(\mathscr{B}), B_e) \cong KK_*(A \rtimes_r G, A)$. If $\mathscr{B}$ is not necessarily saturated, by Proposition II.2.15, $\mathrm{End}^0(L^2(\mathscr{B}))$ is isomorphic to an ideal of A, so we have homomorphisms

$$KK_*^{\hat{G}}(C_r^*(\mathscr{B}), B_e) \leftarrow KK_*^G(A, C_0(G, \underline{A})) \cong KK_*^{\hat{G}}(A \rtimes_r G, A)$$

and $KK_*(C_r^*(\mathscr{B}), B_e) \leftarrow KK_*(A \rtimes_r G, A)$. If $\mathscr{B}$ is fissured, by Theorem II.2.14, $\mathrm{End}^0(L^2(\mathscr{B}))$ is isomorphic to a complemented ideal in A, so there are injections

$$KK_*^{\hat{G}}(C_r^*(\mathscr{B}), B_e) \hookrightarrow KK_*^G(A, C_0(G, \underline{A})) \cong KK_*^{\hat{G}}(A \rtimes_r G, A)$$

and $KK_*(C_r^*(\mathscr{B}), B_e) \hookrightarrow KK_*(A \rtimes_r G, A)$, making $KK_*^{\hat{G}}(C_r^*(\mathscr{B}), B_e)$ and $KK_*(C_r^*(\mathscr{B}), B_e)$ direct summands in each case. Throughout, recall also that, by Remark II.1.10, $C_0(G, \underline{A})$ is G-equivariantly isomorphic to $C_0(G, A)$.

II.2.2.2 Partial cross-sectional bundles

An important method for constructing Fell bundles is to take an existing Fell bundle over a locally compact group G and form a new bundle over G/N for some closed subgroup N. We here give a brief account of these *partial cross-sectional bundles*, as introduced in [FD88b, §VIII.6]; however, we here use the reduced C*-algebraic completion rather than the L^1 completion.

Let N be a closed normal subgroup of a locally compact group G. Recall that the modular function Δ_N of N is the restriction of the modular function Δ_G of G [DE14, Corollary 1.5.5]. We choose Haar measures μ_G, μ_N, and $\mu_{G/N}$ to be normalised such that

$$\int_G f(g) d\mu_G(g) = \int_{G/N} \int_N f(gn) d\mu_N(n) d\mu_{G/N}(g)$$

for all $f \in C_c(G)$. This can always be done; see e.g. [DE14, Theorem 1.5.3]. We note that, for $f \in C_c(N)$ and $g \in G$,

$$\int f(ghg^{-1}) d\mu_N(h) = \frac{\Delta_G(g)}{\Delta_{G/N}(gN)} \int f(h) d\mu_N(h) \tag{II.2.17}$$

by [FD88a, Proposition III.13.20].

Definition II.2.18. Let N be a closed normal subgroup of a locally compact group G. Let $\mathscr{B} = (B_g)_{g\in G}$ be a Fell bundle over G. We define the restricted Fell bundle $\mathscr{B}_N = (B_g)_{g\in N}$. Over each coset $sN \in G/N$ we also define a Banach bundle $\mathscr{B}_{sN} = (B_g)_{g\in sN}$. Let $C_N = C^*_r(\mathscr{B}_N)$ be the reduced cross-sectional C*-algebra. For $\phi \in C_c(\mathscr{B}_{sN})$ and $\psi \in C_c(\mathscr{B}_{tN})$, we define $\phi\psi \in C_c(\mathscr{B}_{stN})$ by

$$(\phi\psi)(g) = \int \phi(sh)\psi(h^{-1}s^{-1}g) d\mu_N(h) = \int \phi(t^{-1}gh)\psi(h^{-1}t) d\mu_N(h) \tag{II.2.19}$$

for $g \in stN$. Again for $\phi \in C_c(\mathscr{B}_{sN})$, we define $\phi^* \in C_c(\mathscr{B}_{s^{-1}N})$ by

$$\phi^*(g) = \phi(g^{-1})^* \frac{\Delta_G(g^{-1})}{\Delta_{G/N}(g^{-1}N)} = \phi(g^{-1})^* \frac{\Delta_G(g^{-1})}{\Delta_{G/N}(s^{-1}N)} \tag{II.2.20}$$

for $g \in s^{-1}N$. For $\phi, \psi \in C_c(\mathscr{B}_{tN})$, using (II.2.17), $\phi^*\psi \in C_c(\mathscr{B}_N)$ is given by

$$\begin{aligned}(\phi^*\psi)(g) &= \int \phi(h^{-1}t)^* \psi(h^{-1}tg) \frac{\Delta_G(h^{-1}t)}{\Delta_{G/N}(tN)} d\mu_N(h) \\ &= \int \phi(th^{-1})^* \psi(th^{-1}g) \frac{\Delta_G(t^{-1})}{\Delta_{G/N}(t^{-1}N)} \frac{\Delta_G(th^{-1})}{\Delta_{G/N}(tN)} d\mu_N(h) \\ &= \int \phi(th)^* \psi(thg) d\mu_N(h)\end{aligned} \tag{II.2.21}$$

for $g \in N$. We define left and right $C_c(\mathscr{B}_N)$-valued inner products on $C_c(\mathscr{B}_{tN})$ by

$${}_{C_N}\langle \phi \mid \psi \rangle = \phi\psi^* \qquad \langle \phi \mid \psi \rangle_{C_N} = \phi^*\psi \qquad (\phi, \psi \in C_c(\mathscr{B}_{tN})).$$

By [RW98, Corollary 3.13], the completion of $C_c(\mathscr{B}_{tN})$ is a partial imprimitivity C_N-C_N-bimodule, which we call C_{tN}. We denote by $\mathscr{C}$ the bundle $(C_{tN})_{tN\in G/N}$ over G/N. Every element $y \in C_c(\mathscr{B})$ gives rise to a cross-section $\tilde{y}$ of $\mathscr{C}$ given by $\tilde{y}(tN) = y|_{tN}$. Applying [FD88a, Theorem II.13.18], we obtain that $\mathscr{C}$ is a Banach bundle and $\tilde{y} \in C_c(\mathscr{C})$. The multiplication and involution on $\mathscr{C}$ defined by (II.2.19) and (II.2.20) make $\mathscr{C}$ a Fell bundle over G/N; the continuity of these maps follows as in [FD88b, §VIII.6.4–5]. We call $\mathscr{C}$ the *reduced partial cross-sectional Fell bundle.*

Let us consider the Hilbert C_N-module $L^2(\mathscr{C})$. For $\xi, \eta \in C_c(\mathscr{C})$, we have

$$\langle \xi \mid \eta \rangle_{C_N} = \int_{G/N} \xi(sN)^* \eta(sN) d\mu_{G/N}(sN).$$

Let $y, z \in C_c(\mathscr{B})$, with their corresponding elements $\tilde{y}, \tilde{z} \in C_c(\mathscr{C})$. The inner product $\langle \tilde{y} \mid \tilde{z} \rangle_{C_N}$ is then an element of $C_c(\mathscr{B}_N)$, given by

$$\begin{aligned}\langle \tilde{y} \mid \tilde{z} \rangle_{C_N}(g) &= \int_{G/N} (\tilde{y}(sN)^* \tilde{z}(sN))(g) d\mu_{G/N}(sN) \\ &= \int_{G/N} \int_N y(sh)^* z(shg) d\mu_N(h) d\mu_{G/N}(sN) \\ &= \int_G y(s)^* z(sg) d\mu_N(h) d\mu_G(s) \\ &= (y^* z)(g)\end{aligned}$$

using (II.2.21). Let $v, \zeta \in C_c(\mathscr{B}_N) \subseteq L^2(\mathscr{B}_N)$; then yv and $z\zeta$ are elements of $C_c(\mathscr{B})$ and

$$\langle \tilde{y} \otimes v \mid \tilde{z} \otimes \zeta \rangle_{B_e} = \langle v \mid \langle \tilde{y} \mid \tilde{z} \rangle_{C_N} \zeta \rangle_{B_e} = (v^*(y^*z)\zeta)(e) = ((yv)^*(z\zeta))(e) = \langle yv \mid z\zeta \rangle_{B_e}.$$

Hence, the inner products on $L^2(\mathscr{C}) \otimes_{C_N} L^2(\mathscr{B}_N)$ and $L^2(\mathscr{B})$ agree on their common subset $C_c(\mathscr{B})$. By definition, $C_c(\mathscr{B})$ is dense in $L^2(\mathscr{B})$; by [FD88a, Theorem II.14.6], $C_c(\mathscr{B})$ is dense in $L^2(\mathscr{C})$. Hence $L^2(\mathscr{C}) \otimes_{C_N} L^2(\mathscr{B}_N)$ is isomorphic as a Hilbert B_e-module to $L^2(\mathscr{B})$. Again using [FD88a, Theorem II.14.6], we thus obtain that the C*-algebras $C_r^*(\mathscr{C})$ and $C_r^*(\mathscr{B})$ are isomorphic, both containing the dense $*$-subalgebra $C_c(\mathscr{B})$; cf. [FD88b, Proposition VIII.6.7].

Example II.2.22. cf. [FD88b, Definition VIII.6.6] Let N be a closed normal subgroup of a locally compact group G. Let $\mathscr{B}$ be the group bundle over G, of Example II.2.12.1. The reduced partial cross-sectional Fell bundle $\mathscr{C}$ over G/N is called the *(reduced) group extension bundle.* Its fibre at $N \in G/N$ is $C_r^*(N)$.

Proposition II.2.23. *cf. [FD88b, Proposition VIII.6.8] Let N be a closed normal subgroup of a locally compact group G. Let $\mathscr{B}$ be a Fell bundle over G and $\mathscr{C}$ the reduced partial cross-sectional bundle over G/N. If $\mathscr{B}$ is fissured, $\mathscr{C}$ and $\mathscr{B}_N$ are fissured and, if $\mathscr{B}$ is saturated, $\mathscr{C}$ and $\mathscr{B}_N$ are saturated.*

Proof. Suppose that $\mathscr{B}$ is fissured. That is, with $\hat{\mathscr{B}}$ the continuous field of C*-algebras over G with fibre

$$\hat{B}_g = B_g \otimes_{B_e} B_{g^{-1}}$$

at $g \in G$, the C*-algebra of its continuous sections $\Gamma_0(\hat{\mathscr{B}})$ is a complemented ideal of $C_0(G, B_e)$. It is immediate that $\mathscr{B}_N$ is fissured, and saturated if $\mathscr{B}$ is saturated. Let $\hat{\mathscr{C}}$ be the continuous field of C*-algebras over G/N with fibre

$$\hat{C}_{sN} = C_{sN} \otimes_{C_N} C_{s^{-1}N}$$

at $sN \in G/N$. We have nondegenerate injective $*$-homomorphisms $\varphi_1 : C_0(G, B_e) \to M(C_0(G, C_N))$ and $\varphi_2 : C_0(G/N, C_N) \to M(C_0(G, C_N))$. Because $\Gamma_0(\hat{\mathscr{B}})$ is a complemented ideal of $C_0(G, B_e)$, there is a projection $p \in M(C_0(G, B_e))$ such that

$$pC_0(G, B_e) = C_0(G, B_e)p = \Gamma_0(\hat{\mathscr{B}}).$$

At $g \in G$, p is given by the projection $p(g) \in M(B_e)$ for which

$$p(g)B_e = B_e p(g) = \hat{B}_g.$$

Recall from the Proof of Theorem II.2.14 that

$$B_g \otimes_{B_e} B_{g^{-1}} \otimes_{B_e} B_{gh} = B_g \otimes_{B_e} B_h = B_{gh} \otimes_{B_e} B_{h^{-1}} \otimes_{B_e} B_h$$

One can check, then, that $\hat{C}_{sN} = p(s)C_N = C_N p(s)$. Proceeding as in the Proof of Theorem II.2.14, we obtain that

$$\varphi_1(p)\varphi_2(C_0(G/N, C_N)) = \varphi_2(C_0(G/N, C_N))\varphi_1(p) = \varphi_2(\Gamma_0(\hat{\mathscr{C}}))$$

which means that $\Gamma_0(\hat{\mathscr{C}})$ is a complemented ideal of $C_0(G/N, C_N)$. If $\mathscr{B}$ is saturated, $p = 1$; hence also $\varphi_1(p) = 1$ and so $\Gamma_0(\hat{\mathscr{C}}) = C_0(G/N, C_N)$. □

II.2.3 Two unbounded Kasparov modules from weights

For the proof of the following Theorem, we take from Lemma A.3.3 a basic fact about the norm on a Hilbert module. For B a C*-algebra and E a Hilbert B-module, the norm of $\xi \in E$ is equal to

$$\|\xi\| = \sup_{\pi} \sup_{\eta \in H_\pi} \frac{\|\xi \otimes \eta\|}{\|\eta\|}$$

where the supremum is over irreducible representations π of B and $\xi \otimes \eta \in E \otimes_\pi H_\pi$.

Theorem II.2.24. *cf. [CNNR11, Proposition 2.9] Let $\mathscr{B}$ be a Fell bundle over a locally compact group G. Let V be a finite-dimensional complex vector space and $\ell : G \to \operatorname{End} V$ be a self-adjoint, proper, translation-bounded weight. If $\mathscr{B}$ is fissured,*

$$(C_r^*(\mathscr{B}), L^2(\mathscr{B}) \otimes V, M_\ell)$$

is an isometrically $\hat{G}$-equivariant unbounded Kasparov $C_r^*(\mathscr{B})$-B_e-module. The Kasparov module is even if V is $\mathbb{Z}/2\mathbb{Z}$-graded and odd otherwise.

We shall denote the class of $(C_r^*(\mathscr{B}), L^2(\mathscr{B}) \otimes V, M_\ell)$ in $KK(C_r^*(\mathscr{B}), B_e)$ by $[M_\ell]$.

Proof. First, recall the formula (II.2.11) for the action of $\hat{G}$ on $C_r^*(\mathscr{B})$ and $L^2(\mathscr{B})$. The fact that M_ℓ acts by multiplication on each fibre of $\mathscr{B}$ implies that it is isometrically equivariant. Next, recall the integrated representation $\Omega : C_r^*(\mathscr{B}) \rtimes \hat{G} \to \operatorname{End}^*(L^2(\mathscr{B}))$. Since, by the properness of ℓ, $(1+\ell^2)^{-1} \in C_0(G) \otimes \operatorname{End} V$, for all $a \in C_r^*(\mathscr{B})$,

$$a(1 + M_\ell^2)^{-1} \in \Omega(C_r^*(\mathscr{B}) \rtimes \hat{G}) \otimes \operatorname{End} V.$$

By fissuration and Theorem II.2.14,

$$\Omega(C_r^*(\mathscr{B}) \rtimes \hat{G}) \otimes \operatorname{End} V = \operatorname{End}^0(L^2(\mathscr{B})) \otimes \operatorname{End} V = \operatorname{End}^0(L^2(\mathscr{B}) \otimes V),$$

meaning that M_ℓ has locally compact resolvent.

For an element $f \in C_c(\mathscr{B})$ and a vector $\xi \in C_c(\mathscr{B}) \otimes V \subseteq L^2(\mathscr{B}) \otimes V$,

$$\begin{aligned}
([M_\ell, f]\xi)(h) &= (M_\ell f \xi)(h) - (f M_\ell \xi)(h) \\
&= \ell(h)(f\xi)(h) - \int_G f(s)(M_\ell \xi)(s^{-1}h)d\mu(s) \\
&= \ell(h) \int_G f(s)\xi(s^{-1}h)d\mu(s) - \int_G f(s)\ell(s^{-1}h)\xi(s^{-1}h)d\mu(s) \\
&= \int_G \left(\ell(h) - \ell(s^{-1}h)\right) f(s)\xi(s^{-1}h)d\mu(s).
\end{aligned}$$

Let $\pi : B_e \to B(H_\pi)$ be an irreducible representation of B_e and let $\eta \in H$, so that $\xi \otimes \eta \in (L^2(\mathscr{B}) \otimes V) \otimes_\pi H_\pi$. First,

$$\begin{aligned}\langle [M_\ell, f]\xi \otimes \eta \mid [M_\ell, f]\xi \otimes \eta\rangle &= \int_G \left\langle \eta \middle| \left\langle ([M_\ell, f]\xi)(h) \middle| ([M_\ell, f]\xi)(h) \right\rangle \eta \right\rangle d\mu(h) \\ &= \int_G \left\langle ([M_\ell, f]\xi)(h) \otimes \eta \middle| ([M_\ell, f]\xi)(h) \otimes \eta \right\rangle d\mu(h)\end{aligned}$$

where $([M_\ell, f]\xi)(h) \otimes \eta \in V \otimes B_h \otimes_\pi H$. Continuing,

$$\begin{aligned}&\langle [M_\ell, f]\xi \otimes \eta \mid [M_\ell, f]\xi \otimes \eta\rangle \\ &= \int_G \int_G \int_G \left\langle \big(\ell(h) - \ell(s^{-1}h)\big) f(s)\xi(s^{-1}h) \otimes \eta \middle| \big(\ell(h) - \ell(t^{-1}h)\big) f(t)\xi(t^{-1}h) \otimes \eta \right\rangle \\ &\qquad\qquad \times d\mu(s)d\mu(t)d\mu(h) \\ &\le \int_G \int_G \left(\int_G \left\| \big(\ell(h) - \ell(s^{-1}h)\big) f(s)\xi(s^{-1}h) \otimes \eta \right\|^2 d\mu(h) \right)^{1/2} \\ &\quad \times \left(\int_G \left\| \big(\ell(h) - \ell(t^{-1}h)\big) f(t)\xi(t^{-1}h) \otimes \eta \right\|^2 d\mu(h) \right)^{1/2} d\mu(s)d\mu(t) \\ &= \left(\int_G \left(\int_G \left\| \big(\ell(h) - \ell(s^{-1}h)\big) f(s)\xi(s^{-1}h) \otimes \eta \right\|^2 d\mu(h) \right)^{1/2} d\mu(s) \right)^2 \\ &\le \left(\int_G \|f(s)\| d\mu(s) \right)^2 \left(\int_G \|\xi(h) \otimes \eta\|^2 d\mu(h) \right) \sup_{h \in G, s \in \operatorname{supp} f} \left\| \ell(h) - \ell(s^{-1}h) \right\|^2 \\ &= \|\xi \otimes \eta\|^2 \|f\|_{L^1}^2 \sup_{h \in G, s \in \operatorname{supp} f} \left\| \ell(h) - \ell(s^{-1}h) \right\|^2,\end{aligned}$$

using the compact support of f and Lemma II.2.4. Hence

$$\begin{aligned}\left\| [M_\ell, f]\xi \right\| &= \sup_\pi \sup_{\eta \in H_\pi} \frac{\|[M_\ell, f]\xi \otimes \eta\|}{\|\eta\|} \\ &\le \sup_\pi \sup_{\eta \in H_\pi} \frac{\|\xi \otimes \eta\|}{\|\eta\|} \|f\|_{L^1} \sup_{h \in G, s \in \operatorname{supp} f} \left\| \ell(h) - \ell(s^{-1}h) \right\| \\ &= \|\xi\| \|f\|_{L^1} \sup_{h \in G, s \in \operatorname{supp} f} \left\| \ell(h) - \ell(s^{-1}h) \right\|,\end{aligned}$$

meaning that $[M_\ell, f]$ is bounded. □

Suppose that we have a fissured Fell bundle $\mathscr{B}$ and a self-adjoint, proper, translation-bounded weight $\ell : G \to \operatorname{End} V$. The unbounded Kasparov module

$$\left(C_r^*(\mathscr{B}), L^2(\mathscr{B}) \otimes V, M_\ell \right)$$

has class in $KK_*^{\hat{G}}(C_r^*(\mathscr{B}), B_e)$. Let $A = C_r^*(\mathscr{B}) \rtimes_r \hat{G}$ as a G-C*-algebra. By Remark II.2.16, there is an inclusion

$$KK_*^{\hat{G}}(C_r^*(\mathscr{B}), B_e) \hookrightarrow KK_*^G(A, C_0(G, \underline{A})) = KK_*^G(A, C_0(G, A))$$

given by

$$\mathbf{x} \mapsto J^{\hat{G}}(\mathbf{x}) \otimes_{B_e} [L^2(\mathscr{B})^*] \otimes_{\operatorname{End}^0(L^2(\mathscr{B}))} [\Omega]$$

where $\Omega : A \to \operatorname{End}^0(L^2(\mathscr{B}))$ is the integrated representation. The following Theorem gives a representative of the image of

$$\left(C_r^*(\mathscr{B}), L^2(\mathscr{B}) \otimes V, M_\ell \right)$$

under this inclusion.

Theorem II.2.25. *Let G be a locally compact group, A a C^*-algebra with an action α of G, and V a finite-dimensional complex vector space. Let $\ell : G \to \operatorname{End} V$ be a self-adjoint, proper, and translation-bounded weight. Then*

$$(A, C_0(G, A \otimes V)_{C_0(G,A)}, \ell)$$

is a G-equivariant unbounded Kasparov A-$C_0(G, A)$-module. The Kasparov module is even if V is $\mathbb{Z}/2\mathbb{Z}$-graded and odd otherwise.

Note that $(A, C_0(G, A \otimes V)_{C_0(G,A)}, \ell) = \sigma_A((\mathbb{C}, C_0(G, V)_{C_0(G)}, \ell))$. We shall write its class in $KK(A, C_0(G, A))$ as $\sigma_A([\ell])$.

Proof. First, for all $a \in A$, $a(1 + \ell^2)^{-1} \in C_0(G, A \otimes \operatorname{End} V)$ and $[\ell, a] = 0$.

For the G-equivariance, observe that, for an implementer U_g of the action and a vector $\xi \in C_c(G, A \otimes V) \subseteq C_0(G, A \otimes V)$,

$$((U_g \ell U_g^* - \ell)\xi)(h) = (\ell(gh) - \ell(h))\xi(h).$$

So

$$\left\| U_g \ell U_g^* - \ell \right\| = \sup_{h \in G} \left\| \ell(gh) - \ell(h) \right\| < \infty$$

by the translation-boundedness of ℓ. Furthermore, by Lemma II.2.4, $g \mapsto U_g \ell U_g^* - \ell$ is an element of $C(G, C_b(G, \operatorname{End} V)_\beta)$ and so $*$-strongly continuous into $\operatorname{End}^*_{C_0(G,A)}(C_0(G, A \otimes V)) = C_b(G, M(A)_\beta \otimes \operatorname{End} V)$. □

Corollary II.2.26. *Let G be a locally compact group and V be a finite-dimensional complex vector space. Let $\ell : G \to \operatorname{End} V$ be a self-adjoint, proper, and translation-bounded, continuous function. Let M_ℓ be the densely defined operator on $L^2(G, V)$ given by multiplication by ℓ. Then*

$$\left(C_r^*(G), L^2(G, V), M_\ell \right)$$

is an isometrically $\hat{G}$-equivariant spectral triple and

$$\left(\mathbb{C}, C_0(G, V)_{C_0(G)}, \ell \right)$$

is a G-equivariant unbounded Kasparov $\mathbb{C}$-$C_0(G)$-module. Both Kasparov modules are even if V is $\mathbb{Z}/2\mathbb{Z}$-graded and odd otherwise.

Remarks II.2.27.

1. If G is discrete, so that $C_r^*(G)$ is unital, and $(1 + \ell^2)^{-1/2} \in \ell^p(G, \operatorname{End} V)$, the spectral triple in Corollary II.2.26 is p-summable; cf. [Con89, Proposition 6], [Rub22, Proposition 6.8]. There is a well-known obstruction to the building of finitely summable spectral triples for discrete groups with Kazhdan's property (T), due to Connes [Con89, Proposition 19].

   However, for groups with property (T), it may nevertheless be possible to build finitely summable Fredholm modules. In [Con89, Proposition 20] a finitely summable Fredholm module is given for the C*-algebra of G a lattice in $Sp(n, 1)$, which has property (T), acting on the symmetric space $Sp(n, 1)/(Sp(n) \times Sp(1))$. We shall build an unbounded lift of this in Corollary II.3.10 and Example II.3.12. The apparent contradiction is resolved by the fact that to obtain a finitely summable Fredholm module, instead of the usual bounded transform $M_\ell \mapsto F_{M_\ell} = M_\ell \langle M_\ell \rangle^{-1}$, the phase $M_\ell |M_\ell|^{-1}$ must be used. This phenomenon is deserving of further study. A similar situation arises for Cuntz–Krieger algebras; see [GM15].

2. Corollary II.2.26 also applies to twisted group C*-algebras; see Example II.2.12.4. Remark also that
$$KK_*^{\hat{G}}(C_r^*(G,\sigma),\mathbb{C}) \cong KK_*^G(C_r^*(G,\sigma)\rtimes\hat{G}, C_0(G)) \cong KK_*^G(\mathbb{C}, C_0(G))$$
because, by the saturation of the twisted group bundle and Theorem II.2.14, $C_r^*(G,\sigma)\rtimes\hat{G}$ is isomorphic to $K(L^2(G))$ and so Morita equivalent to $\mathbb{C}$.
3. Let $\ell : G \to \operatorname{End} V$ be a self-adjoint, proper, translation-bounded weight. Fix $s \in G$. The weight $\ell' : G \to \operatorname{End} V$ given by $\ell'(g) = \ell(gs)$ is still self-adjoint, proper, and translation-bounded. However, the difference $\ell - \ell'$ may not be bounded if G is not abelian. Further, the classes $[\ell]$ and $[\ell']$ need not be the same; they are related by the right action of an element of $KK_*^G(C_0(G), C_0(G)) \cong R(\hat{G})$ on $KK_*^G(\mathbb{C}, C_0(G))$.

Example II.2.28. Continuing Example II.2.6, let G be the compactly generated locally compact abelian group $\mathbb{R}^m \times \mathbb{Z}^n \times K$, for integers m, n, and a compact group K. As before, we let $(v_i)_{i=1}^{m+n}$ be a basis of $\mathbb{R}^{m+n}$, let V be an irreducible Clifford module for $\mathscr{C}\ell_{m+n}$, and define a weight $\ell : G \to \operatorname{End} V$ by
$$\ell(g) = \sum_{j=1}^{m+n} g_j v_j$$
where g_j is a real or integer component of g. This gives rise to an isometrically $\hat{G}$-equivariant spectral triple
$$\Big(C^*(\mathbb{R}^m \times \mathbb{Z}^n \times K), L^2(\mathbb{R}^m \times \mathbb{Z}^n \times K, V), M_\ell\Big).$$
By Pontryagin duality, this spectral triple is unitarily equivalent to the isometrically $\mathbb{R}^m \times \mathbb{T}^n \times \hat{K}$-equivariant spectral triple
$$\Big(C_0(\mathbb{R}^m \times \mathbb{T}^n \times \hat{K}), L^2(\mathbb{R}^m \times \mathbb{T}^n \times \hat{K}, V), D\Big).$$
Here, D is the Fourier dual of M_ℓ, the differential operator
$$D = \sum_{j=1}^{m+n} v_j i\partial_j$$
where ∂_i is the partial derivative on a real line factor or a circle factor. Its square is
$$D^2 = -\sum_{j=1}^{m+n} \langle v_i \mid v_j\rangle \partial_i\partial_j$$
so that D is the Atiyah-Singer Dirac operator and D^2 the Laplacian on $\mathbb{R}^m \times \mathbb{T}^n \times \hat{K}$ with constant (inverse) Riemannian metric $\mathbf{g}^{ij} = \langle v_i \mid v_j\rangle$. The manifold $\mathbb{R}^m \times \mathbb{T}^n \times \hat{K}$ may have infinitely many connected components, depending on the factor K. The class of the spectral triple is therefore nontrivial in $KK_{m+n}(C_0(\mathbb{R}^m \times \mathbb{T}^n \times \hat{K}), \mathbb{C})$.

Example II.2.29. Let K be a compact group. Let ℓ be the zero weight on $V := \mathbb{C} \oplus 0$. We obtain the even spectral triple
$$(C^*(K), L^2(K) \oplus 0, M_\ell = 0),$$
which is simply the left regular representation. The class $1 \in KK_*(\mathbb{C},\mathbb{C}) \cong KK_*^K(\mathbb{C}, C(K,\mathbb{C}))$ is naturally represented by
$$(\mathbb{C}, (C(K) \oplus 0)_{C(K)}, \ell = 0)$$
corresponding, in some sense, to the action of K on a single point.

Example II.2.30. As in Example II.2.8, consider the multiplication group $k^\times$ of a local field k with absolute value $|\cdot|$ and define the weight $\ell : x \mapsto \log|x| \in \mathbb{C}$ for $k^\times$. One can check that the odd spectral triple

$$(C^*(k^\times), L^2(k^\times), M_\ell)$$

has nontrivial class in $KK_1(C^*(k^\times), \mathbb{C})$.

Example II.2.31. Again, as in Example II.2.8, consider $\mathbb{Q}$, with the usual archimedean absolute value, which is not locally compact. Equip $\mathbb{Q}^\times$ with the discrete topology. Let $\ell : \mathbb{Q}^\times \to \mathbb{C}$ be given by $\ell(x) = \log|x|$. The triple

$$(C^*(\mathbb{Q}^\times), \ell^2(\mathbb{Q}^\times), M_\ell)$$

fails to be a spectral triple only because the resolvent fails to be compact. However, consider the crossed product $C_0(\mathbb{R}^\times) \rtimes \mathbb{Q}^\times$, where $\mathbb{Q}^\times$ acts by dilation. There is a representation of $C_0(\mathbb{R}^\times) \rtimes \mathbb{Q}^\times$ on $\ell^2(\mathbb{Q}^\times)$ given by

$$(fu_y\xi)(x) = f(x)\xi(y^{-1}x) \qquad (f \in C_0(\mathbb{R}^\times), u_y \in C^*(\mathbb{Q}^\times), \xi \in \ell^2(\mathbb{Q}^\times)).$$

Because $\mathbb{Q}^\times$ is dense in $\mathbb{R}^\times$, this is a faithful representation. Furthermore, $C_0(\mathbb{R}^\times) \rtimes \mathbb{Q}^\times$ possesses a trace τ, given by

$$\tau\Big(\sum_{y\in\mathbb{Q}^\times} f_y u_y\Big) = \int_{-\infty}^{\infty} f_1(x)\frac{dx}{|x|}$$

for positive $\sum_{y\in\mathbb{Q}^\times} f_y u_y \in C_0(\mathbb{R}^\times) \rtimes \mathbb{Q}^\times$. Let N be the von Neumann enveloping algebra of the representation of $C_0(\mathbb{R}^\times) \rtimes \mathbb{Q}^\times$ on $\ell^2(\mathbb{Q}^\times)$. We remark that, by [Tak03, Theorem 1.7, Example of (iii)], N is of type II_∞. Since $C^*(\mathbb{Q}^\times) \subset N$ and

$$\tau((1+M_\ell^2)^{-p/2}) = \int_{-\infty}^{\infty} (1+(\log|x|)^2)^{-p/2}\frac{dx}{|x|} = 2\int_{-\infty}^{\infty}(1+v^2)^{-p/2}dv < \infty$$

for $p > 1$,

$$(C^*(\mathbb{Q}^\times), \ell^2(\mathbb{Q}^\times), M_\ell)$$

is a semifinite spectral triple; see [CP98, Definition 2.1].

It is likely that a similar technique could be used more generally to obtain semifinite spectral triples for group C*-algebras from weights which are self-adjoint and translation-bounded but not proper. We leave this as a topic for future investigation. We mention here also [GRU19, §2.2.4], in which semifinite spectral triples are built for group C*-algebras from an isometric action of the group on a Hilbert space; in that case the semifiniteness arises from the infinite-dimensionality of the Hilbert space.

II.2.4 Restriction and induction of weights

We remark that, if ℓ is a self-adjoint, proper, or translation-bounded weight for G, its restriction to a closed subgroup H is also. One can use the idea of induction to go the other way, at least when G/H is compact. We refer to Definition A.2.4 for the idea of a cut-off function.

Proposition II.2.32. *Let $\ell : H \to \operatorname{End} V$ be a weight for H, a closed subgroup of a locally compact group G. Let $c \in C_b(G)$ be a cut-off function for the right action of H on G. The formula*

$$\widehat{\ell}(g) = \int_H c(gs)^2 \ell(s^{-1})d\mu_H(s)$$

gives a weight $\widehat{\ell}$ on G. If ℓ is self-adjoint, $\widehat{\ell}$ is self-adjoint. Suppose that ℓ is translation-bounded. Then, for every pair K_1 and K_2 of compact subsets of G,

$$\sup_{g\in K_1, k\in K_2, u\in H} \|\widehat{\ell}(gku) - \widehat{\ell}(ku)\| < \infty.$$

In particular, if G/H is compact, $\widehat{\ell}$ is translation-bounded. Suppose further that ℓ is proper. Then, for every compact subset K of G,

$$(1+\widehat{\ell}^*\widehat{\ell})^{-1}|_{KH} \in C_0(KH, \operatorname{End} V).$$

In particular, if G/H is compact, $\widehat{\ell}$ is proper.

Proof. We first check that $\widehat{\ell} : G \to \operatorname{End} V$ is continuous. For g in a compact subset $K \subseteq G$, the integrand $c(gs)^2\ell(s^{-1})$ is zero for $s \notin H \cap K^{-1} \operatorname{supp} c$, which is a compact set. Hence, restricted to any compact subset, $\widehat{\ell}$ is equal to a convolution of compactly supported functions and so continuous. An application of Lemma A.1.7 shows that $\widehat{\ell}$ is continuous on all of G. Further, because c is positive, if ℓ is self-adjoint, $\widehat{\ell}$ is self-adjoint.

Now suppose that ℓ is translation-bounded. With $g, k \in G$ and $u \in H$, we have

$$\begin{aligned}
\widehat{\ell}(gku) - \widehat{\ell}(ku) &= \int_H c(gkus)^2\ell(s^{-1})d\mu_H(s) - \int_H c(kut)^2\ell(t^{-1})d\mu_H(t) \\
&= \int_H c(gks)^2\ell(s^{-1}u)d\mu_H(s) - \int_H c(kt)^2\ell(t^{-1}u)d\mu_H(t) \\
&= \int_H c(gks)^2(\ell(s^{-1}u) - \ell(u))d\mu_H(s) + \int_H c(kt)^2(\ell(u) - \ell(t^{-1}u))d\mu_H(t).
\end{aligned}$$

So

$$\begin{aligned}
\| \widehat{\ell}(gku) - \widehat{\ell}(ku)\| &\leq \sup_{s\in H\cap k^{-1}g^{-1}\operatorname{supp} c} \|\ell(s^{-1}u) - \ell(u)\| + \sup_{t\in H\cap k^{-1}\operatorname{supp} c} \|\ell(u) - \ell(t^{-1}u)\| \\
&\leq 2 \sup_{s\in H\cap k^{-1}\{g^{-1},e\}\operatorname{supp} c} \|\ell(s^{-1}u) - \ell(u)\|.
\end{aligned}$$

By the support property of the cut-off function c, $H \cap \operatorname{supp} c$ is compact. If K' is a compact subset of G, $H \cap K' \operatorname{supp} c$ is a closed subset of the compact set $(K' \cup \{e\})(H \cap \operatorname{supp} c)$ and so itself compact. So, with K_1 and K_2 compact subsets of G,

$$\sup_{g\in K_1, k\in K_2, u\in H} \| \widehat{\ell}(gku) - \widehat{\ell}(ku)\| \leq 2 \sup_{s\in H\cap K_2^{-1}\{K_1^{-1},e\}\operatorname{supp} c, u\in H} \|\ell(s^{-1}u) - \ell(u)\| < \infty$$

using the compactness of $H \cap K_2^{-1}\{K_1^{-1}, e\} \operatorname{supp} c$ and Lemma II.2.4. If G/H is compact, we can find a compact subset $K_2 \subseteq G$ such that $K_2 H = G$. Then

$$\sup_{g\in K_1, h\in G} \| \widehat{\ell}(gh) - \widehat{\ell}(h)\| < \infty$$

and an application of Lemma II.2.4 says that $\widehat{\ell}$ is translation-bounded.

Suppose now that ℓ is proper and translation-bounded. Let $k \in G$ and $u \in H$. We have

$$\begin{aligned}
\widehat{\ell}(ku)^* \widehat{\ell}(ku) &= \int_H \int_H c(kus)^2 c(kut)^2 \ell(s^{-1})^* \ell(t^{-1}) d\mu_H(s) d\mu_H(t) \\
&= \int_H \int_H c(ks)^2 c(kt)^2 \ell(s^{-1}u)^* \ell(t^{-1}u) d\mu_H(s) d\mu_H(t) \\
&= \int_H c(ks)^2 \ell(s^{-1}u)^* \ell(s^{-1}u) d\mu_H(s) \qquad (\text{II.2.33}) \\
&\quad - \frac{1}{2} \int_H \int_H c(ks)^2 c(kt)^2 (\ell(s^{-1}u) - \ell(t^{-1}u))^* (\ell(s^{-1}u) - \ell(t^{-1}u)) d\mu_H(s) d\mu_H(t).
\end{aligned}$$

Let K be a compact subset of G. For the second term of (II.2.33),

$$\begin{aligned}
&\sup_{k\in K, u\in H}\left\|\frac{1}{2}\int_H\int_H c(ks)^2c(kt)^2(\ell(s^{-1}u)-\ell(t^{-1}u))^*(\ell(s^{-1}u)-\ell(t^{-1}u))d\mu_H(s)d\mu_H(t)\right\| \\
&\quad\le\frac{1}{2}\sup_{s,t\in H\cap K^{-1}\operatorname{supp}c, u\in H}\|\ell(s^{-1}u)-\ell(t^{-1}u)\|^2 \\
&\quad\le\frac{1}{2}\sup_{s,t\in H\cap K^{-1}\operatorname{supp}c, u\in H}\Big(\|\ell(s^{-1}u)-\ell(u)\|+\|\ell(u)-\ell(t^{-1}u)\|\Big)^2 \\
&< \infty.
\end{aligned}$$

For the first term of (II.2.33), write $g = ku$, so that

$$\int_H c(ks)^2\ell(s^{-1}u)^*\ell(s^{-1}u)d\mu_H(s) = \int_H c(gs)^2\ell(s^{-1})^*\ell(s^{-1})d\mu_H(s).$$

Remarking that $T\mapsto\|T^{-1}\|^{-1}$ gives the smallest eigenvalue of a positive invertible matrix T,

$$\begin{aligned}
&\left\|\Big(\int_H c(gs)^2\ell(s^{-1})^*\ell(s^{-1})d\mu_H(s)\Big)^{-1}\right\|^{-1} \\
&\quad\ge\left\|\left(\int_H c(gs)^2d\mu_H(s)\right)^{-1}\right\|^{-1}\inf_{s\in H\cap g^{-1}\operatorname{supp}c}\|(\ell(s^{-1})^*\ell(s^{-1}))^{-1}\|^{-1} \\
&\quad=\inf_{t\in H\cap g^{-1}\operatorname{supp}c}\|(\ell(s^{-1})^*\ell(s^{-1}))^{-1}\|^{-1}.
\end{aligned}$$

Fix $M\ge 0$. By the properness of ℓ, the set Y of $s\in H$ such that $\|(\ell(s^{-1})^*\ell(s^{-1}))^{-1}\|^{-1}\le M$ is compact. Because H is a closed subgroup of G, Y is compact in G. Because, furthermore, H acts properly on G by right translation and $H\cap K^{-1}\operatorname{supp}c$ is compact,

$$\begin{aligned}
&\left\{g\in KH\,\middle|\,\left\|\Big(\int_H c(ks)^2\ell(s^{-1}u)^*\ell(s^{-1}u)d\mu_H(s)\Big)^{-1}\right\|^{-1}\le M\right\} \\
&\quad\subseteq\{g\in KH\mid\inf_{t\in H\cap g^{-1}\operatorname{supp}c}\|(\ell(t^{-1})^*\ell(t^{-1}))^{-1}\|^{-1}\le M\} \\
&\quad=\{g\in KH\mid Y\cap g^{-1}\operatorname{supp}c\ne\emptyset\} \\
&\quad=\{ku\in KH\mid Y\cap u^{-1}k^{-1}\operatorname{supp}c\ne\emptyset\} \\
&\quad\subseteq\{ku\in KH\mid Y\cap u^{-1}K^{-1}\operatorname{supp}c\ne\emptyset\} \\
&\quad\subseteq\{ku\in KH\mid Y\cap u^{-1}(H\cap K^{-1}\operatorname{supp}c)\ne\emptyset\}
\end{aligned}$$

is compact. Hence $(1+\widehat{\ell}^*\widehat{\ell})^{-1}|_{KH}\in C_0(KH,\operatorname{End}V)$. If G/H is compact, we can find a compact subset $K\subseteq G$ such that $KH=G$ and so $\widehat{\ell}$ is proper. □

Remark II.2.34. Let $\ell : H\to\operatorname{End}V$ be a translation-bounded weight for H, a closed subgroup of a locally compact group G. For a cut-off function $c\in C_b(G)$ for the right action of H on G, define the weight $\widehat{\ell} : G\to\operatorname{End}V$ by

$$\widehat{\ell}(g)=\int_H c(gs)^2\ell(s^{-1})d\mu_H(s).$$

Notice that,

$$\begin{aligned}
\sup_{h\in H}\|\widehat{\ell}(h)-\ell(h)\| &= \sup_{h\in H}\left\|\int_H c(hs)^2(\ell(s^{-1})-\ell(h))d\mu_H(s)\right\| \\
&= \sup_{h\in H}\left\|\int_H c(s)^2(\ell(s^{-1}h)-\ell(h))d\mu_H(s)\right\| \\
&\leq \int_H c(s)^2 d\mu_H(s) \sup_{s\in H\cap\operatorname{supp} c, h\in H}\|\ell(s^{-1}h)-\ell(h)\| \\
&= \sup_{s\in H\cap\operatorname{supp} c, h\in H}\|\ell(s^{-1}h)-\ell(h)\| \\
&< \infty
\end{aligned}$$

because $H\cap\operatorname{supp} c$ is compact and ℓ is translation-bounded. Suppose that G/H is compact. By Proposition II.2.32, $\widehat{\ell}$ is translation-bounded. And, if ℓ is also self-adjoint and proper, giving a class in $KK_*^H(A, C_0(H,\underline{A}))$ for some G-C*-algebra A, the corresponding class in $KK_*^G(A, C_0(G,\underline{A}))$ of Corollary II.1.11 is given by $\widehat{\ell}$.

Remark II.2.35. Let K be a compact group. Because the trivial subgroup is cocompact, there is an isomorphism $KK_*^K(A, C(K,A)) \cong KK_*(A,A)$. Let $\ell : K \to \operatorname{End} V$ be a self-adjoint, proper, translation-bounded, even weight. The class of

$$(A, C(K, V\otimes A)_{C(K,A)}, \ell\otimes 1)$$

corresponds to $\operatorname{ind}(\ell(e))1 \in KK_*(A,A)$, where $\operatorname{ind}(\ell(e))$ is the index of the odd matrix $\ell(e)$ with respect to the grading. Note also that, if G is a noncompact locally compact group and $\ell : G \to \operatorname{End} V$ is a proper, even weight, $\operatorname{ind}(\ell(g)) = 0$ for all $g\in G$.

II.3 Directed length functions from actions on CAT(0) spaces

The use of negatively curved manifolds in the operator-theoretic treatment of group representations is generally credited to Miščenko [Miš74]. In a form more reminiscent of Kasparov's γ-element, this project was pursued by Luke [Luk77]. Here, we will give a very general construction, which can be specialised to manifolds, trees, and CAT(0) complexes.

Definition II.3.1. [BH99, Definitions I.1.3] Let (X,d) be a metric space. A *geodesic* from $x\in X$ to $y\in X$ is a map $c:[0,l]\subset\mathbb{R}\to X$ such that

$$c(0)=x \qquad c(l)=y \qquad d(c(t),c(t'))=|t-t'| \qquad (t,t'\in[0,l]).$$

A *geodesic space* is a metric space in which every two points are joined by a geodesic. A subspace C of a metric space X is *convex* if, for every pair of points in C, there is a geodesic between them which is contained in C.

Definition II.3.2. [BH99, Definitions I.1.10,12, II.3.18] A *geodesic triangle* Δ in a metric space (X,d) consists of a triple (x,y,z) of points in X and a triple (c,c',c'') of geodesic segments joining them.

A *comparison triangle* in $\mathbb{R}^2$ for a triple (x,y,z) of distinct points in X is a triangle in the Euclidean plane with vertices $(\overline{x},\overline{y},\overline{z})$ such that

$$d(x,y)=d(\overline{x},\overline{y}) \qquad d(y,z)=d(\overline{y},\overline{z}) \qquad d(z,x)=d(\overline{z},\overline{x}).$$

A comparison triangle is unique up to isometry. The *comparison angle* between x and y at z, denoted $\overline{\angle}_z(x,y)$, is the interior angle of the comparison triangle and $\overline{z}$. The *Alexandrov angle* between two geodesics c and c' in X with $c(0)=c'(0)$ is

$$\angle(c,c')=\limsup_{t,t'\to 0}\overline{\angle}_{c(o)}(c(t),c(t')).$$

The *space of directions* $S_x(X)$ at a point $x \in X$ is the set of geodesics emanating from x, modulo the equivalence relation of zero Alexandrov angle. The Alexandrov angle thus becomes a metric on $S_x(X)$.

For many results and proofs following, for points x and y of a geodesic metric space X, we denote by $v(x, y) \in S_y(X)$ the direction of the geodesic from x to y as it reaches y.

Definition II.3.3. [BH99, Definition II.1.1, Proposition II.1.7] A geodesic space (X, d) is *CAT(0)* if either of the following equivalent conditions apply:

- For every geodesic triangle Δ in X and a comparison triangle $\overline{\Delta}$ in $\mathbb{R}^2$, $d(p, q) \leq \|\overline{p} - \overline{q}\|$ for all $p, q \in \Delta$ and their comparison points $\overline{p}, \overline{q} \in \overline{\Delta}$.
- For every geodesic triangle (c, c', c'') in X with distinct vertices, a triangle in $\mathbb{R}^2$ with two side lengths $d(c)$, $d(c')$ and interior angle $\angle(c, c')$ has its third side no longer than $d(c'')$.

Geodesics are uniquely determined by their endpoints in a CAT(0) space.

A locally compact group G is *CAT(0)* if it acts properly and cocompactly by isometries on a CAT(0) space.

A complete Riemannian manifold is a CAT(0) space if and only if it is simply connected and its sectional curvature is everywhere non-positive [BH99, Appendix to Chapter II.1]; such manifolds are called *Hadamard manifolds.* The geometric realisation of a graph is CAT(0) if and only if that graph is a tree [BH99, Example II.1.15(4)]. Certain polyhedral complexes provide other important examples of CAT(0) spaces although it is harder to formulate conditions ensuring that they are CAT(0); see [BH99, Chapter II.5].

One can also define CAT(κ) spaces for $\kappa < 0$ and $\kappa > 0$ by using comparison triangles in the real hyperbolic plane or sphere, respectively, but we shall not make use of this idea.

Proposition II.3.4. *Let G be a locally compact group acting isometrically on a CAT(0) space (X, d). Suppose that at a point $x_0 \in X$, the space of directions $S_{x_0}(X)$ is isometric to a sphere $\mathbf{S}^{n-1} \subseteq \mathbb{R}^n$. Let V be a Clifford module for the Clifford algebra $\mathscr{C}\ell_n$. Define the function $\ell : G \to \operatorname{End} V$ by*

$$\ell(g) = d(g^{-1} \cdot x_0, x_0) v(g^{-1} \cdot x_0, x_0)$$

where $v(g^{-1} \cdot x_0, x_0) \in S_{x_0}(X) \cong \mathbf{S}^{n-1} \subseteq \mathbb{R}^n \subseteq \mathscr{C}\ell_n$ acts by Clifford multiplication on V. Then ℓ is self-adjoint and translation bounded; indeed, for all $g, h \in G$, $\|\ell(gh) - \ell(h)\| \leq \|\ell(g)\|$. If G acts properly on X, ℓ is proper.

We remark that the choice of isometry between $S_{x_0}(X)$ and $\mathbf{S}^{n-1}$ is not consequential except in the matter of orientation. If $\psi_1, \psi_2 : S_{x_0}(X) \to \mathbf{S}^{n-1}$ are two isometries and $\psi_1 \circ \psi_2^{-1} \in \operatorname{Iso}(\mathbf{S}^{n-1})$ is orientation preserving, the two resulting weights will be unitarily equivalent.

Proof. We have, where the norm is in $\operatorname{End} V$,

$$\big\|\ell(gh) - \ell(h)\big\| = \Big\|d((gh)^{-1} \cdot x_0, x_0) v((gh)^{-1} \cdot x_0, x_0) - d(h^{-1} \cdot x_0, x_0) v(h^{-1} \cdot x_0, x_0)\Big\| =: L.$$

This is the length of the third side of a Euclidean triangle with the other side lengths $d((gh)^{-1} \cdot x_0, x_0)$ and $d(h^{-1} \cdot x_0, x_0)$ and the opposite angle $\arccos\langle v((gh)^{-1} \cdot x_0, x_0), v(h^{-1} \cdot x_0, x_0)\rangle$. We can compare this to the triangle in X with vertices at x_0, $(gh)^{-1} \cdot x_0$, and $h^{-1} \cdot x_0$. The CAT(0) property guarantees that L will be less than the true distance between $(gh)^{-1} \cdot x_0$ and $h^{-1} \cdot x_0$. Because G acts isometrically,

$$L \leq d(h^{-1} g^{-1} \cdot x_0, h^{-1} \cdot x_0) = d(g^{-1} \cdot x_0, x_0) = \|\ell(g)\|$$

and so $\big\|\ell(gh) - \ell(h)\big\| \leq \|\ell(g)\|$ for all $h \in G$.

The final statement follows immediately from Proposition A.2.2. $\square$

Applying Theorems II.2.24 and II.2.25, we obtain

Corollary II.3.5. *Let G be a locally compact group acting isometrically on a CAT(0) space (X, d). Let A be a G-C^*-algebra. Suppose that a point $x_0 \in X$, the space of directions $S_{x_0}(X)$ is isometric to a sphere $\mathbf{S}^{n-1} \subseteq \mathbb{R}^n$. With a Clifford module V for $\mathscr{C}\ell_n$ and the function $\ell : G \to \operatorname{End} V$ of Proposition II.3.4,*

$$(A \rtimes_r G, L^2(G, A \otimes V), M_\ell)$$

is an isometrically $\hat{G}$-equivariant unbounded Kasparov module with class $[M_\ell]$ and

$$(A, C_0(G, A \otimes V)_{C_0(G,A)}, \ell)$$

is a G-equivariant unbounded Kasparov module with class $\sigma(\ell)$. These classes are related by $[M_\ell] = J^G(\sigma_A([\ell])) \otimes_{C_0(G,A) \rtimes_r G} [L^2(G, A)] \in KK_n^{\hat{G}}(A \rtimes_r G, A)$.

Note that, by Remark II.2.27.3, a change in basepoint from x_0 to sx_0 on its orbit may not give the same class $[\ell]$; however, they are related by the right action of an element of $KK_*^G(C_0(G), C_0(G)) \cong R(\hat{G})$ on $KK_*^G(\mathbb{C}, C_0(G))$. The class $r^{\hat{G},1}([M_\ell]) = j_r^G([\ell])$ is, however, unchanged. We mention that it appears that the group invariant $\dim_{\mathbb{Q}}(KK_*^{\hat{G}}(C_r^*(G), \mathbb{C}) \otimes_{R(\hat{G})} \mathbb{Q})$ seems to reflect the structure of a CAT(0) space on which G acts. We do not yet venture to make a precise conjecture.

Example II.3.6. Continuing Examples II.2.6 and II.2.28, let G be the compactly generated locally compact abelian group $\mathbb{R}^m \times \mathbb{Z}^n \times K$, for integers m, n, and a compact group K. Let us equip $\mathbb{R}^{m+n}$ with the Euclidean metric (in terms of the standard basis). As before, let $(v_i)_{i=1}^{m+n}$ be a basis of $\mathbb{R}^{m+n}$. We shall define a proper action of G on $\mathbb{R}^{m+n}$ by translation, with K acting trivially. We write this action additively, as

$$g + x = x + \sum_{i=1}^{m+n} g_j v_j$$

where g_j is a real or integer component of g. The geodesic from $-g + 0$ to 0 is a straight line; we may think of it as the vector

$$\sum_{j=1}^{m+n} g_j v_j$$

in $\mathbb{R}^n$. Let V be a Clifford module for $\mathscr{C}\ell_{m+n}$ and, to simplify notation, write $(v_i)_{i=1}^{m+n}$ for the images of $(v_i)_{i=1}^{m+n}$ in $\mathscr{C}\ell_{m+n}$. In accordance with Proposition II.3.4, then, we define the weight $\ell : G \to \operatorname{End} V$ by

$$\ell(g) = \sum_{j=1}^{m+n} g_j v_j,$$

recovering the weight of Example II.2.6.

Theorem II.3.7. *Let G be a locally compact group acting properly and isometrically on a CAT(0) space (X, d). Let A be a G-C^*-algebra. Suppose that there is a complete subspace Y of X such that*

- *every path component of Y is a convex subset of X;*
- *Y is isometric to a spin^c Riemannian n-manifold; and*
- *Y contains a neighbourhood of a point $x_0 \in X$.*

Let $x_1 \in X$ be a point not in Y but with $S_{x_1}(X)$ isometric to a sphere $\mathbf{S}^{m-1} \subseteq \mathbb{R}^m$. Let V_0 and V_1 be Clifford modules for $\mathscr{C}\ell_n$ and $\mathscr{C}\ell_m$ respectively, with V_0 irreducible. Define the weights

$$\begin{aligned} \ell_0 : G &\to \operatorname{End} V_0 \\ g &\mapsto d(g^{-1}x_0, x_0)v(g^{-1}x_0, x_0) \end{aligned} \qquad \begin{aligned} \ell_1 : G &\to \operatorname{End} V_1 \\ g &\mapsto d(g^{-1} \cdot x_1, x_1)v(g^{-1} \cdot x_1, x_1), \end{aligned}$$

giving rise to $\sigma_A([\ell_0]), \sigma_A([\ell_1]) \in KK_*^G(A, C_0(G,A))$, and $[M_{\ell_0}], [M_{\ell_1}] \in KK_*^{\hat{G}}(A \rtimes_r G, A)$ as in Corollary II.3.5.

For any closed subgroup H of G preserving Y, let $\eta_H : C_0(Y,A)^H \to A$ be the $*$-homomorphism given by evaluating at x_0, giving a class $[\eta_H] \in KK_0(C_0(Y,A)^H, A)$. For $A = \mathbb{C}$, $[\eta_H] \in KK_0(C_0(Y/H), \mathbb{C})$ is nonzero if and only if H acts cocompactly on Y.

If there exists a closed subgroup H of G such that H preserves Y and acts by pin^c automorphisms and $[\eta_H]$ is nonzero then $\sigma_A([\ell_0]) \in KK_n^G(\mathbb{C}, C_0(G))$ is nonzero and not equal to $\sigma_A([\ell_1])$.

If G itself preserves Y, acts by spin^c automorphisms, and $[\eta_G]$ is nonzero then $r^{\hat{G},1}([M_{\ell_0}]) \in KK_n(A \rtimes_r G, A)$ is nonzero and not equal to $r^{\hat{G},1}([M_{\ell_1}])$.

We emphasise that Y may have infinitely many path components (which, since Y is a manifold, are the same as the connected components). Theorem II.3.7 will be a consequence of

Theorem II.3.8. *Let G be a locally compact group acting properly and isometrically on a CAT(0) space (X,d). Let A be a G-C^*-algebra. Suppose that there is a complete subspace Y of X such that*

- *every path component of Y is a convex subset of X;*
- *Y is isometric to a spin^c Riemannian n-manifold;*
- *Y contains a neighbourhood of a point $x_0 \in X$; and*
- *G preserves Y and acts by spin^c automorphisms.*

Let $x_1 \in X$ be a point not in Y but with $S_{x_1}(X)$ isometric to a sphere $\mathbf{S}^{m-1} \subseteq \mathbb{R}^m$. Let V_0 be the Clifford module $\$_{x_0}$ for $\mathscr{C}\ell_n$, with $\$$ the fundamental spinor bundle on Y. Let V_1 be a Clifford module for $\mathscr{C}\ell_m$. Define the weights

$$\begin{aligned} \ell_0 : G &\to \operatorname{End} V_0 & \qquad \ell_1 : G &\to \operatorname{End} V_1 \\ g &\mapsto d(g^{-1}x_0, x_0)v(g^{-1}x_0, x_0) & g &\mapsto d(g^{-1}\cdot x_1, x_1)v(g^{-1}\cdot x_1, x_1), \end{aligned}$$

giving rise to $\sigma_A([\ell_0]), \sigma_A([\ell_1]) \in KK_*^G(A, C_0(G,A))$ and $[M_{\ell_0}], [M_{\ell_1}] \in KK_*^{\hat{G}}(A \rtimes_r G, A)$ as in Corollary II.3.5.

Let $\alpha_Y \in KK_n^G(C_0(Y), \mathbb{C})$ be the Atiyah–Singer Dirac class and let ${}^G C_0(Y,A)$ be the partial imprimitivity $C_0(Y,A)^G$-$C_0(Y,A) \rtimes G$-bimodule of Theorem A.2.6. With $\eta_G : C_0(Y,A)^G \to A$ the $*$-homomorphism given by evaluating at x_0,

$$[{}^G C_0(Y,A)] \otimes_{C_0(Y,A)\rtimes G} j_r^G(\sigma_A(\alpha_Y)) \otimes_{A\rtimes_r G} r^{\hat{G},1}([M_{\ell_0}]) = [\eta_G] \in KK_0(C_0(Y,A)^G, A)$$

and

$$[{}^G C_0(Y,A)] \otimes_{C_0(Y,A)\rtimes G} j_r^G(\sigma_A(\alpha_Y)) \otimes_{A\rtimes_r G} r^{\hat{G},1}([M_{\ell_1}]) = 0 \in KK_{m+n}(C_0(Y,A)^G, A).$$

We use here the idea of the Baum–Connes assembly map; see e.g. [Val02, §6.2]. Indeed, suppose that $A = \mathbb{C}$ and Y/G is compact. Let $\lambda : \mathbb{C} \to C(Y/G)$ the inclusion given by the unit. Then

$$[\lambda] \otimes_{C(Y/G)} [{}^G C_0(Y)] \otimes_{C_0(Y)\rtimes G} j_r^G(\alpha_Y) \in KK_n(\mathbb{C}, C_r^*(G))$$

is the result of the Baum–Connes assembly map applied to α_Y.

We shall complete the Proof of Theorem II.3.8 in §II.3.4, on p. 79, but, for now, let us show how it implies Theorem II.3.7.

Proof of Theorem II.3.7. First, we remark that, since Y is complete, it is closed. Further, since each path component Z of Y is a convex subspace of a CAT(0) space, Z is also CAT(0). Since Z is furthermore complete and (isometric to) a Riemannian manifold, it is a Hadamard manifold.

Next, if $H \neq G$, we may use the injection $\phi : H \hookrightarrow G$, as in Corollary II.1.11, to obtain classes

$$\phi^* r^\phi(\sigma_A([\ell_0])), \phi^* r^\phi(\sigma_A([\ell_1])) \in KK^H(\mathbb{C}, C_0(H)).$$

Of course, if $\phi^* r^\phi(\sigma_A([\ell_0]))$ is nontrivial, $\sigma_A([\ell_0])$ is nontrivial and, if $\phi^* r^\phi(\sigma_A([\ell_0])) \neq \phi^* r^\phi(\sigma_A([\ell_1]))$, $\sigma_A([\ell_0]) \neq \sigma_A([\ell_1])$. Since H still acts properly and isometrically on (X, d), we may assume without loss of generality, for the rest of the proof, that $G = H$.

Second, suppose that G does not preserve the spin^c structure on Y but only the pin^c structure. As discussed in §II.1.1, G has an index 2 subgroup G_+ which preserves the spin^c structure. Let $\iota : G_+ \to G$ be the inclusion map. As above, it will suffice to show that $\iota^* r^\iota(\sigma_A([\ell_0]))$ is nontrivial and $\iota^* r^\iota(\sigma_A([\ell_0])) \neq \iota^* r^\iota(\sigma_A([\ell_1]))$. Since G_+ still acts cocompactly, properly, and isometrically, we may assume without loss of generality, that $G_+ = G$, that is, G acts by spin^c automorphisms on Y.

Let us also assume that $V_0 = \not{S}_{x_0}$, with $\not{S}$ the fundamental spinor bundle on Y; otherwise we reverse the spin^c structure on Y; see §II.1.1. We are now in the situation of Theorem II.3.8. If $[\eta_G] \in KK_0(C_0(Y, A)^G, A)$ is nonzero, $r^{\hat{G},1}([M_{\ell_0}])$ is nonzero and not equal to $r^{\hat{G},1}([M_{\ell_1}])$. □

We also obtain the following Corollary of Theorem II.3.8.

Corollary II.3.9. *Let G be a locally compact group acting properly and isometrically on a CAT(0) space (X, d). Let A be a G-C^*-algebra. Suppose that there is a complete subspace Y of X such that*

- *every path component of Y is a convex subset of X;*
- *Y is isometric to a spin^c Riemannian n-manifold;*
- *Y contains a neighbourhood of a point $x_0 \in X$; and*
- *G preserves Y and acts by spin^c automorphisms.*

Let $\alpha_Y \in KK_n^G(C_0(Y), \mathbb{C})$ be the Atiyah–Singer Dirac class. Let ${}^G C_0(Y, A)$ be the partial imprimitivity $C_0(Y, A)^G$-$C_0(Y, A) \rtimes G$-bimodule of Theorem A.2.6. Let $\eta_G : C_0(Y, A)^G \to A$ the $*$-homomorphism given by evaluating at x_0, defining a class $[\eta_G] \in KK_0(C_0(Y, A)^G, A)$. If $[\eta_G]$ is nonzero then

$$[{}^G C_0(Y, A)] \otimes j_r^G(\sigma_A(\alpha_Y)) \in KK_n(C_0(Y, A)^G, A \rtimes_r G)$$

is nonzero.

In particular, suppose that $A = \mathbb{C}$ and Y/G is compact, so that $[\eta_G] \in KK_0(C(Y/G), \mathbb{C})$ is nonzero. With $\lambda : \mathbb{C} \to C(Y/G)$ the inclusion given by the unit, the result

$$[\lambda] \otimes_{C(Y/G)} [{}^G C_0(Y)] \otimes_{C_0(Y) \rtimes G} j_r^G(\alpha_Y) \in KK_n(\mathbb{C}, C_r^*(G))$$

of the Baum–Connes assembly map applied to α_Y is nonzero.

Instead of proceeding immediately to prove Theorem II.3.8, we give a number of examples showing its application, in §§II.3.1, II.3.2, and II.3.3. In §II.3.4, we give a number of Lemmas for the Proof of Theorem II.3.8, which finally appears on p. 79.

II.3.1 Hadamard manifolds

Recall that a Hadamard manifold is a simply connected complete Riemannian manifold with non-positive sectional curvature. Let us restate Theorem II.3.7 in the context of Hadamard manifolds. For a Riemannian manifold X, we make the natural identification of the space of directions $S_x X$ at $x \in X$ with the unit cotangent sphere at $x \in X$, so that $v(x, y) \in T_y^* X$.

We have the following Corollary of Theorem II.3.7.

Corollary II.3.10. *Let G be a locally compact group acting isometrically on a spin^c Hadamard n-manifold X. Pick a point $x_0 \in X$. Let V be an irreducible Clifford module for the Clifford algebra $\mathscr{C}\ell(T^*_{x_0}X)$ of the cotangent space at x_0. Define the self-adjoint weight $\ell : G \to \operatorname{End} V$ by*

$$\ell(g) = d(g^{-1} \cdot x_0, x_0) v(g^{-1} \cdot x_0, x_0).$$

Then ℓ is translation-bounded.

Suppose further that G acts properly on X. Let A be a G-C^*-algebra. We obtain an isometrically $\hat{G}$-equivariant unbounded Kasparov module

$$(A \rtimes_r G, L^2(G, V \otimes A)_A, M_\ell)$$

representing $[M_\ell] \in KK^{\hat{G}}_n(A \rtimes_r G, A)$ and a G-equivariant unbounded Kasparov module

$$(A, C_0(G, A \otimes V)_{C_0(G,A)}, \ell)$$

representing $\sigma_A([\ell]) \in KK^G_n(A, C_0(G, A))$.

For any closed subgroup H of G, let $\eta_H : C_0(X, A)^H \to A$ be the homomorphism given by evaluating at x_0, giving a class $[\eta_H] \in KK_0(C_0(X, A)^H, A)$. For $A = \mathbb{C}$, $[\eta_H] \in KK_0(C_0(X/H), \mathbb{C})$ is nonzero if and only if H acts cocompactly on X.

If there exists a closed subgroup H of G which acts by pin^c automorphisms and such that $[\eta_H]$ is nonzero then $\sigma_A([\ell_0]) \in KK^G_*(\mathbb{C}, C_0(G))$ is nonzero.

If G itself acts by spin^c automorphisms and $[\eta_G]$ is nonzero then $r^{\hat{G},1}([M_\ell]) \in KK_n(A \rtimes_r G, A)$ is nonzero.

Example II.3.11. Consider the semidirect product $\mathbb{R} \rtimes_\varphi \mathbb{R}$, where, for $t \in \mathbb{R}$, $\varphi(t)$ is the automorphism $x \mapsto e^t x$ of $\mathbb{R}$. The group $\mathbb{R} \rtimes_\varphi \mathbb{R}$ is of course isomorphic to the affine group of the real line. There is an isometric action of $\mathbb{R} \rtimes_\varphi \mathbb{R}$ on the real hyperbolic plane $\mathbb{R}\mathbf{H}^2$. In terms of Poincaré half-plane model, this left action is given by

$$(x, s) \cdot z = x + e^s z \qquad (z \in \mathbb{C}).$$

This is an action by Möbius transformations, which can be seen by the injection $\mathbb{R} \rtimes_\varphi \mathbb{R} \hookrightarrow SL(2, \mathbb{R})$ given by

$$(x, s) \mapsto \begin{pmatrix} e^{s/2} & e^{-s/2}x \\ 0 & e^{-s/2} \end{pmatrix}.$$

To define a weight on $\mathbb{R} \rtimes_\varphi \mathbb{R}$ using Corollary II.3.10, it will actually be easier to view $\mathbb{R}\mathbf{H}^2$ in terms of the Poincaré disc model. Let $C = \left(\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right) \in SL(2, \mathbb{R})$. The Möbius transform defined by C maps the upper half-plane conformally to the unit disc. We compute that

$$C(e^{-s}(-x + i)) = \frac{e^{-s}(-x + i) - i}{e^{-s}(-x + i) + i} = \frac{x - i + e^s i}{x - i - e^s i} = \frac{x^2 + 1 - e^{2s} + 2xe^s i}{x^2 + (1 + e^s)^2}$$

and

$$|C(e^{-s}(-x + i))| = \frac{\sqrt{(x^2 + (1 - e^s)^2)(x^2 + (1 + e^s)^2)}}{x^2 + (1 + e^s)^2} = \sqrt{\frac{x^2 + (1 - e^s)^2}{x^2 + (1 + e^s)^2}}.$$

The distance in the hyperbolic metric from 0 in the Poincaré disc to z is given by $d(0, z) = 2 \operatorname{arctanh} |z|$. Choosing the basepoint 0 in the Poincaré disc, one can use Corollary II.3.10 to produce the weight $\ell : SL(2, \mathbb{R}) \to \mathscr{C}\ell_2 \cong \operatorname{End} \mathbb{C}^2$ given by

$$\ell(x, s) = \frac{2xe^s \gamma_1 - (x^2 + 1 - e^{2s})\gamma_2}{\sqrt{4x^2 e^{2s} + (x^2 + 1 - e^{2s})^2}} 2 \operatorname{arctanh} \sqrt{\frac{x^2 + (1 - e^s)^2}{x^2 + (1 + e^s)^2}}.$$

Example II.3.12. Let G be a reductive Lie group and K its maximal compact subgroup. We assume that G/K admits a G-equivariant spin^c structure. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition, giving a diffeomorphism $\pi \circ \exp : \mathfrak{p} \to G/K$. The Killing form B on $\mathfrak{g}$ restricts to a (positive definite) inner product on $\mathfrak{p}$. By e.g. [Hel01, Theorem IV.3.3(iii)], for any $X \in \mathfrak{p}$, $\pi \circ \exp(tX)$ is a geodesic on G/K passing through eK with speed $\|X\| = B(X, X)^{1/2}$. With $x = eK$ and $g = k\exp(X) \in K\exp(\mathfrak{p})$, $d(g^{-1}K, eK) = \|X\|$ and $v(g^{-1}K, eK) = X\|X\|^{-1} \in \mathfrak{p} \subseteq \mathscr{C}\ell(\mathfrak{p}, B)$. Hence, we can take ℓ to be given by $\ell(k\exp(X)) = X$.

For any G-C*-algebra A, we obtain that $\sigma_A([\ell])$ and $r^{\hat{G},1}([M_\ell]) \in KK_n(A \rtimes_r G, A)$ are nonzero. Let Γ be a closed cocompact subgroup of G and $\ell|_\Gamma$ the restriction of ℓ to Γ. We similarly obtain that $\sigma_A([\ell|_\Gamma])$ and $r^{\hat{\Gamma},1}([M_{\ell|_\Gamma}]) \in KK_n(A \rtimes_r \Gamma, A)$ are nonzero.

The following Remark, as well as placing our construction of directed weights in context, will be important for the proof of Theorem II.3.8.

Remark II.3.13. Let G be a locally compact group acting by spin^c isometries on a spin^c Hadamard n-manifold X. Fix $x_0 \in X$ and let $\rho : X \to [0, \infty)$ be given by $\rho(x) = d(x_0, x)$. Recall from §II.1.1 that the dual Dirac element $\beta \in KK_n^G(\mathbb{C}, C_0(X))$ is represented by the uniformly G-equivariant unbounded Kasparov module

$$(\mathbb{C}, \Gamma_0(X, \not{S})_{C_0(X)}, \rho d\rho)$$

where $\rho d\rho \in \Omega^1 X$ acts on $\Gamma_c(X, \not{S})$ by Clifford multiplication. As a section of T^*X, $d\rho$ is given by $(d\rho)(x) = v(x_0, x) \in T^*_x X$. Hence, also as a section of T^*X, $\rho d\rho$ is given by $(\rho d\rho)(x) = d(x_0, x)v(x_0, x)$.

Suppose that G acts properly on X. Let $\omega : C_0(X) \to C_0(G)$ be the G-equivariant $*$-homomorphism given by

$$\omega(f)(g) = f(g \cdot x_0),$$

i.e. evaluation on the orbit of x_0. We extend ω to a map $\Gamma_0(X, \not{S}) \to C_0(G, \not{S}_{x_0})$ by

$$\omega(\sigma)(g) = g^{-1} \cdot \sigma(g \cdot x_0).$$

Applied to $\rho d\rho$,

$$\omega(\rho d\rho)(g) = d(x_0, g \cdot x_0)g^{-1} \cdot v(x_0, g \cdot x_0) = d(g^{-1} \cdot x_0, x_0)v(g^{-1} \cdot x_0, x_0).$$

That is, with $V = \not{S}_{x_0}$ and ℓ as in Corollary II.3.10, $\omega(\rho d\rho) = \ell$. We thereby obtain that

$$\beta \otimes_{C_0(X)} [\omega] = [\ell],$$

where $[\omega] \in KK_0^G(C_0(X), C_0(G))$.

II.3.2 Trees

Given a tree Γ, we equip its geometric realisation $|\Gamma|$ with the standard metric, in which edge is taken to be isometric to the unit interval $[0, 1]$. With this metric, $|\Gamma|$ is a CAT(0) space. For ease of exposition, we shall conflate Γ with its geometric realisation $|\Gamma|$.

By the *bi-infinite line*, we mean the tree Circ_∞ with vertex set $(\mathrm{Circ}_\infty)^0 = \mathbb{Z}$ and edge set $(\mathrm{Circ}_\infty)^1 = \mathbb{Z}$, with $o(n) = n$ and $t(n) = n + 1$. In other words, Circ_∞ is the Cayley graph of $\mathbb{Z}$ with the generator $1 \in \mathbb{Z}$.

We have the following Corollary of Theorem II.3.7.

Corollary II.3.14. *Let G be a locally compact group acting on a tree Γ. Choose a point $x_0 \in \Gamma$ in the interior of an edge. Identify the space of directions at x_0 with $\mathbf{S}^0 = \{+1, -1\}$. Define the function $\ell : G \to \mathbb{C}$ by*

$$\ell_0(g) = d(g^{-1} \cdot x_0, x_0)v(g^{-1} \cdot x_0, x_0).$$

Then ℓ is translation bounded.

Suppose, further, that G acts properly on Γ or, equivalently, that the stabiliser group of every vertex is compact. Let A be a G-C-algebra. We obtain an isometrically $\hat{G}$-equivariant unbounded Kasparov module*

$$(A \rtimes_r G, L^2(G, A)_A, M_{\ell_0})$$

representing $[M_{\ell_0}] \in KK_1^{\hat{G}}(A \rtimes_r G, A)$ and a G-equivariant unbounded Kasparov module

$$(A, C_0(G, A)_{C_0(G,A)}, \ell_0)$$

representing $\sigma_A([\ell_0]) \in KK_1^G(A, C_0(G, A))$.

Let Y be a subgraph of Γ, containing x_0 and isomorphic to the disjoint union of copies of the bi-infinite line. (Note that this could be anything from one copy to infinitely many copies.) Let $x_1 \in \Gamma$ be a point in the interior of an edge not in Y and define ℓ_1 analogously to ℓ_0.

For any closed subgroup H of G preserving Y, let $\eta_H : C_0(Y, A)^H \to A$ be the $*$-homomorphism given by evaluating at x_0, giving a class $[\eta_H] \in KK_0(C_0(Y, A)^H, A)$. For $A = \mathbb{C}$, $[\eta_H] \in KK_0(C_0(Y/H), \mathbb{C})$ is nonzero if and only if H acts cocompactly on Y.

If there exists a closed subgroup H of G such that H preserves Y and $[\eta_H]$ is nonzero then $\sigma_A([\ell_0]) \in KK_1^G(\mathbb{C}, C_0(G))$ is nonzero and not equal to $\sigma_A([\ell_1])$.

If G itself preserves Y and preserves the orientation on Y, and $[\eta_G]$ is nonzero, then $r^{\hat{G},1}([M_{\ell_0}]) \in KK_1(A \rtimes_r G, A)$ is nonzero and not equal to $r^{\hat{G},1}([M_{\ell_1}])$.

Our construction bears a strong superficial resemblance to the element $\gamma \in KK_0^G(\mathbb{C}, \mathbb{C})$ built in [JV84] for a locally compact group G acting on a tree. The dual Green–Julg map gives an element $\Psi^G(\gamma) \in KK_0^G(C_u^*(G), \mathbb{C})$. Let τ be the quotient map $C_u^*(G) \to C_r^*(G)$. Then, with the class $[\ell_0]$ of Corollary II.3.14, $\tau^*([\ell_0])$ is an element of $KK_1^G(C_u^*(G), \mathbb{C})$. Of course, because the parities are different, $\tau^*([\ell_0])$ is not equal to $\Psi^G(\gamma)$. But we can say more: if G is an infinite discrete group, by Remark II.1.17, the map

$$(\Psi^G)^{-1} \circ \tau_G^* \circ r^{\hat{G},1} : KK_*^{\hat{G}}(C_r^*(G), \mathbb{C}) \to KK_*^G(\mathbb{C}, \mathbb{C})$$

does not have γ in its image. Hence, no spectral triple for the C*-algebra of G built using a weight as in §II.2.3 could represent $\Psi^G(\gamma)$.

In a number of examples, we shall discuss the relationship of Corollary II.3.14 to the Pimsner exact sequences of [Pim86], which we outlined in §II.1.2. Let us first make contact with the extension classes (II.1.5) and (II.1.7). Let G be a locally compact group acting on a tree Γ. Denote by Σ the quotient graph Γ/G. Recall from §II.1.2 that for edges $y \in \Sigma^1$, the injections $\sigma_{\overline{y}} : G_y \to G_{o(y)}$ and $\sigma_y : G_y \to G_{t(y)}$ have open image and give rise to homomorphisms $\sigma_{\overline{y}} : C_r^*(G_y) \to C_r^*(G_{o(y)})$ and $\sigma_y : C_r^*(G_y) \to C_r^*(G_{t(y)})$. Let

$$\sigma^* = \sum_{y \in \Sigma^1} (\sigma_y^* - \sigma_{\overline{y}}^*).$$

If the action of G on the Γ is proper or, equivalently, all stabiliser groups are compact, the six-term exact sequence for K-homology (II.1.8) becomes

$$0 \longrightarrow K^0(C_r^*(G)) \hookrightarrow \bigoplus_{P \in \Sigma^0} K^0(C_r^*(G_P)) \xrightarrow{\sigma^*} \bigoplus_{y \in \Sigma^1} K^0(C_r^*(G_y)) \twoheadrightarrow K^1(C_r^*(G)) \longrightarrow 0 \ , \tag{II.3.15}$$

implying that $K^0(C_r^*(G)) \cong \ker \sigma^*$ and $K^1(C_r^*(G)) \cong \operatorname{coker} \sigma^*$.

Remark II.3.16. Let G be a locally compact group acting on a tree X. Recall from §II.1.2 the six-term exact sequence

$$\begin{array}{ccccc} KK_0^G(C_0(X^1), C_0(G)) & \longleftarrow & KK_0^G(C_0(X^0), C_0(G)) & \longleftarrow & KK_0^G(\mathbb{C}, C_0(G)) \\ \downarrow & & & & \uparrow \\ KK_1^G(\mathbb{C}, C_0(G)) & \longrightarrow & KK_1^G(C_0(X^0), C_0(G)) & \longrightarrow & KK_1^G(C_0(X^1), C_0(G)) \end{array}$$

and the associated extension class, given as a G-equivariant bounded Kasparov module by

$$(\mathbb{C}, C_0(X^1)_{C_0(X^1)}, 2\chi_P - 1).$$

The multiplier $2\chi_P - 1$ is the function on X^1 with value $+1$ on X^1_P and -1 elsewhere. In other words,

$$(2\chi_P - 1)(y) = v(P, y).$$

Suppose that G acts properly on Γ. Let $\omega : C_0(\Gamma^1) \to C_0(G)$ be the G-equivariant $*$-homomorphism given by

$$\omega(f)(g) = f(g \cdot x_0),$$

i.e. evaluation on the orbit of x_0. Applied to $2\chi_P - 1$,

$$\omega(2\chi_P - 1)(g) = v(P, g \cdot x_0) = v(g^{-1} \cdot P, x_0).$$

That is, with $V = \$_{x_0}$ and ℓ as in Corollary II.3.10, $\omega(\rho d\rho)$ is equal to $\ell|\ell|$ up to a difference in $C_0(G, \operatorname{End} V)$. We thereby obtain that

$$[\ell] = [(\mathbb{C}, C_0(X^1)_{C_0(X^1)}, 2\chi_P - 1)] \otimes_{C_0(X)} [\omega]$$

where $[\omega] \in KK_0^G(C_0(X), C_0(G))$.

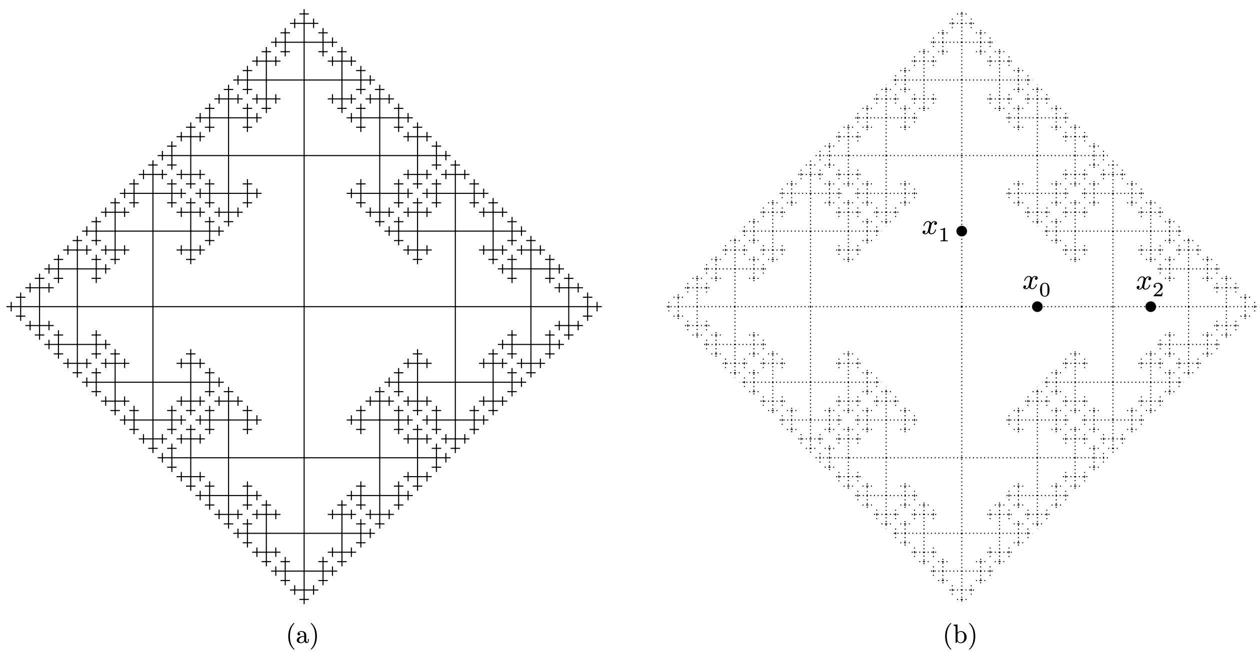


Figure II.1: The Cayley graph of F_2 and some points on it.

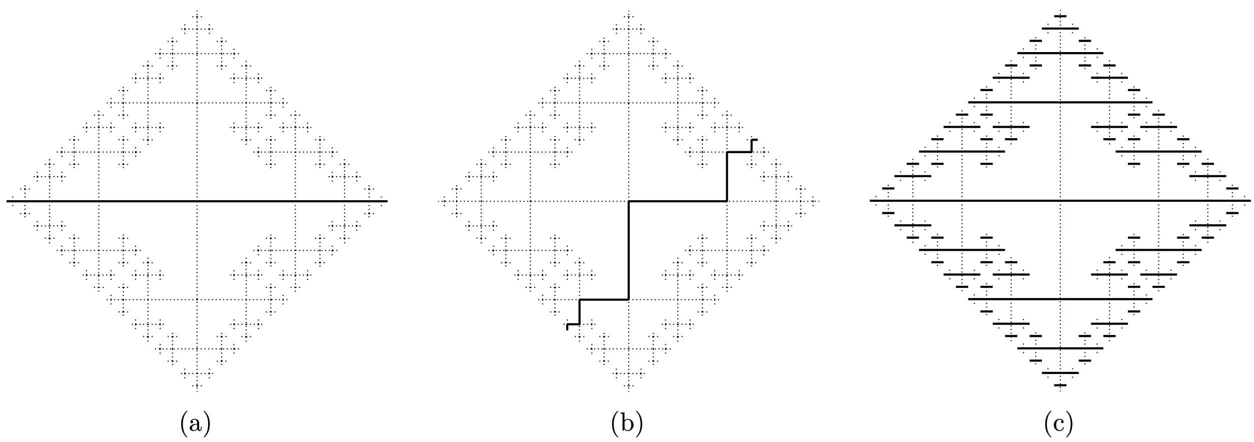


Figure II.2: Subgraphs for F_2

Example II.3.17. Consider the free group F_n on n generators. The Cayley graph of F_n is a tree Γ, on which every vertex has order n. The quotient graph consists of one vertex with n loops. For instance, for F_2, the Cayley graph is pictured in Figure II.1(a) (although not isometrically and only up to a certain resolution). For $G = F_n$ acting on its Cayley graph, the sequence (II.3.15) becomes

$$0 \longrightarrow \mathbb{Z} \hookrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}^n \twoheadrightarrow \mathbb{Z}^n \longrightarrow 0 \ .$$

The group C*-algebra of F_n has K-theory and K-homology

$$K_0(C_r^*(F_n)) = \mathbb{Z} = K^0(C_r^*(F_n)) \qquad K_1(C_r^*(F_n)) = \mathbb{Z}^n = K^1(C_r^*(F_n));$$

see [Cun83, §3(1)].

For simplicity, let us restrict the discussion to F_2, generated by a and b. The graph Γ is pictured in Figure II.1(a); we take a to move the central horizontal line one step rightward and b to move the central vertical line one step upward. Let x_0, x_1, and x_2 be the points on Γ shown in Figure II.1(b) and define ℓ_0, ℓ_1, and ℓ_2 from each, as in Corollary II.3.14. The bi-infinite line in Figure II.2(a) is preserved by the subgroup $\langle a\rangle \cong \mathbb{Z}$. The bi-infinite line in Figure II.2(b) is preserved by the subgroup $\langle ab\rangle \cong \mathbb{Z}$. By taking Y to be, in turn, each of the bi-infinite lines in Figures II.2(a) and II.2(b), Corollary II.3.14 implies that $[\ell_0]$, $[\ell_1]$, and $[\ell_2]$ are all nonzero and distinct. By considering the family of bi-infinite lines in Figure II.2(c), preserved by F_2, we obtain that $r^{\hat{F}_2,1}([M_{\ell_0}])$ is nonzero and not equal to $r^{\hat{F}_2,1}([M_{\ell_1}])$. Indeed, $r^{\hat{F}_2,1}([M_{\ell_0}])$ and $r^{\hat{F}_2,1}([M_{\ell_1}])$ generate $K^0(C_r^*(F_2)) \cong \mathbb{Z}^2$.

The construction of Corollary II.3.14 admits of an interpretation in term of words in the group. Any element of the free group F_2 can be written as a unique shortest word. Let the weights $\ell_a, \ell_b : F_n \to \mathbb{C}$ be given on a word w by

$$\ell_a(w) = \begin{cases} +|w| & w \text{ ends in } a \\ -|w| & \text{otherwise} \end{cases} \qquad \ell_b(w) = \begin{cases} +|w| & w \text{ ends in } b \\ -|w| & \text{otherwise} \end{cases} .$$

One can verify that $\ell_a = \ell_0$ and $\ell_b = \ell_1$.

One could more generally interpret Corollary II.3.14 in terms of words on a *graph of groups* [Ser80, §I.4–5]. We do not pursue this here.

Example II.3.18. Let $m, n > 1$. The free product $\mathbb{Z}/m\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$ acts on the infinite biregular tree Γ of valencies m and n. The group acts transitively on the set of edges. For example, $\mathbb{Z}/4\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ acts on the tree pictured in Figure II.3(a). The generating subgroups $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ stabilise the points thus marked in the Figure. For $\mathbb{Z}/m\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$, the sequence (II.3.15) becomes

$$0 \longrightarrow \mathbb{Z}^{m+n-1} \hookrightarrow \mathbb{Z}^m \oplus \mathbb{Z}^n \xrightarrow{(1,-1)} \mathbb{Z} \twoheadrightarrow 0 \longrightarrow 0 \ .$$

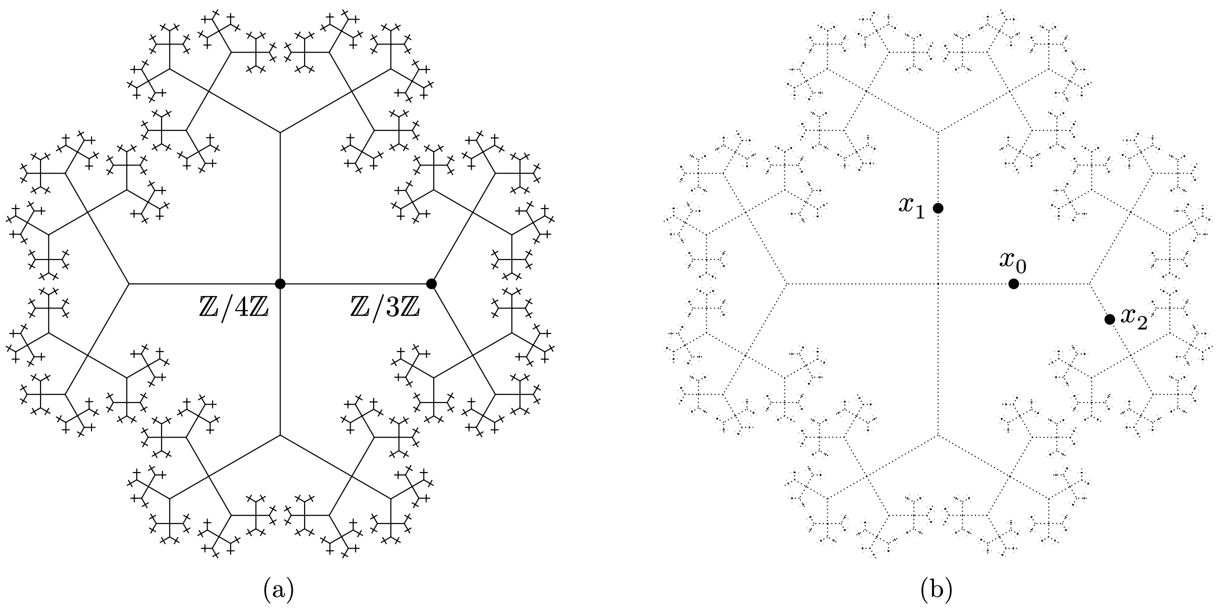


Figure II.3: The tree associated with $\mathbb{Z}/4\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ and some points on it.

The group C*-algebra of $\mathbb{Z}/m\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$ has K-theory and K-homology

$$K_0(C_r^*(\mathbb{Z}/m\mathbb{Z} * \mathbb{Z}/n\mathbb{Z})) = \mathbb{Z}^{m+n-1} = K^0(C_r^*(\mathbb{Z}/m\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}))$$
$$K_1(C_r^*(\mathbb{Z}/m\mathbb{Z} * \mathbb{Z}/n\mathbb{Z})) = 0 = K^1(C_r^*(\mathbb{Z}/m\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}));$$

see [Cun83, §3(2)].

For simplicity, let us restrict the discussion to $\mathbb{Z}/4\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, generated by $a \in \mathbb{Z}/4\mathbb{Z}$ and $b \in \mathbb{Z}/3\mathbb{Z}$. The tree Γ is pictured in Figure II.3(a); we take a to rotate anticlockwise around the point marked $\mathbb{Z}/4\mathbb{Z}$ and b to rotate anticlockwise around the point marked $\mathbb{Z}/3\mathbb{Z}$. Let x_0, x_1, and x_2 be the points on Γ shown in Figure II.3(b) and define ℓ_0, ℓ_1, and ℓ_2 from each, as in Corollary II.3.14. The bi-infinite line in Figure II.4(a) is preserved by the subgroup $\langle ab\rangle \cong \mathbb{Z}$. The bi-infinite line in Figure II.4(b) is preserved by the subgroup $\langle ab^2\rangle \cong \mathbb{Z}$. The bi-infinite line in Figure II.4(c) is preserved by the subgroup $\langle a^2b\rangle \cong \mathbb{Z}$. By taking Y to be, in turn, each of these bi-infinite lines, Corollary II.3.14 implies that $[\ell_0]$, $[\ell_1]$, and $[\ell_2]$ are all nonzero and distinct.

Because the action of $\mathbb{Z}/4\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ on its tree is transitive on the edges, we cannot use Corollary II.3.14 to detect the nontriviality of $r^{\mathbb{Z}/4\widehat{\mathbb{Z}*\mathbb{Z}}/3\mathbb{Z},1}([M_{\ell_0}]) \in KK_1(C_r^*(\mathbb{Z}/4\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}), \mathbb{C})$. Indeed, since

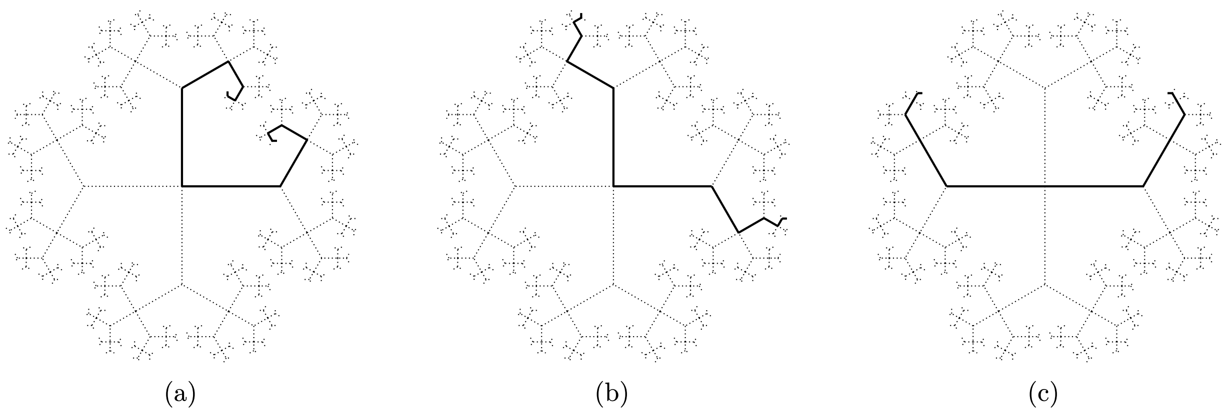


Figure II.4: Subgraphs for $\mathbb{Z}/4\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

$KK_1(C_r^*(\mathbb{Z}/4\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}), \mathbb{C}) = 0$, we conclude that

$$r^{\widehat{\mathbb{Z}/4\mathbb{Z}*\mathbb{Z}/3\mathbb{Z}},1}([M_{\ell_0}]) = r^{\widehat{\mathbb{Z}/4\mathbb{Z}*\mathbb{Z}/3\mathbb{Z}},1}([M_{\ell_1}]) = r^{\widehat{\mathbb{Z}/4\mathbb{Z}*\mathbb{Z}/3\mathbb{Z}},1}([M_{\ell_2}]) = 0.$$

Example II.3.19. Take $\mathbb{Z}/2\mathbb{Z}$ to be the subgroup $\{+1, -1\}$ of the quaternion group Q_8 and consider the amalgamated product $Q_8 *_{\mathbb{Z}/2\mathbb{Z}} Q_8$. The amalgamated product $Q_8 *_{\mathbb{Z}/2\mathbb{Z}} Q_8$ acts on the tree in Figure II.1(a). The quaternion group Q_8 has 5 irreducible complex representations: four of dimension one and one of dimension two. Hence

$$C^*(Q_8) \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \qquad K^0(C^*(Q_8)) \cong \mathbb{Z}^5.$$

Further,

$$C^*(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{C} \oplus \mathbb{C} \qquad K^0(C^*(\mathbb{Z}/2\mathbb{Z})) \cong \mathbb{Z}^2.$$

All the one dimensional representations of Q_8 restrict to the trivial representation of $\mathbb{Z}/2\mathbb{Z}$. The two dimensional representation of Q_8 restricts to two copies of the (only) nontrivial representation of $\mathbb{Z}/2\mathbb{Z}$. Hence, with

$$\sigma^*(x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5) = (x_1 + x_2 + x_3 + x_4 - y_1 - y_2 - y_3 - y_4, 2x_5 - 2y_5),$$

we have the exact sequence

$$0 \longrightarrow \mathbb{Z}^8 \hookrightarrow \mathbb{Z}^5 \oplus \mathbb{Z}^5 \xrightarrow{\sigma^*} \mathbb{Z}^2 \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

in K-homology. Because the action of $Q_8 *_{\mathbb{Z}/2\mathbb{Z}} Q_8$ on the tree is transitive on the edges, we cannot use Corollary II.3.14 to detect the nontriviality of $r^{\widehat{Q_8 *_{\mathbb{Z}/2\mathbb{Z}} Q_8},1}([M_{\ell_0}]) \in KK_1(C_r^*(Q_8 *_{\mathbb{Z}/2\mathbb{Z}} Q_8), \mathbb{C})$. By Remark II.3.16, it is in fact a torsion class, generating $K^1(C_r^*(Q_8 *_{\mathbb{Z}/2\mathbb{Z}} Q_8)) \cong \mathbb{Z}/2\mathbb{Z}$.

II.3.3 CAT(0) cell complexes

Another context in which the conditions of Proposition II.3.4 may be naturally satisfied is when X is a CAT(0) cell complex. If we choose x_0 to be in an open n-cell then $S_{x_0}X$ is isometric to $\mathbf{S}^{n-1}$ [BH99, §7.14]. We shall consider two well-studied families of CAT(0) cell complexes: buildings and CAT(0) cube complexes. Throughout this section, we assume every cell complex has a bound on the dimension of its cells.

Apart from trees, the first family of CAT(0) complexes to be extensively studied were *buildings*. These arise in the study of reductive Lie groups over nonarchimedean local fields, for which the *Bruhat–Tits* building is the natural analogue of the symmetric space G/K for a connected Lie group and its maximal compact subgroup K. For more details on buildings, we refer to [Bro89, Tho18].

Definition II.3.20. e.g. [Tho18, Definition 1.18] Let I be a finite set, of size n, and $(m_{ij})_{i,j\in I}$ a symmetric matrix with values in $\mathbb{N}_{>0} \cup \{\infty\}$ such that $m_{ij} = 1$ if and only if $i = j$. The weighed graph associated with $(m_{ij})_{i,j\in I}$ is a *Coxeter diagram*; conventionally one does not include the edges with $m_{ij} \in \{1, 2\}$ as this is sufficient to recontruct $(m_{ij})_{i,j\in I}$. The *Coxeter group* of $(m_{ij})_{i,j\in I}$ is the group with presentation

$$W = \left\langle \{s_i\}_{i\in I} \,\middle|\, \forall i, j \in I, (s_i s_j)^{m_{ij}} = 1 \right\rangle.$$

Here, if $m_{ij} = \infty$, we mean that no additional constraint is to be placed on $s_i s_j$. Denote by $S = \{s_i\}_{i\in I}$ the set of generators. The pair (W, S) is a *Coxeter system.*

A Coxeter system is an abstract generalisation of a group of reflections. Associated to a Coxeter system (W, S) is its *Coxeter complex*, a connected simplicial complex of dimension $n - 1$ on which W acts with fundamental domain a single top-dimensional simplex [Cas23, Corollary 4.0.7]. However, W does not necessarily act properly on the Coxeter complex; when it does, (W, S) is called a *simplicial*

Coxeter system [Cas23, Definition 3.2.11]. Lannér's Theorem states that, when (W, S) is a simplicial Coxeter system, its Coxeter complex is a tiling of $\mathbf{S}^{n-1}$, $\mathbb{R}^{n-1}$, or $\mathbb{R}\mathbf{H}^{n-1}$ and W acts by reflections [Cas23, Theorem 3.2.12]. In each of these three cases, (W, S) is called *spherical*, *Euclidean*, or *hyperbolic*, respectively. Spherical, Euclidean, and hyperbolic Coxeter systems have a well-known classification; see e.g. [Cas23, §2.4, Tables 2.2–4].

Definition II.3.21. [Bro89, §IV.1] [Tho18, Definition 6.1] A *building of type* (W, S) is a simplicial complex Δ which is the union of *apartments*, subcomplexes isomorphic to the Coxeter complex of (W, S), such that

- Any two cells of Δ lie in a common apartment; and
- For any two apartments A and B, there is an isomorphism between them fixing their intersection.

The maximal simplices of Δ are called *chambers*.

Suppose that (W, S) is Euclidean or hyperbolic, so that the apartments are (tilings of) Euclidean or hyperbolic space. The geometric realisation $|\Delta|$ of Δ can be equipped with a metric in which every apartment is isometric to Euclidean or hyperbolic space, making $|\Delta|$ a CAT(0) space [Tho18, Theorem 7.14].

Euclidean buildings of dimension 1 are trees without valence-1 vertices, with the apartments the bi-infinite lines [Tho18, Example 6.5]. Buildings of dimension higher than 1 are difficult to visualise [JV87, Examples] but a gallant attempt is made in [BS22].

Groups acting on Euclidean buildings have been studied in the context of the Baum–Connes conjecture in [JV87, Jul89, KS91].

For ease of exposition, we shall conflate Δ with its geometric realisation $|\Delta|$. We have the following Corollary of Theorem II.3.7.

Corollary II.3.22. *Let G be a locally compact group acting on a Euclidean or hyperbolic building Δ of dimension n. Choose a point $x_0 \in \Delta$ which lies in the interior of a chamber C_0, so that its space of directions is isometric to $\mathbf{S}^{n-1}$. Let V be an irreducible Clifford module for $\mathscr{C}\ell_n$. Define the self-adjoint weight $\ell : G \to \operatorname{End} V$ by*

$$\ell_0(g) = d(g^{-1} \cdot x_0, x_0)v(g^{-1} \cdot x_0, x_0).$$

Then ℓ_0 is translation-bounded.

Suppose further that G acts properly on Δ. Let A be a G-C^*-algebra. We obtain an isometrically $\hat{G}$-equivariant unbounded Kasparov module

$$(A \rtimes_r G, L^2(G, V \otimes A)_A, M_{\ell_0})$$

representing $[M_{\ell_0}] \in KK_n^{\hat{G}}(A \rtimes_r G, A)$ and a G-equivariant unbounded Kasparov module

$$(A, C_0(G, A \otimes V)_{C_0(G,A)}, \ell_0)$$

representing $\sigma_A([\ell_0]) \in KK_n^G(A, C_0(G, A))$.

Let Y be a subcomplex of Δ, containing C_0 and consisting of the union of a collection of mutually disjoint apartments of Δ. Note that Y could be just a single apartment containing x_0.

Let x_1 be a point of Δ not in Y and define ℓ_1 analogously to ℓ_0.

For any closed subgroup H of G preserving Y, let $\eta_H : C_0(Y, A)^H \to A$ be the $*$-homomorphism given by evaluating at x_0, giving a class $[\eta_H] \in KK_0(C_0(Y, A)^H, A)$. For $A = \mathbb{C}$, $[\eta_H] \in KK_0(C_0(Y/H), \mathbb{C})$ is nonzero if and only if H acts cocompactly on Y.

If there exists a closed subgroup H of G such that H preserves Y and $[\eta_H]$ is nonzero then $\sigma_A([\ell_0]) \in KK_n^G(A, C_0(G, A))$ is nonzero and not equal to $\sigma_A([\ell_1])$.

Choose an orientation on Y, that is, an orientation on each apartment in Y. If G itself preserves Y and its orientation, and $[\eta_G]$ is nonzero, then $r^{\hat{G},1}([M_{\ell_0}]) \in KK_n(A \rtimes_r G, A)$ is nonzero and not equal to $r^{\hat{G},1}([M_{\ell_1}])$.

Example II.3.23. Let G be a reductive Lie group over a nonarchimedean local field k; see e.g. [RTW15]. Choose a maximal k-split torus $T = (k^\times)^n$, where n is the k-rank. Associated with G is its *Bruhat–Tits* building Δ, a Euclidean building with n-dimensional apartments whose Coxeter system is determined by the normaliser of T in G. The group G acts properly and isometrically on its Bruhat–Tits building. Further, the action is *strongly transitive*, meaning that G acts transitively on the set of apartments and, for every apartment B, the stabiliser group G_B acts transitively on the chambers in B [Tho18, Definition 9.5].

Let $x_0 \in \Delta$ be a point in the interior of a chamber C_0. Let B_0 be an apartment containing C_0. By strong transitivity, G_{B_0} acts cocompactly on B_0 and we thus obtain from Corollary II.3.22 that $[\ell_0] \in KK_n^G(\mathbb{C}, C_0(G))$ is nontrivial. If x_1 is a point in the interior of a chamber C_1 not in B_0, $[\ell_0] \neq [\ell_1]$. Because the action of G on Δ is strongly transitive, we cannot use Corollary II.3.22 to determine whether $r^{\hat{G},1}([M_{\ell_0}])$ is nontrivial.

CAT(0) cube complexes are another important family of CAT(0) cell complexes; for further details we refer to [NR98].

Definition II.3.24. [NR98, §2.2] A *cube complex* is a metric cell complex Δ in which every cell is isometric to a unit Euclidean cube and the glueing maps are isometries. We call a maximal cell of Δ a *maximal cube.* A *flat* of a cube complex is an isometrically embedded copy of Euclidean space $\mathbb{R}^n$ for some n.

The conditions making a cube complex CAT(0) are quite tractable; we refer to [NR98, §2.2]. Groups acting on CAT(0) cube complexes have been studied in the context of the Baum–Connes conjecture in [BGH19, BGHN20].

We have the following Corollary of Theorem II.3.7.

Corollary II.3.25. *Let G be a locally compact group acting on a CAT(0) cube complex Δ. Choose a point $x_0 \in \Delta$ which lies in the interior of a maximal cube C_0, of dimension n, so that its space of directions is isometric to $\mathbf{S}^{n-1}$. Let V be an irreducible Clifford module for $\mathscr{C}\ell_n$. Define the self-adjoint weight $\ell : G \to \operatorname{End} V$ by*

$$\ell_0(g) = d(g^{-1} \cdot x_0, x_0)v(g^{-1} \cdot x_0, x_0).$$

Then ℓ_0 is translation-bounded.

Suppose further that G acts properly on Δ. Let A be a G-C^*-algebra. We obtain an isometrically $\hat{G}$-equivariant unbounded Kasparov module

$$(A \rtimes_r G, L^2(G, V \otimes A)_A, M_{\ell_0})$$

representing $[M_{\ell_0}] \in KK_n^{\hat{G}}(A \rtimes_r G, A)$ and a G-equivariant unbounded Kasparov module

$$(A, C_0(G, A \otimes V)_{C_0(G,A)}, \ell_0)$$

representing $\sigma_A([\ell_0]) \in KK_n^G(A, C_0(G, A))$.

Let Y be a subcomplex of Δ, containing C_0 and consisting of the union of a collection of mutually disjoint flats of Δ. Note that Y could be just a single flat containing C_0. Let x_1 be a point of Δ not in Y and define ℓ_1 analogously to ℓ_0.

For any closed subgroup H of G preserving Y, let $\eta_H : C_0(Y, A)^H \to A$ be the $*$-homomorphism given by evaluating at x_0, giving a class $[\eta_H] \in KK_0(C_0(Y, A)^H, A)$. For $A = \mathbb{C}$, $[\eta_H] \in KK_0(C_0(Y/H), \mathbb{C})$ is nonzero if and only if H acts cocompactly on Y.

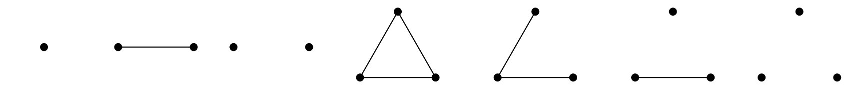
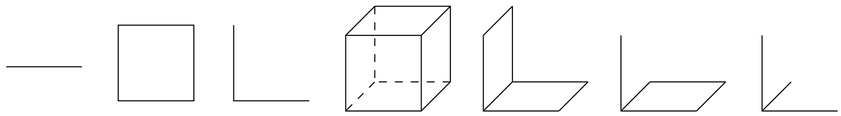
If there exists a closed subgroup H of G such that H preserves Y and $[\eta_H]$ is nonzero then $\sigma_A([\ell_0]) \in KK_n^G(A, C_0(G, A))$ is nonzero and not equal to $\sigma_A([\ell_1])$.

Choose an orientation on Y, that is, an orientation on each of the disjoint flats making up Y. If G itself preserves Y and its orientation, and $[\eta_G]$ is nonzero, then $r^{\hat{G},1}([M_{\ell_0}]) \in KK_n(A \rtimes_r G, A)$ is nonzero and not equal to $r^{\hat{G},1}([M_{\ell_1}])$.

A well-studied family of CAT(0) groups are the *right-angled Artin groups*; for more details we refer to [Cha07].

Definition II.3.26. [Cha07, §3.1,6] Let Γ be a finite graph without loops. The *right-angled Artin group* A_Γ associated with Γ is the group with generators $(s_i)_{i\in\Gamma^0}$ and relations $s_is_j = s_js_i$ for $(i,j)\in\Gamma^1$. The *Salvetti complex* S_Γ is the space constructed from tori $\mathbb{T}^{\#C^0}$ for each clique $K\subseteq\Gamma$, glued according to the partial order on the cliques. By construction, the fundamental group of S_Γ is A_Γ. The universal cover $\tilde{S}_\Gamma$ is a CAT(0) cube complex. The group A_Γ acts properly and cocompactly on $\tilde{S}_\Gamma$ (for properness, see e.g. [Mun14, Theorem 81.5]).

When the graph Γ has no edges, A_Γ is the free group on the generators $(s_i)_{i\in\Gamma^0}$ and $\tilde{S}_\Gamma$ is its Cayley graph. When the graph Γ is complete, A_Γ is the free abelian group on the generators $(s_i)_{i\in\Gamma^0}$ and $\tilde{S}_\Gamma$ is the cubical tiling of $\mathbb{R}^{\#\Gamma^0}$. For graphs of size up to three, we have the following classification.

Γ							
A_Γ	$\mathbb{Z}$	$\mathbb{Z}^2$	F_2	$\mathbb{Z}^3$	$\mathbb{Z}\times F_2$	$\mathbb{Z}*\mathbb{Z}^2$	F_3
Fundamental domain for $\tilde{S}_\Gamma$							

Example II.3.27. Let Γ be a finite graph without loops and let A_Γ be the associated right-angled Artin group. Let $\Delta = \tilde{S}_\Gamma$ be the universal cover of the Salvetti complex. Let $x_0\in\tilde{S}_\Gamma$ be a point in the interior of a maximal cube C_0, which will be a cell corresponding to a maximal clique K of Γ. Let $n = \dim C_0 = \#K^0$. Define $\ell_0 : G\to \operatorname{End} V$ as in Corollary II.3.25.

By [Cha07, §3.2], the clique K gives rise to a subgroup $A_K\leq A_\Gamma$ isomorphic to $\mathbb{Z}^n$. There is a subgroup conjugate to $A_K\cong\mathbb{Z}^n$ which preserves the flat F containing C_0, which is isometric to $\mathbb{R}^n$. Applying Corollary II.3.25, we see that $[\ell_0]\in KK^G_n(\mathbb{C}, C_0(G))$ is nonzero.

Let Y be the orbit of the flat F containing C_0. This is a collection of copies of $\mathbb{R}^n$ on which G acts cocompactly and preserving the orientation. Applying Corollary II.3.25, $r^{\hat{G},1}([M_{\ell_0}])\in KK_1(C^*_r(G),\mathbb{C})$ is nonzero.

II.3.4 Pairing with a Dirac class

In this section, we prove Theorem II.3.8, on p. 79. The strategy is to use the constructive unbounded Kasparov product, then to reduce to the case of a connected manifold, and finally to use Remark II.3.13 to complete the pairing.

Lemma II.3.28. *Let (X,d) be a CAT(0) space. Let $x_0\in X$ be such that $S_{x_0}(X)$ is isometric to $\mathbf{S}^{m-1}\subset\mathbb{R}^m$. Let V be a Clifford module for $\mathscr{C}\ell_m$ and define $\tilde\ell : X\to\operatorname{End}V$ by*

$$\tilde\ell : x\mapsto d(x,x_0)v(x,x_0).$$

We have, for $x,y\in X$,

$$\left\|\tilde\ell(x)-\tilde\ell(y)\right\|\leq d(x,y).$$

Hence ℓ is Lipschitz continuous from X to $\operatorname{End}V$. Also, the absolute value $|\tilde\ell|$ is a Lipschitz continuous function on X. If $\gamma_1,\dots,\gamma_m$ are the generators of $\mathscr{C}\ell_m$,

$$\tilde\ell(x) = \sum_{i=1}^m\langle\tilde\ell(x)\mid\gamma_i\rangle\gamma_i.$$

For every $i\in[1..m]$, the functions $\langle\tilde\ell\mid\gamma_i\rangle$ are also Lipschitz continuous.

Of course, for any subset Y of X, equipped with the restricted metric, the restrictions of $|\tilde{\ell}|$ and $\langle \tilde{\ell} \mid \gamma_i \rangle$ to Y are Lipschitz continuous functions on Y.

Proof. First, $|\tilde{\ell}(x)| = d(x, x_0)$, so $|\ell|$ is clearly Lipschitz continuous. Next, we have, where the norm is in $\operatorname{End} V$,

$$\left\|\tilde{\ell}(x) - \tilde{\ell}(y)\right\| = \left\|d(x, x_0)v(x, x_0) - d(y, x_0)v(y, x_0)\right\|.$$

This is the length of the third side of a Euclidean triangle with the other side lengths $d(x, x_0)$ and $d(y, x_0)$ and the opposite angle $\arccos\langle v(x, x_0), v(y, x_0)\rangle$. We can compare this to the triangle in X with vertices at x_0, x, and y. The CAT(0) property guarantees that

$$\left\|\tilde{\ell}(x) - \tilde{\ell}(y)\right\| = \left\|d(x, x_0)v(x, x_0) - d(y, x_0)v(y, x_0)\right\| \le d(x, y).$$

Hence, $\tilde{\ell}$ is a Lipschitz continuous function from X to $\operatorname{End} V$. For each $i \in [1..n]$, the function

$$w \mapsto \frac{1}{2}(w\gamma_i + \gamma_i w)$$

is Lipschitz continuous from $\operatorname{End} V$ to $\mathbb{C}$. So $\langle \tilde{\ell} \mid \gamma_i \rangle = \frac{1}{2}(\ell\gamma_i + \gamma_i\ell)$ is Lipschitz continuous. □

Next, we have the following as a special case of Proposition A.2.9.

Proposition II.3.29. *Let G be a locally compact group acting properly, isometrically, and by spin^c automorphisms on a Riemannian n-manifold Y. Let A be a G-C^*-algebra. Let*

$$(C_0(Y), L^2(Y, \$), \not{D})$$

be the Atiyah–Singer Dirac spectral triple representing the class $\alpha \in KK_n^G(C_0(Y), \mathbb{C})$. Let ${}^G C_0(Y, A)$ be the partial imprimitivity $C_0(Y, A)^G$-$C_0(Y, A) \rtimes_r G$-bimodule of Corollary A.2.7. Define a right action of $C_c(G, A) \subseteq A \rtimes_r G$ on $\Gamma_c(Y, \$ \otimes A)$ by

$$(\xi f)(x) = \int_G g \cdot \left(\xi(g^{-1}x)f(g^{-1})\right)\Delta_G(g^{-1})^{1/2} d\mu(g) \qquad (\xi \in \Gamma_c(Y, \$ \otimes A), f \in C_c(G, A))$$

and a $C_c(G, A)$-valued inner product by

$$\langle \xi_1 \mid \xi_2 \rangle(g) = \int_Y \left\langle \xi_1(x) \mid g \cdot \xi_2(g^{-1}x)\right\rangle_{\$ \otimes A} \Delta_G(g^{-1})^{1/2} \mathrm{vol}_{\mathrm{g}}(x) \qquad (\xi_1, \xi_2 \in \Gamma_c(Y, \$ \otimes A)).$$

The completion of $\Gamma_c(Y, \$ \otimes A)$ is a Hilbert $A \rtimes_r G$-module $\overline{\Gamma_c(Y, \$ \otimes A)}$. The Kasparov product

$$[{}^G C_0(Y, A)] \otimes_{C_0(Y, A) \rtimes_r G} j_r^G(\sigma_A(\alpha)) \in KK_n(C_0(Y, A)^G, A \rtimes_r G)$$

is represented by

$$(C_0(Y, A)^G, \overline{\Gamma_c(Y, \$ \otimes A)}_{A \rtimes_r G}, \not{D} \otimes 1)$$

where, by a slight abuse of notation, we denote by $\not{D} \otimes 1$ its closure on $\Gamma_c^\infty(Y, \$ \otimes A)$.

Proposition II.3.30. *Let G be a locally compact group acting properly and isometrically on a CAT(0) space (X, d). Let A be a G-C^*-algebra. Suppose that there is a complete subspace Y of X such that*

- *every path component of Y is a convex subset of X;*
- *Y is isometric to a spin^c Riemannian n-manifold; and*
- *G preserves Y and acts by spin^c automorphisms.*

Let $x_0 \in X$ be such that $S_{x_0}(X)$ is isometric to $\mathbf{S}^{m-1} \subset \mathbb{R}^m$. Let V be a Clifford module for $\mathscr{C}\ell_m$ and define

$$\ell : G \to \operatorname{End} V \qquad \tilde{\ell} : X \to \operatorname{End} V$$
$$g \mapsto d(g^{-1}x_0, x_0)v(g^{-1}x_0, x_0) \qquad x \mapsto d(x, x_0)v(x, x_0).$$

Let $[M_\ell] \in KK_m^{\hat{G}}(A \rtimes_r G, A)$ be the class of

$$(A \rtimes_r G, L^2(G, V \otimes A)_A, M_\ell \otimes 1).$$

Let

$$(C_0(Y), L^2(Y, \not{S}), \not{D})$$

be the Atiyah–Singer Dirac spectral triple representing the class $\alpha_Y \in KK_n^G(C_0(Y), \mathbb{C})$. Let ${}^GC_0(Y, A)$ be the partial imprimitivity $C_0(Y, A)^G$-$C_0(Y, A) \rtimes G$-bimodule of Theorem A.2.6. Then the Kasparov product

$$[{}^GC_0(Y, A)] \otimes_{C_0(Y,A)\rtimes G} j_r^G(\sigma_A(\alpha_Y)) \otimes_{A\rtimes_r G} r^{\hat{G},1}([M_\ell])$$

in $KK_{m+n}(C_0(Y, A)^G, A)$ is represented by

$$(C_0(Y, A)^G, L^2(Y, \not{S} \tilde{\otimes} V \otimes A)_A, \not{D} \tilde{\otimes} 1 \otimes 1 + 1 \tilde{\otimes} M_{\tilde{\ell}} \otimes 1).$$

Proof. First, by Proposition II.3.29, $[{}^GC_0(Y, A)] \otimes_{C_0(Y,A)\rtimes G} j_r^G(\sigma_A(\alpha_Y))$ is represented by

$$(C_0(Y, A)^G, \overline{\Gamma_c(Y, \not{S} \otimes A)}_{A\rtimes_r G}, \not{D} \otimes 1).$$

Let us begin by examining the internal tensor product Hilbert module $\overline{\Gamma_c(Y, \not{S} \otimes A)} \tilde{\otimes}_{A\rtimes_r G} L^2(G, V \otimes A)$. For $\zeta \in \Gamma_c(Y, \not{S} \otimes A)$ and $\xi \in C_c(G, V \otimes A)$, by the balancing over $C_c(G, A)$, we may consider $\zeta \tilde{\otimes} \xi \in \Gamma_c(Y, \not{S} \otimes A) \otimes_{C_c(G,A)} C_c(G, V \otimes A)$ as an element of $\Gamma_c(Y, \not{S} \tilde{\otimes} V \otimes A)$, given by

$$(\zeta \tilde{\otimes} \xi)(x) = \int_G g \cdot \zeta(g^{-1}x) \tilde{\otimes} \alpha_g(\xi(g^{-1}))\Delta_G(g^{-1})^{1/2} d\mu(g).$$

Indeed, this shows that $\Gamma_c(Y, \not{S} \otimes A) \otimes_{C_c(G,A)} C_c(G, V \otimes A) = \Gamma_c(Y, \not{S} \tilde{\otimes} V \otimes A)$. For $\zeta_1, \zeta_2 \in \Gamma_c(Y, \not{S} \otimes A)$, recall from Proposition II.3.29 that their $A \rtimes_r G$-valued inner product is given by

$$\langle \zeta_1 \mid \zeta_2 \rangle(g) = \int_Y \big\langle \zeta_1(x) \mid g \cdot \zeta_2(g^{-1}x) \big\rangle_{\not{S}\otimes A} \Delta_G(g^{-1})^{1/2} \mathrm{vol}_{\mathbf{g}}(x).$$

For $\xi_1, \xi_2 \in C_c(G, V \otimes A) \subseteq L^2(G, V \otimes A)$, we have

$$\begin{aligned}
\langle \zeta_1 \tilde{\otimes} \xi_2 \mid \zeta_2 \tilde{\otimes} \xi_2 \rangle &= \big\langle \xi_1 \big| \langle \zeta_1 \mid \zeta_2 \rangle \xi_1 \big\rangle \\
&= \int_G \alpha_{g^{-1}}\Big(\big\langle \xi_1(g) \big| (\langle \zeta_1 \mid \zeta_2 \rangle \xi_1)(g) \big\rangle_{V\otimes A}\Big) d\mu(g) \\
&= \int_G \int_G \alpha_{g^{-1}}\Big(\big\langle \xi_1(g) \big| \langle \zeta_1 \mid \zeta_2 \rangle(h) \alpha_h(\xi_1(h^{-1}g)) \big\rangle_{V\otimes A}\Big) d\mu(g) d\mu(h) \\
&= \int_G \int_G \int_Y \alpha_{g^{-1}}\Big(\Big\langle \xi_1(g) \Big| \big\langle \zeta_1(x) \mid h \cdot \zeta_2(h^{-1}x) \big\rangle_{\not{S}\otimes A} \alpha_h(\xi_1(h^{-1}g)) \Big\rangle_{V\otimes A}\Big) \\
&\qquad \times \Delta_G(h^{-1})^{1/2} \mathrm{vol}_{\mathbf{g}}(x) d\mu(g) d\mu(h) \\
&= \int_G \int_G \int_Y \alpha_{g^{-1}}\Big(\Big\langle \xi_1(g) \Big| \big\langle \zeta_1(x) \mid gh \cdot \zeta_2((gh)^{-1}x) \big\rangle_{\not{S}\otimes A} \alpha_{gh}(\xi_1(h^{-1})) \Big\rangle_{V\otimes A}\Big) \\
&\qquad \times \Delta_G((gh)^{-1})^{1/2} \mathrm{vol}_{\mathbf{g}}(x) d\mu(g) d\mu(h) \\
&= \int_G \int_G \int_Y \alpha_{g^{-1}}\Big(\Big\langle \xi_1(g) \Big| \big\langle \zeta_1(gx) \mid gh \cdot \zeta_2(h^{-1}x) \big\rangle_{\not{S}\otimes A} \alpha_{gh}(\xi_1(h^{-1})) \Big\rangle_{V\otimes A}\Big) \\
&\qquad \times \Delta_G((gh)^{-1})^{1/2} \mathrm{vol}_{\mathbf{g}}(x) d\mu(g) d\mu(h)
\end{aligned}$$

$$= \int_G \int_G \int_Y \alpha_{g^{-1}}\Big(\Big\langle \xi_1(g) \Big| \alpha_g\Big(\Big\langle g^{-1}\cdot\zeta_1(gx) \mid h\cdot\zeta_2(h^{-1}x)\Big\rangle_{\not{S}\otimes A}\Big)\alpha_{gh}(\xi_1(h^{-1}))\Big\rangle_{V\otimes A}\Big)$$
$$\times \Delta_G((gh)^{-1})^{1/2}\mathrm{vol}_{\mathbf{g}}(x)d\mu(g)d\mu(h)$$
$$= \int_G \int_G \int_Y \Big\langle \alpha_{g^{-1}}(\xi_1(g)) \Big| \Big\langle g^{-1}\cdot\zeta_1(gx) \mid h\cdot\zeta_2(h^{-1}x)\Big\rangle_{\not{S}\otimes A}\alpha_h(\xi_1(h^{-1}))\Big\rangle_{V\otimes A}$$
$$\times \Delta_G((gh)^{-1})^{1/2}\mathrm{vol}_{\mathbf{g}}(x)d\mu(g)d\mu(h)$$
$$= \int_G \int_G \int_Y \Big\langle \alpha_g(\xi_1(g^{-1})) \Big| \Big\langle g\cdot\zeta_1(g^{-1}x) \mid h\cdot\zeta_2(h^{-1}x)\Big\rangle_{\not{S}\otimes A}\alpha_h(\xi_1(h^{-1}))\Big\rangle_{V\otimes A}$$
$$\times \Delta_G((gh)^{-1})^{1/2}\mathrm{vol}_{\mathbf{g}}(x)d\mu(g)d\mu(h)$$
$$= \int_G \int_G \int_Y \Big\langle g\cdot\zeta_1(g^{-1}x)\,\tilde{\otimes}\,\alpha_g(\xi_1(g^{-1})) \Big| h\cdot\zeta_2(h^{-1}x)\,\tilde{\otimes}\,\alpha_h(\xi_1(h^{-1}))\Big\rangle_{\not{S}\tilde{\otimes}V\otimes A}$$
$$\times \Delta_G((gh)^{-1})^{1/2}\mathrm{vol}_{\mathbf{g}}(x)d\mu(g)d\mu(h)$$
$$= \int_Y \Big\langle (\zeta_1\,\tilde{\otimes}\,\xi_2)(x) \Big| (\zeta_2\,\tilde{\otimes}\,\xi_2)(x)\Big\rangle \mathrm{vol}_{\mathbf{g}}(x).$$

So $\overline{\Gamma_c(Y,\not{S}\otimes A)}\,\tilde{\otimes}_{A\rtimes_r G}\, L^2(G,V\otimes A)$ is isomorphic to $L^2(Y,\not{S}\,\tilde{\otimes}\,V\otimes A)$.

We shall show that

$$(C_0(Y,A)^G, L^2(Y,\not{S}\,\tilde{\otimes}\,V\otimes A)_A, \not{D}\,\tilde{\otimes}\,1\otimes 1 + 1\,\tilde{\otimes}\,M_{\tilde{\ell}}\otimes 1)$$

is the constructive unbounded Kasparov product for

$$(C_0(Y,A)^G, \overline{\Gamma_c(Y,\not{S}\otimes A)}_{A\rtimes_r G}, \not{D}\otimes 1)\otimes_{A\rtimes_r G}(A\rtimes_r G, L^2(G,V)\otimes A_A, M_\ell\otimes 1).$$

To apply [LM19, Theorem 7.4], we need to check the connection condition, that $1\,\tilde{\otimes}\,M_{\tilde{\ell}}\otimes 1$ has bounded commutators, and that $\not{D}\,\tilde{\otimes}\,1\otimes 1$ and $1\,\tilde{\otimes}\,M_{\tilde{\ell}}\otimes 1$ weakly anticommute.

First, $1\,\tilde{\otimes}\,M_{\tilde{\ell}}\otimes 1$ commutes with the representation of $C_0(Y,A)^G$. Second, let $\zeta\in\Gamma_c(Y,\not{S}\otimes A)$ and consider the operator $T_\zeta\in\mathrm{Hom}^*(L^2(G,V\otimes A), L^2(Y,\not{S}\,\tilde{\otimes}\,V\otimes A))$ given by

$$(T_\zeta\xi)(x) = (\zeta\,\tilde{\otimes}\,\xi)(x) = \int_G g\cdot\zeta(g^{-1}x)\,\tilde{\otimes}\,\alpha_g(\xi(g^{-1}))\Delta_G(g^{-1})^{1/2}d\mu(g)$$

for $\xi\in C_c(G,V\otimes A)$. We have

$$\Big(\Big((1\,\tilde{\otimes}\,M_{\tilde{\ell}}\otimes 1)T_\zeta - T_\zeta(M_\ell\otimes 1)\Big)\xi\Big)(x) = \int_G g\cdot\zeta(g^{-1}x)\,\tilde{\otimes}\,(\tilde{\ell}(x)-\ell(g^{-1}))\alpha_g(\xi(g^{-1}))\Delta_G(g^{-1})^{1/2}d\mu(g).$$

Let $\pi : A\to B(H_\pi)$ be an irreducible representation of A and let $\eta\in H_\pi$, so that $\xi\otimes\eta\in L^2(G,V\otimes A)\otimes_\pi H_\pi$. With $\xi_2\in C_c(Y,\not{S}\,\tilde{\otimes}\,V\otimes A)\subseteq L^2(Y,\not{S}\,\tilde{\otimes}\,V\otimes A)$, we compute

$$\Big|\Big\langle \xi_2\otimes\eta \Big| \Big((1\,\tilde{\otimes}\,M_{\tilde{\ell}}\otimes 1)T_\zeta - T_\zeta(M_\ell\otimes 1)\Big)\xi\otimes\eta\Big\rangle\Big|$$
$$\leq \int_Y\int_N \Big|\Big\langle \xi_2(x)\otimes\eta \Big| g\cdot\zeta(g^{-1}x)\,\tilde{\otimes}\,(\tilde{\ell}(x)-\ell(g^{-1}))\alpha_g(\xi(g^{-1}))\otimes\eta\Big\rangle\Big|\Delta_G(g^{-1})^{1/2}d\mu(g)\mathrm{vol}_{\mathbf{g}}(x)$$
$$\leq \int_G\int_N \|\xi_2(x)\otimes\eta\|\Big\|\tilde{\ell}(x)-\ell(g^{-1})\Big\|\|g\cdot\zeta(g^{-1}x)\|\|\alpha_g(\xi(g^{-1}))\otimes\eta\|\Delta_G(g^{-1})^{1/2}d\mu(g)\mathrm{vol}_{\mathbf{g}}(x).$$

Next, by Lemma II.3.28,

$$c := \sup_{x\in\mathrm{supp}\,\zeta, g\in G}\Big\|\tilde{\ell}(g^{-1}x)-\ell(g^{-1})\Big\| \leq \sup_{x\in\mathrm{supp}\,\zeta, g\in G} d(g^{-1}\cdot x, g^{-1}\cdot x_0) = \sup_{x\in\mathrm{supp}\,\zeta} d(x,x_0),$$

which is finite by the compactness of $\operatorname{supp}\zeta$. Furthermore, we note that, if for some $g, h \in G$, we know that $g^{-1}x, h^{-1}x \in \operatorname{supp}\zeta$. In particular, $x \in g(\operatorname{supp}\zeta) \cap h(\operatorname{supp}\zeta)$ which implies that

$$g(\operatorname{supp}\zeta) \cap h(\operatorname{supp}\zeta) \neq \emptyset.$$

By the properness of the action of G on Y and the compactness of $\operatorname{supp}\zeta$, we obtain that $g^{-1}h$ is guaranteed to be in some compact subset K of G, independent of x. We denote by χ_K the characteristic function of K. Putting $\xi_2 = ((1 \tilde{\otimes} M_{\tilde{\ell}} \otimes 1)T_\zeta - T_\zeta(M_\ell \otimes 1))\xi$, we estimate that

$$\begin{aligned}
&\left\|\left((1 \tilde{\otimes} M_{\tilde{\ell}} \otimes 1)T_\zeta - T_\zeta(M_\ell \otimes 1)\right)\xi \otimes \eta\right\|^2 \\
&\quad\leq \int_Y\int_G\int_G \left\|\tilde{\ell}(x) - \ell(h^{-1})\right\| \|h\cdot\zeta(h^{-1}x)\| \|\alpha_h(\xi(h^{-1}))\otimes\eta\| \Delta_G(h^{-1})^{1/2} \\
&\qquad\times \left\|\tilde{\ell}(x) - \ell(g^{-1})\right\| \|g\cdot\zeta(g^{-1}x)\| \|\alpha_g(\xi(g^{-1}))\otimes\eta\| \Delta_G(g^{-1})^{1/2} d\mu(g)d\mu(h)\mathrm{vol}_{\mathbf{g}}(x) \\
&\quad\leq \int_G\int_G \left(\int_X \|g\cdot\zeta(g^{-1}x)\| \|h\cdot\zeta(h^{-1}x)\| \mathrm{vol}_{\mathbf{g}}(x)\right) \\
&\qquad\times \|\alpha_g(\xi(g^{-1}))\otimes\eta\| \|\alpha_h(\xi(h^{-1}))\otimes\eta\| \chi_K(g^{-1}h)\Delta_G(g^{-1}h^{-1})^{1/2} d\mu(g)d\mu(h) \\
&\qquad\times \sup_{x\in Y, g\in G | g^{-1}x\in\operatorname{supp}\zeta} \left\|\tilde{\ell}(x) - \ell(g^{-1})\right\|^2 \\
&\quad\leq c^2 \int_N\int_N \left(\int_X \|g\cdot\zeta(g^{-1}x)\|^2 \mathrm{vol}_{\mathbf{g}}(x)\right)^{1/2} \left(\int_Y \|h\cdot\zeta(h^{-1}x)\|^2 \mathrm{vol}_{\mathbf{g}}(x)\right)^{1/2} \\
&\qquad\times \|\alpha_g(\xi(g^{-1}))\otimes\eta\| \|\alpha_h(\xi(h^{-1}))\otimes\eta\| \chi_K(g^{-1}h)\Delta_G(g^{-1}h^{-1})^{1/2} d\mu(g)d\mu(h) \\
&\quad= c^2 \left(\int_X \|\zeta(x)\|^2 \mathrm{vol}_{\mathbf{g}}(x)\right) \\
&\qquad\times \int_G\int_G \|\alpha_g(\xi(g^{-1}))\otimes\eta\| \|\alpha_h(\xi(h^{-1}))\otimes\eta\| \chi_K(g^{-1}h)\Delta_G(g^{-1}h^{-1})^{1/2} d\mu(g)d\mu(h) \\
&\quad= c^2 \left(\int_X \|\zeta(x)\|^2 \mathrm{vol}_{\mathbf{g}}(x)\right) \\
&\qquad\times \int_G\int_G \|\alpha_g(\xi(g^{-1}))\otimes\eta\| \|\alpha_{gh}(\xi(h^{-1}g^{-1}))\otimes\eta\| \chi_K(h)\Delta_G(g^{-1})\Delta_G(h^{-1})^{1/2} d\mu(g)d\mu(h) \\
&\quad= c^2 \left(\int_X \|\zeta(x)\|^2 \mathrm{vol}_{\mathbf{g}}(x)\right) \\
&\qquad\times \int_G\int_G \|\alpha_{g^{-1}}(\xi(g))\otimes\eta\| \|\alpha_{g^{-1}h}(\xi(h^{-1}g))\otimes\eta\| \chi_K(h)\Delta_G(h^{-1})^{1/2} d\mu(g)d\mu(h) \\
&\quad\leq c^2 \int_G \left(\int_G \|\alpha_{g^{-1}}(\xi(g))\otimes\eta\|^2 d\mu(g)\right)^{1/2} \left(\int_G \|\alpha_{g^{-1}h}(\xi(h^{-1}g))\otimes\eta\| d\mu(g)\right)^{1/2} \\
&\qquad\times \chi_K(h)\Delta_G(h^{-1})^{1/2} d\mu(h) \left(\int_X \|\zeta(x)\|^2 \mathrm{vol}_{\mathbf{g}}(x)\right) \\
&\quad= c^2 \|\xi\otimes\eta\|^2 \int_G \chi_K(h)\Delta_G(h^{-1})^{1/2} d\mu(h) \left(\int_X \|\zeta(x)\|^2 \mathrm{vol}_{\mathbf{g}}(x)\right) \\
&\quad\leq c^2 \|\xi\otimes\eta\|^2 \int_G \mu(K) \sup_{h\in K} \Delta_G(h^{-1})^{1/2} \left(\int_X \|\zeta(x)\|^2 \mathrm{vol}_{\mathbf{g}}(x)\right) \\
&\quad= c'^2 \|\xi\otimes\eta\|^2
\end{aligned}$$

for $0 \leq c' < \infty$ independent of ξ and η. Now, by Lemma A.3.3,

$$\left\|(1 \tilde{\otimes} M_{\tilde{\ell}} \otimes 1)T_\zeta - T_\zeta(M_\ell \otimes 1)\right\| \leq c' < \infty.$$

This gives us the condition [LM19, Theorem 7.4(i)].

We finally check that $\not{D}\,\tilde{\otimes}\,1\otimes 1$ and $1\,\tilde{\otimes}\,M_{\tilde{\ell}}\otimes 1$ weakly anticommute. If $\gamma_1, \dots, \gamma_m$ are the generators of $\mathscr{C}\ell_m$,

$$\tilde{\ell}(x) = \sum_{i=1}^{m} \langle \tilde{\ell}(x) \mid \gamma_i \rangle \gamma_i.$$

The components $\langle \tilde{\ell} \mid \gamma_i \rangle$ are Lipschitz continuous functions on Y by Lemma II.3.28. We may therefore say that

$$(\not{D}\,\tilde{\otimes}\,1\otimes 1)(1\,\tilde{\otimes}\,M_{\tilde{\ell}}\otimes 1) + (1\,\tilde{\otimes}\,M_{\tilde{\ell}}\otimes 1)(\not{D}\,\tilde{\otimes}\,1\otimes 1) = \sum_{i=1}^{m} [\not{D}, M_{\langle \tilde{\ell}|\gamma_i\rangle}]\,\tilde{\otimes}\,\gamma_i \otimes 1$$

is bounded below, and we are done. □

We now make a basic observation about the fixed point algebra of a manifold with multiple path components.

Lemma II.3.31. *Let G be a locally compact group acting properly on a manifold Y. Let A be a G-C^*-algebra. Let Z be a path component of Y and let G_Z be its stabiliser group. There is an isomorphism between $C_0(Z, A)^{G_Z}$ and $C_0(GZ, A)^G$. Consequently, $C_0(Y, A)^G$ is isomorphic to $C_0(Z, A)^{G_Z} \oplus C_0(Y \setminus GZ, A)^G$.*

Proof. First, the action of G takes path components of Y to path components. If for some $g \in G$ there exist $x, y \in Z$ such that $gx = y$ then g must be in the stabiliser group G_Z of Z. Hence $(GZ)/G$ is the same as Z/G_Z. Remark also that the orbit GZ and its complement in Y are manifolds.

Let $f \in C_0(Z, A)^{G_Z} \subseteq C_b(Z, A)$. By definition of the fixed point algebra, we have

$$f(sz) = \alpha_s(f(z))$$

for $s \in G_Z$ and $z \in Z$. We first extend f to a function $\tilde{f} \in C_b(GZ, A)$ by the formula

$$\tilde{f}(gz) = \alpha_g(f(z))$$

for $g \in G$ and $z \in Z$. To see that this is well-defined, suppose that $gz = hy$ for $g, h \in G$ and $y, z \in Z$, and note that

$$\alpha_g(f(z)) = \alpha_g(f((hg^{-1})^{-1}y)) = \alpha_{hg^{-1}}(\alpha_g(f(y))) = \alpha_h(f(y)).$$

Note also that, for any $h \in G$,

$$\tilde{f}(hgz) = \alpha_{hg}(f(z)) = \alpha_h(\tilde{f}(gz)).$$

Recalling that $(GZ)/G = Z/G_Z$ and that $x \mapsto \|f(x)\|$ gives an element of $C_0(Z/G_Z)$, we obtain that $\tilde{f} \in C_0(GZ, A)^G$.

On the other hand, if $f \in C_0(GZ, A)^G \subseteq C_b(GZ, A)$, its restriction to Z gives an element $f|_Z \in C_0(Z, A)^{G_Z}$. □

Proposition II.3.32. *Let G be a locally compact group acting properly and isometrically on a CAT(0) space (X, d). Let A be a G-C^*-algebra. Suppose that there is a complete subspace Y of X such that*

- *every path component of Y is a convex subset of X;*
- *Y is isometric to a spin^c Riemannian n-manifold; and*
- *G preserves Y and acts by spin^c automorphisms.*

Let $x_0 \in X$ be such that $S_{x_0}(X)$ is isometric to $\mathbf{S}^{m-1} \subset \mathbb{R}^m$. Let V be a Clifford module for $\mathscr{C}\ell_m$ and define

$$\begin{aligned} \ell : G &\to \operatorname{End} V & \tilde{\ell} : X &\to \operatorname{End} V \\ g &\mapsto d(g^{-1}x_0, x_0)v(g^{-1}x_0, x_0) & x &\mapsto d(x, x_0)v(x, x_0). \end{aligned}$$

The class of

$$(C_0(Y, A)^G, L^2(Y, \not{S} \tilde{\otimes} V \otimes A)_A, \not{D} \tilde{\otimes} 1 \otimes 1 + 1 \tilde{\otimes} M_{\tilde{\ell}} \otimes 1)$$

in $KK_{m+n}(C_0(Y, A)^G, A)$ is equal to zero if $x_0 \notin Y$ and, if $x_0 \in Y$, to the class of

$$\kappa_{Y_0}^*((C_0(Y_0, A)^{G_{Y_0}}, L^2(Y_0, \not{S}|_{Y_0} \tilde{\otimes} V \otimes A)_A, \not{D} \otimes 1 \otimes 1 + 1 \otimes M_{\tilde{\ell}|_{Y_0} \otimes 1}))$$

for the connected component Y_0 of Y containing x_0, where the homomorphism $\kappa_{Y_0} : C_0(Y, A)^G \to C_0(Y_0, A)^{G_{Y_0}}$ is given by restricting to Y_0.

Proof. Suppose that $x_0 \in Y_0$. We first notice that

$$\begin{aligned} &(C_0(Y, A)^G, L^2(Y, \not{S} \tilde{\otimes} V \otimes A)_A, \not{D} \tilde{\otimes} 1 \otimes 1 + 1 \tilde{\otimes} M_{\tilde{\ell}} \otimes 1) \\ &\quad = (C_0(Y, A)^G, L^2(Y_0, \not{S} \tilde{\otimes} V \otimes A)_A, \not{D} \tilde{\otimes} 1 \otimes 1 + 1 \tilde{\otimes} M_{\tilde{\ell}} \otimes 1) \\ &\qquad \oplus (C_0(Y, A)^G, L^2(Y \setminus Y_0, \not{S} \tilde{\otimes} V \otimes A)_A, \not{D} \tilde{\otimes} 1 \otimes 1 + 1 \tilde{\otimes} M_{\tilde{\ell}} \otimes 1) \end{aligned}$$

and, since the image of the representation of $C_0(Y, A)^G$ on $L^2(Y_0, \not{S} \tilde{\otimes} V \otimes A)$ is equal to $C_0(Y_0, A)^{G_{Y_0}}$,

$$\begin{aligned} &(C_0(Y, A)^G, L^2(Y_0, \not{S} \tilde{\otimes} V \otimes A)_A, \not{D} \tilde{\otimes} 1 \otimes 1 + 1 \tilde{\otimes} M_{\tilde{\ell}} \otimes 1) \\ &\quad = \kappa_{Y_0}^*((C_0(Y_0, A)^{G_{Y_0}}, L^2(Y_0, \not{S}|_{Y_0} \tilde{\otimes} V \otimes A)_A, \not{D} \otimes 1 \otimes 1 + 1 \otimes M_{\tilde{\ell}|_{Y_0} \otimes 1})). \end{aligned}$$

Therefore, letting $Y' = Y \setminus Y_0$ if $x_0 \in Y_0$ or $Y' = Y$ if $x_0 \notin Y$, it will suffice to show that

$$(C_0(Y, A)^G, L^2(Y', \not{S} \tilde{\otimes} V \otimes A)_A, \not{D} \tilde{\otimes} 1 \otimes 1 + 1 \tilde{\otimes} M_{\tilde{\ell}} \otimes 1)$$

represents the zero class in $KK_{m+n}(C_0(Y, A)^G, A)$.

First, $\ell|_{Y'}$ is a nonvanishing continuous function from Y' to $\mathscr{C}\ell_n$. Indeed, since Y is closed in X, $|\ell|_{Y'}|$ is bounded below by some $\varepsilon > 0$; cf. [BH99, Proposition II.2.4]. The operator $s = M_{\operatorname{sgn} \ell|_{Y'}}$ is self-adjoint and unitary and commutes with the left action of $C_0(Y, A)^G$ on $L^2(Y', \not{S}|_{Y'} \tilde{\otimes} V)$. Let $f : \mathbb{R} \to \mathbb{R}$ be any Lipschitz continuous function which takes $x \mapsto x^{-1}$ for $x \geq \varepsilon$. If $\gamma_1, \dots, \gamma_m$ are the generators of $\mathscr{C}\ell_m$,

$$s = \sum_{i=1}^{m} \langle s \mid \gamma_i \rangle \gamma_i = \sum_{i=1}^{m} |\tilde{\ell}|^{-1} \langle \tilde{\ell} \mid \gamma_i \rangle \gamma_i = \sum_{i=1}^{m} f(|\tilde{\ell}|) \langle \tilde{\ell} \mid \gamma_i \rangle \gamma_i.$$

We see by Lemma II.3.28 that each $\langle s \mid \gamma_i \rangle$ is a Lipschitz continuous function on Y'. Hence

$$\begin{aligned} &(\not{D} \tilde{\otimes} 1 + 1 \tilde{\otimes} M_{\tilde{\ell}|_{Y'}})(1 \otimes s) + (1 \otimes s)(\not{D} \tilde{\otimes} 1 + 1 \tilde{\otimes} M_{\tilde{\ell}|_Z}) \\ &\quad = \sum_{i=1}^{m} [\not{D}, \langle s \mid \gamma_i \rangle] \tilde{\otimes} \gamma_i + 1 \tilde{\otimes} 2M_{|\tilde{\ell}|_Z|} \end{aligned}$$

is semi-bounded below. Therefore

$$(C_0(Y, A)^G, L^2(Y', \not{S} \tilde{\otimes} V \otimes A)_A, \not{D} \tilde{\otimes} 1 \otimes 1 + 1 \tilde{\otimes} M_{\tilde{\ell}} \otimes 1)$$

is positively degenerate, in the sense of Definition I.1.12, and so represents $0 \in KK_{m+n}(C_0(Y, A)^G, A)$. $\square$

Lemma II.3.33. *Let X be a locally compact Hausdorff space with a proper action of a locally compact group G. Let A be a G-C^*-algebra. Let ${}^G C_0(X,A)$ be the partial imprimitivity $C_0(X,A)^G$-$C_0(X,A)\rtimes_r G$-bimodule of Corollary A.2.7. Fix $x_0 \in X$ and let $\omega : C_0(X) \to C_0(G)$ be the G-equivariant $*$-homomorphism given by $\omega(f)(g) = f(g \cdot x_0)$. Then*

$$[{}^G C_0(X,A)] \otimes_{C_0(X,A)\rtimes_r G} j_r^G(\sigma_A([\omega])) \otimes [L^2(G,A)] \in KK_0(C_0(X,A)^G, A)$$

is equal to the class of the homomorphism $\eta : C_0(X,A) \to A$ given by evaluation at x_0.

Proof. Write

$$\tilde{\omega} : C_0(X,A) \rtimes_r G \to C_0(G,A) \rtimes G$$

for the homomorphism induced by ω, given by

$$\tilde{\omega}(f)(h,g) = f(h, g \cdot x_0) \qquad (f \in C_c(G \times X, A) \subseteq C_0(X,A) \rtimes G).$$

Composing $\tilde{\omega}$ with the integrated representation of $C_0(G,A) \rtimes G$ on $L^2(G,A)$ produces a representation π of $C_0(X,A) \rtimes G$ on $L^2(G,A)$, given by

$$(\pi(f)\xi)(g) = \int_G \alpha_{g^{-1}}(\tilde{\omega}(f)(h,g))\xi(h^{-1}g)d\mu(h) = \int_G \alpha_{g^{-1}}(f(h, g \cdot x_0))\xi(h^{-1}g)d\mu(h)$$

for $f \in C_c(G \times X, A)$ and $\xi \in C_c(G,A)$. The image of π is in $\mathrm{End}_A^0(L^2(G,A)) \cong C_0(G,A) \rtimes G$, so

$$(C_0(X,A) \rtimes G, {}_\pi(L^2(G,A) \oplus 0)_A, 0)$$

is an even unbounded Kasparov $C_0(X,A)\rtimes G$-A-module, representing the class $[\pi] = j_r^G([\omega]) \otimes_{C_0(G,A)\rtimes G} [L^2(G,A)] \in KK_0(C_0(X,A)\rtimes G, A)$. Let us consider the induced left action of $C_0(X,A)^G \subseteq C_b(X,A) \subseteq M(C_0(X,A) \rtimes G)$ on $L^2(G,A)$. For $f \in C_0(X,A)^G$,

$$(\pi(f)\xi)(g) = \alpha_{g^{-1}}(f(g \cdot x_0))\xi(g) = f(x_0)\xi(g)d\mu(s).$$

Let c be a cut-off function for the action of G on X. Define $p_c \in M(C_0(X,A) \rtimes_r G)$, as in (A.2.5), by

$$p_c(g,x) = c(x)c(g^{-1}x)\Delta_G(g^{-1})^{1/2}.$$

By Corollary A.2.7, ${}^G C_0(X,A) \cong p_c(C_0(X,A) \rtimes_r G)$. Following [EE11, Lemma 3.9], define the vector $c_{x_0} \in C_c(C,A) \subseteq L^2(G,A)$ by $c_{x_0}(g) = c(g \cdot x_0)\Delta(g^{-1})^{1/2}$. For $\xi \in C_c(G,A)$, we have

$$\begin{aligned}
(\pi(p_c)\xi)(g) &= \int_G p_c(s, g \cdot x_0)\xi(s^{-1}g)d\mu(s) \\
&= \int_G c(g \cdot x_0)c(s^{-1}g \cdot x_0)\Delta(s^{-1})^{1/2}\xi(s^{-1}g)d\mu(s) \\
&= \int_G c(g \cdot x_0)c(s^{-1} \cdot x_0)\Delta(s^{-1}g^{-1})^{1/2}\xi(s^{-1})d\mu(s) \\
&= c_{x_0}(g) \int_G c(s \cdot x_0)\Delta(s^{-1})^{1/2}\xi(s)d\mu(s).
\end{aligned}$$

In other words, $\pi(p_c)$ projects $L^2(G,A)$ down to the right Hilbert A-submodule $c_{x_0}A$. Hence

$${}^G C_0(X,A) \otimes_\pi L^2(G,A) \cong p_c(C_0(X,A) \rtimes_r G) \otimes_\pi L^2(G,A) = \pi(p_c)L^2(G,A) = c_{x_0}A.$$

The left action of $C_0(X,A)^G$ on $c_{x_0}A$ is given by $f(x_0)$, as required. ∎

Let us recall and finally prove

Theorem II.3.8. *Let G be a locally compact group acting properly and isometrically on a CAT(0) space (X,d). Let A be a G-C^*-algebra. Suppose that there is a complete subspace Y of X such that*

- *every path component of Y is a convex subset of X;*
- *Y is isometric to a spin^c Riemannian n-manifold;*
- *Y contains a neighbourhood of a point $x_0 \in X$; and*
- *G preserves Y and acts by spin^c automorphisms.*

Let $x_1 \in X$ be a point not in Y but with $S_{x_1}(X)$ isometric to a sphere $\mathbf{S}^{m-1} \subseteq \mathbb{R}^m$. Let V_0 be the Clifford module $\$_{x_0}$ for $\mathscr{C}\ell_n$, with $\$$ the fundamental spinor bundle on Y. Let V_1 be a Clifford module for $\mathscr{C}\ell_m$. Define the weights

$$\begin{aligned} \ell_0 : G &\to \operatorname{End} V_0 \\ g &\mapsto d(g^{-1}x_0, x_0)v(g^{-1}x_0, x_0) \end{aligned} \qquad \begin{aligned} \ell_1 : G &\to \operatorname{End} V_1 \\ g &\mapsto d(g^{-1}\cdot x_1, x_1)v(g^{-1}\cdot x_1, x_1), \end{aligned}$$

giving rise to $\sigma_A([\ell_0]), \sigma_A([\ell_1]) \in KK_*^G(A, C_0(G,A))$ and $[M_{\ell_0}], [M_{\ell_1}] \in KK_*^{\hat{G}}(A \rtimes_r G, A)$ as in Corollary II.3.5.

Let $\alpha_Y \in KK_n^G(C_0(Y), \mathbb{C})$ be the Atiyah–Singer Dirac class and let ${}^G C_0(Y,A)$ be the partial imprimitivity $C_0(Y,A)^G$-$C_0(Y,A) \rtimes G$-bimodule of Theorem A.2.6. With $\eta_G : C_0(Y,A)^G \to A$ the $*$-homomorphism given by evaluating at x_0,

$$[{}^G C_0(Y,A)] \otimes_{C_0(Y,A)\rtimes G} j_r^G(\sigma_A(\alpha_Y)) \otimes_{A\rtimes_r G} r^{\hat{G},1}([M_{\ell_0}]) = [\eta_G] \in KK_0(C_0(Y,A)^G, A)$$

and

$$[{}^G C_0(Y,A)] \otimes_{C_0(Y,A)\rtimes G} j_r^G(\sigma_A(\alpha_Y)) \otimes_{A\rtimes_r G} r^{\hat{G},1}([M_{\ell_1}]) = 0 \in KK_{m+n}(C_0(Y,A)^G, A).$$

Proof. Let Y_0 be the path component of Y containing x_0, G_{Y_0} its stabiliser group, and η_{G,Y_0} : $C_0(Y_0,A)^{G_{Y_0}} \to A$ the $*$-homomorphism given by evaluating at x_0. Let $\alpha_Y \in KK_n^G(C_0(Y),\mathbb{C})$ be the Atiyah–Singer Dirac class for Y and let $\alpha_{Y_0} \in KK_n^{G_{Y_0}}(C_0(Y_0),\mathbb{C})$ be the Atiyah–Singer Dirac class for Y_0. We remark that $\alpha_{Y_0} = \iota^*(\alpha_Y)$ for the inclusion $\iota : Y_0 \hookrightarrow Y$. Recall that, by Lemma II.3.31, $C_0(Y,A)^G$ is isomorphic to $C_0(Y_0,A)^{G_{Y_0}} \oplus C_0(Y \setminus GY_0, A)^G$. Let $\kappa_{Y_0} : C_0(Y,A)^G \to C_0(Y_0,A)^{G_{Y_0}}$ be the $*$-homomorphism given by restricting to the first term of the direct sum. Remark that $\eta_{G,Y_0} \circ \kappa_{Y_0} = \eta_G$.

Proposition II.3.30 tells us that, for $j \in \{0,1\}$,

$$[{}^G C_0(Y,A)] \otimes_{C_0(Y,A)\rtimes G} j_r^G(\sigma_A(\alpha_Y)) \otimes_{A\rtimes_r G} [M_{\ell_j}] \in KK_{m+n}(C_0(Y,A)^G, A)$$

is represented by

$$(C_0(Y,A)^G, L^2(Y, \$ \mathbin{\tilde{\otimes}} V \otimes A)_A, \not{D} \mathbin{\tilde{\otimes}} 1 \otimes 1 + 1 \mathbin{\tilde{\otimes}} M_{\tilde{\ell}_j} \otimes 1).$$

Proposition II.3.32 tells us that

$$(C_0(Y,A)^G, L^2(Y, \$ \mathbin{\tilde{\otimes}} V \otimes A)_A, \not{D} \mathbin{\tilde{\otimes}} 1 \otimes 1 + 1 \mathbin{\tilde{\otimes}} M_{\tilde{\ell}_0} \otimes 1)$$

and

$$\kappa_{Y_0}^*((C_0(Y_0,A)^{G_{Y_0}}, L^2(Y_0, \$|_{Y_0} \mathbin{\tilde{\otimes}} V \otimes A)_A, \not{D} \otimes 1 \otimes 1 + 1 \otimes M_{\tilde{\ell}_0|_{Y_0} \otimes 1}))$$

have the same class in $KK_0(C_0(Y,A)^G, A)$ and that

$$(C_0(Y,A)^G, L^2(Y, \$ \mathbin{\tilde{\otimes}} V \otimes A)_A, \not{D} \mathbin{\tilde{\otimes}} 1 \otimes 1 + 1 \mathbin{\tilde{\otimes}} M_{\tilde{\ell}_1} \otimes 1)$$

has trivial class in $KK_{m+n}(C_0(Y,A)^G, A)$. That is,

$$\begin{aligned}[][^GC_0(Y,A)] &\otimes_{C_0(Y,A)\rtimes G} j_r^G(\sigma_A(\alpha_Y)) \otimes_{A\rtimes_r G} [M_{\ell_0}] \\ &= [\kappa_{Y_0}] \otimes_{C_0(Y_0,A)^{G_{Y_0}}} [^{G_{Y_0}}C_0(Y_0,A)] \otimes_{C_0(Y,A)\rtimes G} j_r^G(\sigma_A(\alpha_{Y_0})) \otimes_{A\rtimes_r G} [M_{\ell_0}].\end{aligned}$$

and

$$[^GC_0(Y,A)] \otimes_{C_0(Y,A)\rtimes G} j_r^G(\sigma_A(\alpha_Y)) \otimes_{A\rtimes_r G} [M_{\ell_1}] = 0.$$

Let $\omega : C_0(Y_0) \to C_0(G)$ be the G_{Y_0}-equivariant $*$-homomorphism given by

$$\omega(f)(g) = f(g \cdot x_0).$$

Since Y_0 is a spin^c Hadamard manifold, with G_{Y_0} acting by spin^c isometries, there is a dual Dirac element $\beta_{Y_0} \in KK_n^{G_{Y_0}}(\mathbb{C}, C_0(Y_0))$ for which, by definition,

$$\alpha_{Y_0} \otimes_{\mathbb{C}} \beta_{Y_0} = 1 \in KK_0^{G_{Y_0}}(C_0(Y_0), C_0(Y_0)).$$

Further, by Remark II.3.13, $\beta_{Y_0} \otimes_{C_0(Y_0)} [\omega] = [\ell_0]$. By Lemma II.3.33,

$$[^GC_0(X,A)] \otimes_{C_0(X,A)\rtimes_r G} j_r^G(\sigma_A([\omega])) \otimes [L^2(G,A)] = [\eta_{G,Y_0}] \in KK_0(C_0(X,A)^G, A)$$

where, as above, $\eta_{G,Y_0} : C_0(Y_0,A)^{G_{Y_0}} \to A$ is the $*$-homomorphism given by evaluating at x_0. Putting all this together,

$$\begin{aligned}[][^{G_{Y_0}}C_0(Y_0,A)] &\otimes_{C_0(Y,A)\rtimes G} j_r^G(\sigma_A(\alpha_{Y_0})) \otimes_{A\rtimes_r G} [M_{\ell_0}] \\ &= [^{G_{Y_0}}C_0(Y_0,A)] \otimes_{C_0(Y,A)\rtimes G} j_r^G(\sigma_A(\alpha_{Y_0})) \otimes_{A\rtimes_r G} j_r^G(\sigma_A([\ell_0])) \otimes_{C_0(G,A)\rtimes_r G} [L^2(G,A)] \\ &= [^{G_{Y_0}}C_0(Y_0,A)] \otimes_{C_0(Y,A)\rtimes G} j_r^G(\sigma_A(\alpha_{Y_0} \otimes_{\mathbb{C}} \beta_{Y_0} \otimes_{C_0(Y_0)} [\omega])) \otimes_{C_0(G,A)\rtimes_r G} [L^2(G,A)] \\ &= [^{G_{Y_0}}C_0(Y_0,A)] \otimes_{C_0(Y,A)\rtimes G} j_r^G(\sigma_A([\omega])) \otimes_{C_0(G,A)\rtimes_r G} [L^2(G,A)] \\ &= [\eta_{G,Y_0}]\end{aligned}$$

and so

$$\begin{aligned}[][^GC_0(Y,A)] &\otimes_{C_0(Y,A)\rtimes G} j_r^G(\sigma_A(\alpha_Y)) \otimes_{A\rtimes_r G} [M_{\ell_0}] \\ &= [\kappa_{Y_0}] \otimes_{C_0(Y_0,A)^{G_{Y_0}}} [^{G_{Y_0}}C_0(Y_0,A)] \otimes_{C_0(Y,A)\rtimes G} j_r^G(\sigma_A(\alpha_{Y_0})) \otimes_{A\rtimes_r G} [M_{\ell_0}] \\ &= [\kappa_{Y_0}] \otimes_{C_0(Y_0,A)^{G_{Y_0}}} [\eta_{G,Y_0}] \\ &= [\eta_G],\end{aligned}$$

as required. □

II.4 The Kasparov product for group extensions

The understanding of group extensions in the framework of this Chapter is a microcosm of the more general problem of the constructive unbounded Kasparov product. In particular, the constructive product will often fail when approached naïvely, as we shall see in §§II.4.1 and II.4.2.

The external Kasparov product gives a map

$$KK^H(\mathbb{C}, C_0(H)) \otimes KK^N(\mathbb{C}, C_0(N)) \to KK^{H\times N}(\mathbb{C}, C_0(H \times N))$$

for a direct product $H \times N$ of groups. This map is constructive for unbounded Kasparov modules. Indeed, if one has self-adjoint, proper, translation-bounded weights $\ell_N : N \to \operatorname{End} V_N$ and $\ell_H : H \to \operatorname{End} V_H$ then

$$\ell_{H\times N} := \pi_1^*(\ell_H) \tilde{\otimes} 1 + 1 \tilde{\otimes} \pi_2^*(\ell_N) : H \times N \to V_{H\times N} := V_H \tilde{\otimes} V_N$$

is a self-adjoint, proper, translation-bounded weight, where $\pi_1 : H \times N \to H$ and $\pi_2 : H \times N \to N$ are the projection homomorphisms.

More generally, let us consider short exact sequences

$$0 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} H \longrightarrow 0,$$

of locally compact groups, that is, ι is a homeomorphism onto $\iota(N)$, a closed normal subgroup of G, and H is isomorphic to $G/\iota(N)$ as a topological group; see [Wil07, §1.2.4] for more details.

Proposition II.4.1. *Let*

$$0 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} H \longrightarrow 0$$

be an exact sequence of locally compact groups and let A be a G-C^*-algebra. Suppose that we have a self-adjoint, proper, translation-bounded weight $\ell_N : N \to \operatorname{End} V_N$. Define the weight $\widehat{\ell_N} : G \to \operatorname{End} V_N$ by

$$\widehat{\ell_N}(g) = \int_H c(gs)^2 \ell_N(s^{-1}) d\mu_N(s)$$

where $c \in C_b(G)$ is a cut-off function for the right action of N on G. Then

$$\left(C_0(H, A), C_0(G, A \otimes V_N)_{C_0(G,A)}, \widehat{\ell_N}\right)$$

is a uniformly G-equivariant unbounded Kasparov module.

More generally, if $\widetilde{\ell_N} : G \to V_N$ is a self-adjoint weight such that

$$\sup_{n \in N} \|\widetilde{\ell_N}(\iota(n)) - \ell_N(n)\| < \infty$$

then

$$\left(C_0(H, A), C_0(G, A \otimes V_N)_{C_0(G,A)}, \widetilde{\ell_N}\right)$$

is a uniformly G-equivariant unbounded Kasparov module.

Proof. By Proposition II.2.32, $\widehat{\ell_N}(g)$ is a self-adjoint weight. Further, the algebra $C_0(H, A)$ commutes with the operator $\widehat{\ell_N}$.

For the G-equivariance, observe that, for an implementer U_g of the action, an algebra element $f \in C_c(H, A)$, and a vector $\xi \in C_c(G, A \otimes V_N)$,

$$((U_g \ell U_g^* - \ell) f \xi)(h) = (\ell(gh) - \ell(h)) f(\pi(h)) \xi(h).$$

So

$$\left\|(U_g \widehat{\ell_N} U_g^* - \widehat{\ell_N}) f\right\| \leq \|f\| \sup_{ku \in \operatorname{supp} f} \left\|\widehat{\ell_N}(gku) - \widehat{\ell_N}(ku)\right\| < \infty$$

by Proposition II.2.32. Furthermore, if we apply Lemma II.2.4 to the function from $G \times G$ to $\operatorname{End} V_N$ given by $(g, h) \mapsto (\ell(gh) - \ell(h)) f(h)$, we obtain that $g \mapsto (U_g \ell U_g^* - \ell) f$ is an element of $C(G, C_b(G, \operatorname{End} V_N)_\beta)$ and so $*$-strongly continuous into $\operatorname{End}^*(C_0(G, A \otimes V)) \cong C_b(G, M(A)_\beta \otimes \operatorname{End} V_N)$.

Again let $f \in C_c(H, A)$. By Proposition II.2.32,

$$(1 + \widehat{\ell}^2)^{-1}|_{\pi^{-1}(\operatorname{supp}(f))} \in C_0(\pi^{-1}(\operatorname{supp}(f)), \operatorname{End} V_N) \subseteq C_0(G, \operatorname{End} V_N).$$

So, for all $a \in A$, $af(1 + \ell^2)^{-1} \in C_0(G, A \otimes \operatorname{End} V_N)$. $\square$

Theorem II.4.2. *Let*
$$0 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} H \longrightarrow 0$$
be an exact sequence of locally compact groups and let A be a G-C^*-algebra. Suppose that we have self-adjoint, proper, translation-bounded weights $\ell_N : N \to \operatorname{End} V_N$ and $\ell_H : H \to \operatorname{End} V_H$. Define the weight $\widehat{\ell_N} : G \to \operatorname{End} V_N$ by
$$\widehat{\ell_N}(g) = \int_H c(gs)^2 \ell_N(s^{-1}) d\mu_N(s)$$
where $c \in C_b(G)$ is a cut-off function for the right action of N on G. Then
$$\ell_G := \pi^*(\ell_H) \mathbin{\tilde{\otimes}} 1 + 1 \mathbin{\tilde{\otimes}} \widehat{\ell_N} : G \to V_G := V_H \mathbin{\tilde{\otimes}} V_N$$
is a self-adjoint proper weight. If $\widehat{\ell_N}$ is translation-bounded, then so is ℓ_G and
$${}^G\Big(A, C_0(G, A \otimes V_G)_{C_0(G,A)}, \ell_G\Big)$$
represents the Kasparov product
$${}^G\Big(A, C_0(H, A \otimes V_H)_{C_0(H,A)}, \ell_H\Big) \otimes_{C_0(H,A)} {}^G\Big(C_0(H,A), C_0(G, A \otimes V_N)_{C_0(G,A)}, \widehat{\ell_N}\Big). \tag{II.4.3}$$
More generally, suppose that we have a self-adjoint, translation-bounded weight $\widetilde{\ell_N} : G \to V_N$ such that
$$\sup_{n \in N} \|\widetilde{\ell_N}(\iota(n)) - \ell_N(n)\| < \infty.$$
Then $\ell_G := \pi^*(\ell_H) \mathbin{\tilde{\otimes}} 1 + 1 \mathbin{\tilde{\otimes}} \widetilde{\ell_N} : G \to V_G$ is a self-adjoint, proper, translation-bounded weight and
$${}^G\Big(A, C_0(G, A \otimes V_G)_{C_0(G,A)}, \ell_G\Big)$$
represents the Kasparov product (II.4.3).

Proof. It is immediate that ℓ_G is self-adjoint. Instead of proving the properness of ℓ_G directly we shall make use of the constructive unbounded Kasparov product. First, for $\xi \in C_c(H, A \otimes V_H)$ and the corresponding operator $T_\xi \in \operatorname{Hom}^*(C_0(G, A \otimes V_N), C_0(G, A \otimes V_G))$,
$$(1 \mathbin{\tilde{\otimes}} \widehat{\ell_N}) T_\xi - T_\xi \widehat{\ell_N} = 0.$$
Second, $\pi^*(\ell_H) \mathbin{\tilde{\otimes}} 1$ and $1 \mathbin{\tilde{\otimes}} \widehat{\ell_N}$ anticommute on the common core $C_c(G, A \otimes V_G)$, which they also preserve. Hence, by [LM19, Theorem 7.4],
$$a(1 + \ell_G^2)^{-1} \in \operatorname{End}^0(G, A \otimes V_G) = C_0(G, A \otimes \operatorname{End} V_G)$$
for all $a \in A$. By considering the case $A = \mathbb{C}$, we obtain that ℓ_G is proper. Now, suppose that $\widehat{\ell_N}$ is translation-bounded, making ℓ_G translation-bounded as well. We obtain an unbounded Kasparov module
$${}^G\Big(A, C_0(G, A \otimes V_G)_{C_0(G,A)}, \ell_G\Big)$$
which represents the Kasparov product.

Now, instead, suppose we have a weight $\widetilde{\ell_N}$ as in the statement. The self-adjointness of $\ell_G := \pi^*(\ell_H) \mathbin{\tilde{\otimes}} 1 + 1 \mathbin{\tilde{\otimes}} \widetilde{\ell_N} : G \to V_G$ is immediate. Next, for all $g \in G$,
$$\begin{aligned}
\sup_{h \in G} \Big\|\ell_G(gh) - \ell_G(h)\Big\|^2 &= \sup_{h \in G} \Big\|\Big(\ell_H(\pi(gh)) - \ell_H(\pi(h))\Big) \mathbin{\tilde{\otimes}} 1 + 1 \mathbin{\tilde{\otimes}} \Big(\widetilde{\ell_N}(gh) - \widetilde{\ell_N}(h)\Big)\Big\|^2 \\
&\le \sup_{s \in H} \Big\|\ell_H(\pi(g)s) - \ell_H(s)\Big\|^2 + \sup_{h \in G} \Big\|\widetilde{\ell_N}(gh) - \widetilde{\ell_N}(h)\Big\|^2 \\
&< \infty.
\end{aligned}$$

Remark that

$$\begin{aligned}\widetilde{\ell_N}(g) - \widehat{\ell_N}(g) &= \int_H c(gs)^2(\widetilde{\ell_N}(g) - \ell_N(s^{-1}))d\mu_N(s)\\ &= \int_H c(gs)^2\Big(\widetilde{\ell_N}(g) - \widetilde{\ell_N}(\iota(s^{-1})) + \widetilde{\ell_N}(\iota(s^{-1})) - \ell_N(s^{-1})\Big)d\mu_N(s)\end{aligned}$$

and so

$$\begin{aligned}\sup_{g\in G}\left\|\widetilde{\ell_N}(g) - \widehat{\ell_N}(g)\right\| &\leq \sup_{g\in G}\sup_{s\in N\cap g^{-1}\operatorname{supp} c}\left\|\widetilde{\ell_N}(g) - \widetilde{\ell_N}(\iota(s^{-1}))\right\| + \sup_{s\in N}\left\|\widetilde{\ell_N}(\iota(s^{-1})) - \ell_N(s^{-1})\right\|\\ &\leq \sup_{g\in G}\sup_{s\in N\cap\operatorname{supp} c}\left\|\widetilde{\ell_N}(g) - \widetilde{\ell_N}(\iota(s^{-1})g)\right\| + \sup_{s\in N}\left\|\widetilde{\ell_N}(\iota(s^{-1})) - \ell_N(s^{-1})\right\|\\ &< \infty\end{aligned}$$

by the compactness of $N \cap \operatorname{supp} c$ and the assumed bound. Hence, $\widetilde{\ell_N} - \widehat{\ell_N} \in C_b(G, \operatorname{End} V)$. As we have seen, $\pi^*(\ell_H) \tilde{\otimes} 1 + 1 \tilde{\otimes} \widehat{\ell_N}$ is proper (without any assumption on $\widehat{\ell_N}$) and therefore so is $\ell_G = \pi^*(\ell_H) \tilde{\otimes} 1 + 1 \tilde{\otimes} \widetilde{\ell_N}$. The boundedness of the difference implies that the class in KK-theory is the same. □

Remark II.4.4. By Remark II.2.34, we can indeed take $\widetilde{\ell_N} = \widehat{\ell_N}$ in Theorem II.4.2, except that it may not be translation-bounded. Implicit is the fact that, if any $\widetilde{\ell_N}$ exists fulfilling all the conditions of Theorem II.4.2, then the collection of all such weights $\widetilde{\ell_N}$ forms an affine space containing $\widehat{\ell_N}$. This is not surprising; indeed, such weights are connections for the constructive unbounded Kasparov product; cf. [Mes12].

We now turn to the reduced partial cross-sectional Fell bundles of §II.2.2.2.

Proposition II.4.5. *Let*

$$0 \longrightarrow N \xrightarrow{\ \iota\ } G \xrightarrow{\ \pi\ } H \longrightarrow 0$$

be an exact sequence of locally compact groups. Let $\mathscr{B}$ be a fissured Fell bundle over G and $\mathscr{C}$ the reduced partial-cross sectional Fell bundle over $H = G/N$. If $\ell_H : H \to \operatorname{End} V_H$ is a self-adjoint, proper, translation-bounded weight then

$$\Big(C_r^*(\mathscr{C}), (L^2(\mathscr{C}) \otimes V_H)_{C_N}, M_{\ell_H}\Big)$$

is an isometrically $\hat{G}$-equivariant unbounded Kasparov module.

Proof. By Proposition II.2.23, $\mathscr{C}$ is fissured. Applying Theorem II.2.24,

$$\Big(C_r^*(\mathscr{C}), (L^2(\mathscr{C}) \otimes V_H)_{C_N}, M_{\ell_H}\Big)$$

is an isometrically $\hat{H}$-equivariant unbounded Kasparov module. However, since $C_r^*(\mathscr{C})$ is isomorphic to $C_r^*(\mathscr{B})$, it possesses a $\hat{G}$-action. Further, since $C_N = C_r^*(\mathscr{B}_N)$, it too possesses a $\hat{G}$-action, pulled back from the $\hat{N}$-action. The formula (II.2.11) gives a $\hat{G}$-action on $L^2(\mathscr{C})$, for the dense subset $C_c(\mathscr{B})$. It is routine to check that these actions of $\hat{G}$ are compatible. The fact that M_ℓ acts by multiplication by $\ell_H(\pi(g))$ on each fibre B_g of $\mathscr{B}$ implies that it is isometrically $\hat{G}$-equivariant. □

Theorem II.4.6. *Let*

$$0 \longrightarrow N \xrightarrow{\ \iota\ } G \xrightarrow{\ \pi\ } H \longrightarrow 0$$

be an exact sequence of locally compact groups. Let $\mathscr{B}$ be a fissured Fell bundle over G and $\mathscr{C}$ the reduced partial-cross sectional Fell bundle over $H = G/N$.

Let $\ell_N : N \to \operatorname{End} V_N$ and $\ell_H : H \to \operatorname{End} V_H$ be self-adjoint, proper, translation-bounded weights. Suppose that we have a self-adjoint, translation-bounded weight $\widetilde{\ell_N} : G \to V_N$ such that

$$\sup_{n\in N} \|\widetilde{\ell_N}(\iota(n)) - \ell_N(n)\| < \infty. \tag{II.4.7}$$

Let $\ell_G := \pi^*(\ell_H) \tilde{\otimes} 1 + 1 \tilde{\otimes} \widetilde{\ell_N} : G \to V_G := V_H \tilde{\otimes} V_N$. The unbounded Kasparov module

$${}^{\hat{G}}\Big(C_r^*(\mathscr{B}), (L^2(\mathscr{B}) \otimes V_G)_{B_e}, M_{\ell_G}\Big)$$

represents the Kasparov product

$${}^{\hat{G}}\Big(C_r^*(\mathscr{C}), (L^2(\mathscr{C}) \otimes V_H)_{C_N}, M_{\ell_H}\Big) \otimes_{C_r^*(\mathscr{B}_N)} {}^{\hat{G}}\Big(C_r^*(\mathscr{B}_N), (L^2(\mathscr{B}_N) \otimes V_N)_{B_e}, M_{\ell_N}\Big).$$

Recall that, by definition, $C_N = C_r^*(\mathscr{B}_N)$ and that there is an isomorphism $C_r^*(\mathscr{C}) \cong C_r^*(\mathscr{B})$, so the statement is sensible.

Proof. First, recall from §II.2.2.2 the isomorphism

$$L^2(\mathscr{C}) \otimes_{C_N} L^2(\mathscr{B}_H) \cong L^2(\mathscr{B})$$

of Hilbert B_e-modules. Second, $\pi^*(\ell_H) \tilde{\otimes} 1$ and $1 \tilde{\otimes} \widetilde{\ell_N}$ anticommute, so the positivity condition of Theorem I.4.3 is satisfied. We therefore need only to check the connection condition.

Let $\zeta \in C_c(\mathscr{B}) \otimes V_H \subseteq L^2(\mathscr{C}) \otimes V_H$ and consider the operator $T_\zeta \in \operatorname{Hom}^*(L^2(\mathscr{B}_N) \otimes V_N, L^2(\mathscr{B}) \otimes V_G)$ given by

$$(T_\zeta \xi)(g) = \int_N \zeta(gn) \tilde{\otimes} \xi(n^{-1}) d\mu_N(n)$$

for $\xi \in C_c(\mathscr{B}_N) \otimes V_N$ and $g \in G$. We have

$$M_{\ell_G} T_\zeta - T_\zeta M_{\ell_N} = (1 \tilde{\otimes} M_{\widetilde{\ell_N}}) T_\zeta - T_\zeta M_{\iota^*(\widetilde{\ell_N})} + T_\zeta M_{\iota^*(\widetilde{\ell_N}) - \ell_N} + (M_{\pi^*(\ell_H)} \tilde{\otimes} 1) T_\zeta.$$

The final two terms are both bounded, by the assumption (II.4.7), so it will suffice for the connection condition to show that the remainder is also bounded. Still with $\xi \in C_c(\mathscr{B}_N) \otimes V_N$, for $g \in G$,

$$\Big(\big((1 \tilde{\otimes} M_{\widetilde{\ell_N}}) T_\zeta - T_\zeta M_{\iota^*(\widetilde{\ell_N})}\big)\xi\Big)(g) = \int_N \zeta(gn) \tilde{\otimes} \Big(\widetilde{\ell_N}(g) - \widetilde{\ell_N}(n^{-1})\Big)\xi(n^{-1}) d\mu_N(n).$$

Let $\pi : B_e \to B(H_\pi)$ be an irreducible representation of B_e and let $\eta \in H_\pi$, so that $\xi \otimes \eta \in (L^2(\mathscr{B}_N) \otimes V_N) \otimes_\pi H_\pi$. With $\xi_2 \in C_c(\mathscr{B}) \otimes V_G \subseteq L^2(\mathscr{B}) \otimes V_G$, we compute

$$\begin{aligned}
&\Big|\Big\langle \xi_2 \otimes \eta \Big| \big((1 \tilde{\otimes} M_{\widetilde{\ell_N}}) T_\zeta - T_\zeta M_{\iota^*(\widetilde{\ell_N})}\big) \xi \otimes \eta \Big\rangle\Big| \\
&\quad \leq \int_G \int_N \Big|\Big\langle \xi_2(g) \otimes \eta \Big| \zeta(gn) \tilde{\otimes} \Big(\widetilde{\ell_N}(g) - \widetilde{\ell_N}(n^{-1})\Big)\xi(n^{-1}) \otimes \eta \Big\rangle\Big| d\mu_N(n) d\mu_G(g) \\
&\quad \leq \int_G \int_N \|\xi_2(g) \otimes \eta\| \Big\|\widetilde{\ell_N}(g) - \widetilde{\ell_N}(n^{-1})\Big\| \|\zeta(gn)\| \|\xi(n^{-1}) \otimes \eta\| d\mu_N(n) d\mu_G(g).
\end{aligned}$$

Next, we let

$$c = \sup_{g \in \operatorname{supp} \zeta, n \in H} \Big\|\widetilde{\ell_N}(gn) - \widetilde{\ell_N}(n)\Big\|,$$

which is finite by the compactness of $\operatorname{supp}\zeta$ and Lemma II.2.4. Putting $\xi_2 = ((1\tilde{\otimes} M_{\widetilde{\ell_N}})T_\zeta - T_\zeta M_{\iota^*(\widetilde{\ell_N})})\xi$, we estimate that

$$
\begin{aligned}
&\left\|\left((1 \tilde{\otimes} M_{\widetilde{\ell_N}})T_\zeta - T_\zeta M_{\iota^*(\widetilde{\ell_N})}\right)\xi \otimes \eta\right\|^2 \\
&\quad\le \int_G \int_N \int_N \left\|\widetilde{\ell_N}(g) - \widetilde{\ell_N}(s^{-1})\right\| \|\zeta(gs)\| \|\xi(s^{-1}) \otimes \eta\| \\
&\qquad\times \left\|\widetilde{\ell_N}(g) - \widetilde{\ell_N}(n^{-1})\right\| \|\zeta(gn)\| \|\xi(n^{-1}) \otimes \eta\| d\mu_N(s) d\mu_N(n) d\mu_G(g) \\
&\quad\le \int_N \int_N \left(\int_G \|\zeta(gs)\|\|\zeta(gn)\| d\mu_G(g)\right) \\
&\qquad\times \|\xi(s^{-1}) \otimes \eta\| \|\xi(n^{-1}) \otimes \eta\| \chi_{(\operatorname{supp}\zeta)^{-1}(\operatorname{supp}\zeta)}(s^{-1}n) d\mu_N(s) d\mu_N(n) \\
&\qquad\times \sup_{g\in G, n\in H | gn \in \operatorname{supp}\zeta} \left\|\widetilde{\ell_N}(g) - \widetilde{\ell_N}(n^{-1})\right\|^2 \\
&\quad\le c^2 \int_N \int_N \left(\int_G \|\zeta(gs)\|^2 d\mu_G(g)\right)^{1/2} \left(\int_G \|\zeta(gn)\|^2 d\mu_G(g)\right)^{1/2} \\
&\qquad\times \|\xi(s^{-1}) \otimes \eta\| \|\xi(n^{-1}) \otimes \eta\| \chi_{(\operatorname{supp}\zeta)^{-1}(\operatorname{supp}\zeta)}(s^{-1}n) d\mu_N(s) d\mu_N(n) \\
&\quad= c^2 \left(\int_G \|\zeta(g)\|^2 d\mu_G(g)\right) \int_N \int_N \Delta_G(s^{-1})^{1/2} \Delta_G(n^{-1})^{1/2} \\
&\qquad\times \|\xi(s^{-1}) \otimes \eta\| \|\xi(n^{-1}) \otimes \eta\| \chi_{(\operatorname{supp}\zeta)^{-1}(\operatorname{supp}\zeta)}(s^{-1}n) d\mu_N(s) d\mu_N(n) \\
&\quad= \int_N \int_N \Delta_G(s^{-1}) \Delta_G(n^{-1})^{1/2} \|\xi(s^{-1}) \otimes \eta\| \|\xi(n^{-1}s^{-1}) \otimes \eta\| \chi_{(\operatorname{supp}\zeta)^{-1}(\operatorname{supp}\zeta)}(n) d\mu_N(n) d\mu_N(s) \\
&\qquad\times c^2 \left(\int_G \|\zeta(g)\|^2 d\mu_G(g)\right) \\
&\quad= \int_N \int_N \|\xi(s) \otimes \eta\| \|\xi(n^{-1}s) \otimes \eta\| d\mu_N(s) \chi_{(\operatorname{supp}\zeta)^{-1}(\operatorname{supp}\zeta)}(n) \Delta_G(n^{-1})^{1/2} d\mu_N(n) \\
&\qquad\times c^2 \left(\int_G \|\zeta(g)\|^2 d\mu_G(g)\right) \\
&\quad\le c^2 \int_N \left(\int_N \|\xi(s) \otimes \eta\|^2 d\mu_N(s)\right)^{1/2} \left(\int_N \|\xi(n^{-1}s) \otimes \eta\|^2 d\mu_N(s)\right)^{1/2} \\
&\qquad\times \chi_{(\operatorname{supp}\zeta)^{-1}(\operatorname{supp}\zeta)}(n) \Delta_G(n^{-1})^{1/2} d\mu_N(n) \left(\int_G \|\zeta(g)\|^2 d\mu_G(g)\right) \\
&\quad= c^2 \|\xi \otimes \eta\|^2 \int_N \chi_{(\operatorname{supp}\zeta)^{-1}(\operatorname{supp}\zeta)}(n) \Delta_G(n^{-1})^{1/2} d\mu_N(n) \left(\int_G \|\zeta(g)\|^2 d\mu_G(g)\right) \\
&\quad\le c^2 \|\xi \otimes \eta\|^2 \mu_N((\operatorname{supp}\zeta)^{-1}(\operatorname{supp}\zeta)) \sup_{n \in (\operatorname{supp}\zeta)^{-1}(\operatorname{supp}\zeta)} \Delta_G(n^{-1})^{1/2} \left(\int_G \|\zeta(g)\|^2 d\mu_G(g)\right) \\
&\quad= c'^2 \|\xi \otimes \eta\|^2
\end{aligned}
$$

for $0 \le c' < \infty$ independent of ξ and η. Now, taking the supremum over irreducible representations $\pi : B_e \to B(H_\pi)$, Lemma A.3.3 yields

$$\left\|(1 \tilde{\otimes} M_{\widetilde{\ell_N}})T_\zeta - T_\zeta M_{\iota^*(\widetilde{\ell_N})}\right\| \le c' < \infty$$

and we are done. □

Example II.4.8. The universal cover of $SL(2,\mathbb{R})$ fits into the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ \iota\ } \widetilde{SL}(2,\mathbb{R}) \xrightarrow{\ \pi\ } SL(2,\mathbb{R}) \longrightarrow 0 \ .$$

As this is a central extension, we may represent elements of $\widetilde{SL}(2,\mathbb{R})$ as pairs in $SL(2,\mathbb{R})\times\mathbb{Z}$, with the multiplication law

$$(g_1,n_1)(g_2,n_2)=(g_1g_2,n_1+n_2+W(g_1,g_2))$$

for some 2-cocycle W. A remarkably simple explicit form for W is given in [Asa70, Theorem 2]. If we define $\operatorname{sgn}: SL(2,\mathbb{R})\to\{-1,0,+1\}$ by

$$\operatorname{sgn}\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\begin{cases}\operatorname{sgn}(c) & c\neq 0\\ \operatorname{sgn}(a)=\operatorname{sgn}(d) & c=0\end{cases},$$

we may let

$$W(g_1,g_2)=\begin{cases}+1 & +1=\operatorname{sgn}(g_1)=\operatorname{sgn}(g_2)=-\operatorname{sgn}(g_1g_2)\\ -1 & -1=\operatorname{sgn}(g_1)=\operatorname{sgn}(g_2)=-\operatorname{sgn}(g_1g_2)\\ 0 & \text{otherwise}\end{cases}.$$

To produce a weight $\widetilde{\ell_{\mathbb{Z}}}$, we cannot simply take $(g,n)\mapsto n$ as this is not continuous. To remedy the situation, we appeal to the Iwasawa decomposition of $SL(2,\mathbb{R})$. Following [Asa70, §1-1], any element of $SL(2,\mathbb{Z})$ can be uniquely decomposed as

$$g=\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}y^{1/2} & \\ & y^{-1/2}\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}=y^{-1/2}\begin{pmatrix}x\sin\theta+y\cos\theta & x\cos\theta-y\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

where $x+iy$ is a point in the upper-half plane and $\theta\in[-\pi,\pi)$. We have

$$\operatorname{sgn}(g)=\begin{cases}\operatorname{sgn}(\theta) & \theta\in(-\pi,0)\cup(0,\pi)\\ +1 & \theta=0\\ -1 & \theta=-\pi\end{cases}=\begin{cases}+1 & \theta\in[0,\pi)\\ -1 & \theta\in[-\pi,0)\end{cases}.$$

By [Asa70, §1-2], the function $\widetilde{\ell_{\mathbb{Z}}}:\widetilde{SL}(2,\mathbb{R})\to\mathbb{C}$ given by

$$\widetilde{\ell_{\mathbb{Z}}}(g,n)=\frac{\theta}{2\pi}+n$$

is continuous. By [Asa70, (5)], with $\arg:\mathbb{C}\setminus\{0\}\to[-\pi,\pi)$,

$$\widetilde{\ell_{\mathbb{Z}}}((g_1,n_1)(g_2,n_2))=\frac{\theta_2}{2\pi}+n_2+\frac{1}{2\pi}\arg((x_2+iy_2)\sin\theta_1+\cos\theta_1)+n_1$$

and so

$$\widetilde{\ell_{\mathbb{Z}}}((g_1,n_1)(g_2,n_2))-\widetilde{\ell_{\mathbb{Z}}}(g_2,n_2)=\frac{1}{2\pi}\arg((x_2+iy_2)\sin\theta_1+\cos\theta_1)+n_1$$

is bounded by $|n_1|+\frac{1}{2}$. (This latter is just a consequence of the boundedness of the cocycle W.) For the open neighbourhood

$$U=\{(g,0)\mid\theta\in(-\pi,\pi)\}$$

of the identity $(e,0)\in\widetilde{SL}(2,\mathbb{R})$,

$$\sup_{(g_1,n_1)\in U,(g_2,n_2)\in\widetilde{SL}(2,\mathbb{R})}\left\|\widetilde{\ell_{\mathbb{Z}}}((g_1,n_1)(g_2,n_2))-\widetilde{\ell_{\mathbb{Z}}}(g_2,n_2)\right\|\le\frac{1}{2}.$$

Hence $\widetilde{\ell_{\mathbb{Z}}}$ is translation bounded. Also, for e the identity in $SL(2,\mathbb{R})$,

$$\sup_{n\in\mathbb{Z}}\|\widetilde{\ell_{\mathbb{Z}}}(e,n)-\ell_{\mathbb{Z}}(n)\|=0,$$

so we may apply Theorems II.4.2 and II.4.6.

Remark II.4.9. In low dimensions, real and complex Lie algebras and groups can be classified up to isomorphism. For 3-dimensional real Lie groups, this is the Bianchi classification. Unless otherwise specified, we use the notation of [OV94, §§7.1.1–2]. In [TB16], the sectional curvatures of Riemannian metrics of a Lie group for each Lie algebra in the Bianchi classification are computed. The following table summarises the details relevant to us.

Lie group G	Lie algebra $\mathfrak{g}$	G/K	$\mathrm{CAT}(\cdot)$	$\dim G/K$	Type
$\mathbb{R}^3$	$\mathbb{R}^3$	$\mathbb{R}^3$	0	3	abelian
$\mathbb{R}^2 \times \mathbb{T}$		$\mathbb{R}^2$	0	2	
$\mathbb{R} \times \mathbb{T}^2$		$\mathbb{R}$	0	1	
$\mathbb{T}^3$		$\{\mathrm{pt}\}$	$-$	0	
H_3	$\mathfrak{h}_3(\mathbb{R})$	H_3	$+$	3	nilpotent
$\mathsf{H}_3/Z(\mathsf{H}_3(\mathbb{Z}))$		$\mathbb{R}^2$	0	2	
$\widetilde{SL}(2,\mathbb{R})$	$\mathfrak{sl}(2,\mathbb{R})$	$\widetilde{SL}(2,\mathbb{R})$	$+$	3	semisimple
$SL_{k/2}(2,\mathbb{R})$ $(k\in\mathbb{N})$		$\mathbb{R}\mathbf{H}^2$	$-$	2	
$SU(2)$	$\mathfrak{su}(2)$	$\{\mathrm{pt}\}$	$-$	0	semisimple
$SO(3)$					
R_3	$\mathfrak{r}_3(\mathbb{R})$	R_3	$-$	3	solvable
$R_{3,1}$	$\mathfrak{r}_{3,1}(\mathbb{R})$	$\mathbb{R}\mathbf{H}^3$	$-$	3	solvable
$R_{3,\lambda}$ $(\lambda\in(0,+1))$	$\mathfrak{r}_{3,\lambda}(\mathbb{R})$	$R_{3,\lambda}$			
$R_{3,\lambda}$ $(\lambda\in[-1,0))$			$+$		
$R_2 \times \mathbb{R}$	$\mathfrak{r}_2(\mathbb{R}) \oplus \mathbb{R}$	$\mathbb{R}\mathbf{H}^2 \times \mathbb{R}$	0		
$R_2 \times \mathbb{T}$		$\mathbb{R}\mathbf{H}^2$	$-$	2	
$R'_{3,\lambda}$ $(\lambda\in\mathbb{R}\setminus\{0\})$	$\mathfrak{r}'_{3,\lambda}$	$\mathbb{R}\mathbf{H}^3$	$-$	3	solvable
$\widetilde{E}(2)$	$\mathfrak{e}(2)$	$\mathbb{R}^3$	0		
$E_k(2)$ $(k\in\mathbb{N})$		$\mathbb{R}^2$	0	2	

All the CAT(0) (and *a fortoiri* CAT(-1)) entries fall under the aegis of Proposition II.3.4. Only three entries in the table fail to be CAT(0): the Heisenberg group, the universal cover of $SL(2,\mathbb{R})$, and the 1-parameter family $R_{3,\lambda}$ for $\lambda \in [-1,0)$. We have just seen that $\widetilde{SL}(2,\mathbb{R})$ can be readily dealt with. In the next section, we will build spectral triples for semidirect products $\mathbb{R}^n \rtimes \mathbb{R}$, of which $R_{3,\lambda}$, and indeed several other entries in the table, are a special case. In §II.4.2, we will build a spectral triple for the Heisenberg group.

II.4.1 A family of semidirect products

Fix $X \in \mathfrak{gl}(n,\mathbb{R})$ and define a homomorphism $\varphi : \mathbb{R} \to GL(n,\mathbb{R})$ by $\varphi(t) = \exp(tX)$. The semidirect product $\mathbb{R}^n \rtimes_\varphi \mathbb{R}$ consists of elements of the Cartesian product $\mathbb{R}^n \times \mathbb{R}$ with the product law

$$(x,s)(y,t) = (x + \varphi(s)y, s+t).$$

There is an exact sequence

$$0 \longrightarrow \mathbb{R}^n \xrightarrow{\iota} \mathbb{R}^n \rtimes_\varphi \mathbb{R} \xrightarrow{\pi} \mathbb{R} \longrightarrow 0\ .$$

Let $(v_i)_{i=1}^n$ be a basis of $\mathbb{R}^n$. To simplify notation, we will also write $(v_i)_{i=1}^n$ for their images in $\mathscr{C}\ell_n$. Let $V_{\mathbb{R}^n}$ be a Clifford module for $\mathscr{C}\ell_n$ and define a weight $\ell_{\mathbb{R}^n} : \mathbb{R}^n \to \operatorname{End} V_{\mathbb{R}^n}$ by

$$\ell_{\mathbb{R}^n}(x) = \sum_{j=1}^n \langle x \mid e_j \rangle v_j$$

for $x \in \mathbb{R}^n$, where $(e_j)_{j=1}^n$ is the standard basis of $\mathbb{R}^n$. A possible candidate for $\widetilde{\ell_{\mathbb{R}^n}} : \mathbb{R}^n \rtimes_\varphi \mathbb{R} \to V_{\mathbb{R}^n}$ would be given by

$$\widetilde{\ell_{\mathbb{R}^n}}(x, s) = \ell_{\mathbb{R}^n}(x).$$

Indeed,

$$\sup_{x \in \mathbb{R}^n} \|\widetilde{\ell_{\mathbb{R}^n}}(\iota(x)) - \ell_{\mathbb{R}^n}(x)\| = 0. \tag{II.4.10}$$

However, when we come to check the translation-boundedness of $\widetilde{\ell_{\mathbb{R}^n}}$, we find that

$$\widetilde{\ell_{\mathbb{R}^n}}((x,s)(y,t)) - \widetilde{\ell_{\mathbb{R}^n}}(y,t) = \sum_{j=1}^n \langle x + (\varphi(s) - 1)y \mid e_j \rangle v_j$$

which will not be bounded in $(y, t) \in \mathbb{R}^n \rtimes_\varphi \mathbb{R}$ unless $X = 0$. Thinking more carefully (or taking the crossed product on the left), we could define

$$\widetilde{\ell_{\mathbb{R}^n}}(x, s) = \ell_{\mathbb{R}^n}(\varphi(-s)x).$$

As in (II.4.10), we still have

$$\sup_{x \in \mathbb{R}^n} \|\widetilde{\ell_{\mathbb{R}^n}}(\iota(x)) - \ell_{\mathbb{R}^n}(x)\| = 0,$$

but now

$$\widetilde{\ell_{\mathbb{R}^n}}((x,s)(y,t)) - \widetilde{\ell_{\mathbb{R}^n}}(y,t) = \sum_{j=1}^n \langle \varphi(-s-t)x \mid e_j \rangle v_j.$$

Hence

$$\begin{aligned}
\left\|\widetilde{\ell_{\mathbb{R}^n}}((x,s)(y,t)) - \widetilde{\ell_{\mathbb{R}^n}}(y,t)\right\|^2 &= \sum_{i,j=1}^n \langle \varphi(-s-t)x \mid e_i \rangle \langle \varphi(-s-t)x \mid e_j \rangle \langle v_i \mid v_j \rangle \\
&= \sum_{i,j=1}^n \langle \varphi(-s-t)x \mid e_i \rangle \langle v_i \mid v_j \rangle \langle e_j \mid \varphi(-s-t)x \rangle \\
&= \|T\varphi(-s-t)x\|^2 \\
&\leq \|T\|^2 \|\varphi(-t)\|^2 \|\varphi(-s)x\|^2
\end{aligned}$$

where $T : \mathbb{R}^n \to \mathbb{R}^n$ is the linear map taking $e_i \mapsto v_i$. Now $\|\varphi(-t)\| = \|\exp(-tX)\|$. If $X \in \mathfrak{o}(n, \mathbb{R})$, $\sup_{t \in \mathbb{R}} \|\varphi(-t)\| = 1$ and $\widetilde{\ell_{\mathbb{R}^n}}$ is translation bounded; otherwise $\sup_{t \in \mathbb{R}} \|\varphi(-t)\| = \infty$ and $\widetilde{\ell_{\mathbb{R}^n}}$ is not translation bounded. Indeed, since the C*-algebra of a semidirect product can be written as a crossed product C*-algebra, this is just Definition IV.3.12 applied to this example.

To deal with this latter case, we will 'logarithmically dampen' the weight $\ell_{\mathbb{R}^n}$, for which we need the following Lemma. This is inspired by [GMR19]; we shall give a fuller account in §III.1.4.

Lemma II.4.11. *Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be given by*

$$L(x_1, \dots, x_n) = \frac{\log(1 + \|x\|)}{1 + \|x\|}(x_1, \dots, x_n).$$

For $x, y \in \mathbb{R}^n$ and $\alpha > 0$,

$$\|L(x + \alpha y) - L(y)\| \leq 2(\alpha + 1 + \alpha^{-1})(1 + \|x\|)^2.$$

The estimate is by no means optimal; the important thing is that it is independent of y and continuous in x and α.

Proof. First,

$$\begin{aligned}
\left\| \frac{x+\alpha y}{1+\|x+\alpha y\|} - \frac{y}{1+\|y\|} \right\| &= \left\| \frac{\alpha^{-1}x+y}{\alpha^{-1}+\|\alpha^{-1}x+y\|} - \frac{y}{1+\|y\|} \right\| \\
&= \left\| \frac{\alpha^{-1}x}{\alpha^{-1}+\|\alpha^{-1}x+y\|} + y\frac{1+\|y\|-\alpha^{-1}-\|\alpha^{-1}x+y\|)}{(\alpha^{-1}+\|\alpha^{-1}x+y\|)(1+\|y\|)} \right\| \\
&\leq \left\| \frac{\alpha^{-1}x}{\alpha^{-1}+\|\alpha^{-1}x+y\|} \right\| + \left| \frac{1+\|y\|-\alpha^{-1}-\|\alpha^{-1}x+y\|}{\alpha^{-1}+\|\alpha^{-1}x+y\|} \right| \\
&= \frac{\alpha^{-1}\|x\|}{\alpha^{-1}+\|\alpha^{-1}x+y\|} + \left| \frac{1+\|y\|}{\alpha^{-1}+\|\alpha^{-1}x+y\|} - 1 \right| \\
&= \frac{\alpha^{-1}\|x\| + \left|1+\|y\|-\alpha^{-1}-\|\alpha^{-1}x+y\|\right|}{\alpha^{-1}+\|\alpha^{-1}x+y\|} \\
&\leq \frac{2\alpha^{-1}\|x\|+|1-\alpha^{-1}|}{\alpha^{-1}+\|\alpha^{-1}x+y\|}.
\end{aligned}$$

Using this, we have

$$\begin{aligned}
\|L(x+\alpha y)-L(y)\| &= \left\| \frac{\log(1+\|x+\alpha y\|)}{1+\|x+\alpha y\|}(x+\alpha y) - \frac{\log(1+\|y\|)}{1+\|y\|}y \right\| \\
&\leq \left| \log(1+\|x+\alpha y\|) - \log(1+\|y\|) \right| \frac{\|x+\alpha y\|}{1+\|x+\alpha y\|} \\
&\quad + \log(1+\|y\|) \left\| \frac{x+\alpha y}{1+\|x+\alpha y\|} - \frac{y}{1+\|y\|} \right\| \\
&\leq \left| \log(\alpha) + \log\frac{\alpha^{-1}+\|\alpha^{-1}x+y\|}{1+\|y\|} \right| \\
&\quad + \|y\| \frac{2\alpha^{-1}\|x\|+|1-\alpha^{-1}|}{\alpha^{-1}+\|\alpha^{-1}x+y\|} \\
&\leq |\log\alpha| + \max\left\{ \log\frac{\alpha^{-1}+\alpha^{-1}\|x\|+\|y\|}{1+\|y\|}, \log\frac{1+\alpha^{-1}\|x\|+\|\alpha^{-1}x+y\|}{\alpha^{-1}+\|\alpha^{-1}x+y\|} \right\} \\
&\quad + (\|\alpha^{-1}x+y\|+\alpha^{-1}\|x\|)\frac{2\alpha^{-1}\|x\|+|1-\alpha^{-1}|}{\alpha^{-1}+\|\alpha^{-1}x+y\|} \\
&\leq |\log\alpha| + \max\left\{ 0, \log(\alpha^{-1}+\alpha^{-1}\|x\|), \log(\alpha+\|x\|) \right\} \\
&\quad + \max\{1, 2\|x\|+|\alpha-1|\} + \alpha^{-1}\|x\|(2\|x\|+|\alpha-1|) \\
&\leq 2|\log\alpha| + \log(1+\|x\|) + 2\|x\| + \alpha + 1 + 2\alpha^{-1}\|x\|^2 + \|x\| + \alpha^{-1}\|x\| \\
&\leq 2\alpha + 2 + \alpha^{-1} + (4+\alpha^{-1})\|x\| + 2\alpha^{-1}\|x\|^2 \\
&\leq 2(\alpha+1+\alpha^{-1})(1+\|x\|)^2
\end{aligned}$$

as required. □

We now define a new weight $\ell'_{\mathbb{R}^n} : \mathbb{R}^n \to \operatorname{End} V_{\mathbb{R}^n}$ by

$$\ell'_{\mathbb{R}^n}(x) = L(\ell_{\mathbb{R}^n}(x)) = L\left(\sum_{j=1}^{n} \langle x \mid e_j \rangle v_j \right)$$

where by a slight abuse of notation we consider each v_j as an element of $\mathbb{R}^n$ and $\ell'_{\mathbb{R}^n}(x)$ as an element of $\mathscr{C}\ell_n$. The self-adjointness, properness, and translation-boundedness of $\ell'_{\mathbb{R}^n}$ follow from the corresponding properties of $\ell_{\mathbb{R}^n}$, Lemma II.4.11, and the fact that $\|L(x)\| \to \infty$ as $\|x\| \to \infty$. Now let $\widetilde{\ell'_{\mathbb{R}^n}} : \mathbb{R}^n \rtimes_\varphi \mathbb{R} \to V_{\mathbb{R}^n}$ be given by

$$\widetilde{\ell'_{\mathbb{R}^n}}(x,s) = \ell'_{\mathbb{R}^n}(\|\varphi(-s)\|^{-1}\varphi(-s)x) = L\left(\|\varphi(-s)\|^{-1}\sum_{j=1}^{n}\langle\varphi(-s)x \mid e_j\rangle v_j\right). \tag{II.4.12}$$

Our dividing by the norm of $\varphi(-s)$ can be compared to [KK20, Definition 4.6]. Again, as in (II.4.10),

$$\sup_{x\in\mathbb{R}^n} \|\widetilde{\ell'_{\mathbb{R}^n}}(\iota(x)) - \ell'_{\mathbb{R}^n}(x)\| = 0.$$

Applying Lemma II.4.11 and recalling that $\varphi(t) = \exp(tX)$,

$$\begin{aligned}
&\left\|\widetilde{\ell'_{\mathbb{R}^n}}((x,s)(y,t)) - \widetilde{\ell'_{\mathbb{R}^n}}(y,t)\right\| \\
&= \left\|L\left(\|\varphi(-s-t)\|^{-1}\sum_{j=1}^{n}\langle\varphi(-s-t)x + \varphi(-t)y \mid e_j\rangle v_j\right) - L\left(\|\varphi(-t)\|^{-1}\sum_{j=1}^{n}\langle\varphi(-t)y \mid e_j\rangle v_j\right)\right\| \\
&\le 2\left(\frac{\|\varphi(-t)\|}{\|\varphi(-s-t)\|} + 1 + \frac{\|\varphi(-s-t)\|}{\|\varphi(-t)\|}\right)(1 + \|\varphi(-s-t)\|^{-1}\|T\varphi(-s-t)x\|)^2 \\
&\le 2\big(\|\varphi(s)\| + 1 + \|\varphi(-s)\|\big)(1 + \|T\|\|x\|)^2 \\
&\le 2(1 + 2e^{|s|\|X\|})(1 + \|T\|\|x\|)^2,
\end{aligned}$$

where $T : \mathbb{R}^n \to \mathbb{R}^n$ is again the linear map taking $e_i \mapsto v_i$. Hence

$$\sup_{(y,t)\in\mathbb{R}^n\rtimes_\varphi\mathbb{R}} \left\|\widetilde{\ell'_{\mathbb{R}^n}}((x,s)(y,t)) - \widetilde{\ell'_{\mathbb{R}^n}}(y,t)\right\| < \infty$$

for all $(x,s) \in \mathbb{R}^n \rtimes_\varphi \mathbb{R}$. For the neighbourhood $U = \{(x,s) \in \mathbb{R}^n \rtimes_\varphi \mathbb{R} \mid \|x\| < 1, |s| < 1\}$ of the identity $(0,0)$,

$$\sup_{(x,s)\in U,(y,t)\in\mathbb{R}^n\rtimes_\varphi\mathbb{R}} \left\|\widetilde{\ell'_{\mathbb{R}^n}}((x,s)(y,t)) - \widetilde{\ell'_{\mathbb{R}^n}}(y,t)\right\| \le 2(1 + 2e^{\|X\|})(1 + \|T\|)^2 < \infty.$$

Hence $\widetilde{\ell'_{\mathbb{R}^n}}$ is translation-bounded and we can apply Theorems II.4.2 and II.4.6.

This construction is, of course, quite ugly. It may well be the case, for a particular φ, that $\mathbb{R}^n \rtimes_\varphi \mathbb{R}$ admits a left invariant Riemannian metric of nonpositive sectional curvature, in which case we can build a directed length function as in §II.3.

Example II.4.13. Let us consider the special case $n = 1$, making the arbitrary choice $X = 1$ so that $\varphi(t) = e^t$. Define the weight $\ell_{\mathbb{R}} : \mathbb{R} \to \mathbb{C}$ by $x \mapsto x$. Combining the weight (II.4.12) with $\pi * (\ell_{\mathbb{R}})$ produces the self-adjoint, proper, translation-bounded weight $\ell' : \mathbb{R} \rtimes_\varphi \mathbb{R} \to \mathscr{C}\ell_2 \cong \operatorname{End}\mathbb{C}_2$ given by

$$\ell'(x,s) = s\gamma_2 + \frac{x}{1+|x|}\log(1+|x|)\gamma_1.$$

However, already in Example II.3.11, we constructed a weight $\ell : \mathbb{R} \rtimes_\varphi \mathbb{R} \to \mathscr{C}\ell_2 \cong \operatorname{End}\mathbb{C}_2$, given by

$$\ell(x,s) = \frac{2xe^s\gamma_1 - (x^2+1-e^{2s})\gamma_2}{\sqrt{4x^2e^{2s} + (x^2+1-e^{2s})^2}} 2\operatorname{arctanh}\sqrt{\frac{x^2+(1-e^s)^2}{x^2+(1+e^s)^2}},$$

from the action of $\mathbb{R} \rtimes_\varphi \mathbb{R}$ on $\mathbb{R}\mathbf{H}^2$. One can check that $\ell(0,s) = s\gamma_2$ and

$$\ell(x,0) = \frac{2\,\mathrm{sgn}(x)\gamma_1 - |x|\gamma_2}{\sqrt{4+x^2}} \log(1 + \tfrac{1}{2}x^2).$$

In particular, ℓ and ℓ' agree for $x = 0$ and both have a 'logarithmic' behaviour in the x direction. However, the difference $\ell - \ell'$ is not bounded and, furthermore, the difference $\mathrm{sgn}(\ell) - \mathrm{sgn}(\ell')$ does not vanish at infinity, so it is not clear that they will give the same class in $KK_0^{\mathbb{R}\rtimes_\varphi\mathbb{R}}(\mathbb{C}, C_0(\mathbb{R} \rtimes_\varphi \mathbb{R}))$. However, one can also check that, for any $t \in [0,1]$, $t\ell + (1-t)\ell'$ is a self-adjoint, proper, translation-bounded weight. Hence, $(C_0(\mathbb{R} \rtimes_\varphi \mathbb{R}), C_0(\mathbb{R} \rtimes_\varphi \mathbb{R}, \mathbb{C}^2), \ell)$ and $(C_0(\mathbb{R} \rtimes_\varphi \mathbb{R}), C_0(\mathbb{R} \rtimes_\varphi \mathbb{R}, \mathbb{C}^2), \ell')$ are homotopic and indeed do have the same class.

Even though, by the homotopy, we know $(C_0(\mathbb{R} \rtimes_\varphi \mathbb{R}), C_0(\mathbb{R} \rtimes_\varphi \mathbb{R}, \mathbb{C}^2), \ell)$ represents the Kasparov product

$$(\mathbb{C}, C_0(\mathbb{R})_{C_0(\mathbb{R})}, \ell_{\mathbb{R}}) \otimes_{C_0(\mathbb{R})} (C_0(\mathbb{R}), C_0(\mathbb{R} \rtimes_\varphi \mathbb{R})), \widehat{\ell_{\mathbb{R}}})), \tag{II.4.14}$$

it does not satisfy the Kucerovsky conditions. The positivity condition of Theorem I.4.3 fails, in particular, because

$$s\frac{-(x^2+1-e^{2s})}{\sqrt{4x^2e^{2s} + (x^2+1-e^{2s})^2}}$$

is not bounded below. However, we can factorise $(C_0(\mathbb{R} \rtimes_\varphi \mathbb{R}), C_0(\mathbb{R} \rtimes_\varphi \mathbb{R}, \mathbb{C}^2), \ell)$ as

$$(\mathbb{C}, C_0(\mathbb{R})_{C_0(\mathbb{R})}, L(\ell_{\mathbb{R}})) \otimes_{C_0(\mathbb{R})} (C_0(\mathbb{R}), C_0(\mathbb{R} \rtimes_\varphi \mathbb{R})), \pi^*(\ell_{\mathbb{R}})).$$

Here, the left-hand Kasparov module is for the normal subgroup $\mathbb{R}$ of $\mathbb{R} \rtimes_\varphi \mathbb{R}$, rather than the quotient $\mathbb{R} = (\mathbb{R} \rtimes_\varphi \mathbb{R})/\mathbb{R}$ as in II.4.14. We remark that one cannot write the left hand module as

$$(\mathbb{C}, C_0(\mathbb{R})_{C_0(\mathbb{R})}, \ell_{\mathbb{R}}),$$

as one would expect, because it is not $\mathbb{R} \rtimes_\varphi \mathbb{R}$-equivariant. The explanation of this requires the idea of conformal equivariance; see §III.2 and, in particular, Example III.2.1. Now the conditions of Theorem I.4.3 admit almost of a visual demonstration. For the positivity condition,

$$\ell(x,s)L(\ell_{\mathbb{R}}(x))\gamma_1 + L(\ell_{\mathbb{R}}(x))\gamma_1\ell(x,s) = 2L(x)\frac{2xe^s}{\sqrt{4x^2e^{2s} + (x^2+1-e^{2s})^2}}2\,\mathrm{arctanh}\sqrt{\tfrac{x^2+(1-e^s)^2}{x^2+(1+e^s)^2}} \geq 0.$$

This is just the statement that geodesics in the Poincaré half-plane model from $e^{-s}(-x+i)$ to i always travel from left to right. On the other hand, the connection condition follows from the fact that

$$\|\ell(x,s) - \ell_{\mathbb{R}}(s)\gamma_2\| = \|\ell((x,0)(0,s)) - \ell(0,s)\| \leq \|\ell(x,0)\|$$

is uniformly bounded in s for x in any compact subset of $\mathbb{R}$.

II.4.2 The Heisenberg group

Let H_3 be the 3-dimensional Heisenberg group. In the 3×3-matrix presentation, we can write

$$\mathsf{H}_3 = \left\{ g \in M_3(\mathbb{R}) : g = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

The group H_3 is a central extension of $\mathbb{R}^2$ by $\mathbb{R}$, fitting into the exact sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{\iota} \mathsf{H}_3 \xrightarrow{\pi} \mathbb{R}^2 \longrightarrow 0.$$

Define the Euclidean weights

$$\begin{aligned} \ell_{\mathbb{R}} &: \mathbb{R} \to \mathbb{C} & \qquad \ell_{\mathbb{R}^2} &: \mathbb{R}^2 \to \mathscr{C}\ell_2 = \operatorname{End}\mathbb{C}^2 \\ c &\mapsto c & (a,b) &\mapsto a\gamma_1 + b\gamma_2 \end{aligned}$$

Let us naïvely define a weight $\widetilde{\ell_{\mathbb{R}}} : \mathsf{H}_3 \to \mathbb{C}$ by $\widetilde{\ell_{\mathbb{R}}}(g) = c$. First,

$$\sup_{c \in \mathbb{R}} \|\widetilde{\ell_{\mathbb{R}}}(\iota(c)) - \ell_{\mathbb{R}}(c)\| = 0.$$

Alas, with

$$g = \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & a' & c' \\ & 1 & b' \\ & & 1 \end{pmatrix} \qquad gh = \begin{pmatrix} 1 & a+a' & c+c'+ab' \\ & 1 & b+b' \\ & & 1 \end{pmatrix}$$

one can see that

$$\widetilde{\ell_{\mathbb{R}}}(gh) - \widetilde{\ell_{\mathbb{R}}}(h) = c + ab'$$

is not bounded in h. This cannot be remedied by a procedure similar to the one in §II.4.1. Indeed, there we had a semidirect product and here we have a central extension, presenting two very different behaviours.

However, consider the weight $\ell : \mathsf{H}_3 \to \operatorname{End}\mathbb{C}^2$ given by

$$\ell_{\mathsf{H}_3} : \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \mapsto (a\gamma_1 + b\gamma_2)(a^2+b^2)^{1/2} + c\gamma_3.$$

When $c = 0$,

$$\ell_{\mathsf{H}_3} \begin{pmatrix} 1 & a & 0 \\ & 1 & b \\ & & 1 \end{pmatrix} = (a\gamma_1 + b\gamma_2)(a^2+b^2)^{1/2} = \ell_{\mathbb{R}^2}|\ell_{\mathbb{R}^2}|.$$

For this reason, we may consider ℓ_{H_3} to be a '2nd order' weight. The development of a framework to handle such higher order weights will have to wait until §IV.3.1. Although ℓ_{H_3} is self-adjoint and proper, indeed

$$(1+\ell(h)^2)^{1/2} = \left(1 + ({a'}^2+{b'}^2)^2 + {c'}^2\right)^{1/2},$$

it is not translation-bounded. We can, however, compute that

$$\begin{aligned} \ell(gh) - \ell(h) &= \Big((a+a')\gamma_1 + (b+b')\gamma_2\Big)\Big((a+a')^2 + (b+b')^2\Big)^{1/2} + (c+c'+ab')\gamma_3 \\ &\qquad - (a'\gamma_1 + b'\gamma_2)({a'}^2+{b'}^2)^{1/2} + c'\gamma_3 \\ &= (a'\gamma_1 + b'\gamma_2)\Big(((a+a')^2 + (b+b')^2)^{1/2} - ({a'}^2+{b'}^2)^{1/2}\Big) \\ &\qquad + (a\gamma_1 + b\gamma_2)((a+a')^2 + (b+b')^2)^{1/2} + (c+ab')\gamma_3. \end{aligned}$$

Hence $(\ell(gh) - \ell(h))\,(1+\ell(h)^2)^{-1/4}$ is uniformly bounded in $h \in G$. A slight generalisation of Theorem II.2.24 then shows that, for $f \in C_c(\mathsf{H}_3)$, the operator $[M_\ell, f](1+M_\ell^2)^{-1/4} = [M_\ell, f]\langle M_\ell\rangle^{-1/2}$ is bounded where M_ℓ is multiplication by ℓ. We arrive at the order-2 spectral triple

$$(C^*(\mathsf{H}_3), L^2(\mathsf{H}_3, \mathbb{C}^2), M_\ell),$$

which one can check has nontrivial class in $KK_1(C^*(\mathsf{H}_3), \mathbb{C})$.

In Example III.2.10 we shall examine the conformal geometry of this order-2 spectral triple. In Example IV.1.13 we shall place it in context and in §IV.3.1 make a similar construction, with nontrivial class in KK-theory, for all simply connected nilpotent Lie groups and their cocompact closed subgroups which, in particular, include all finitely generated, torsion-free, nilpotent groups.

Chapter III

Conformal noncommutative geometry

In this Chapter, we extend unbounded KK-theory to encompass conformal group and quantum group equivariance. This new framework allows us to treat conformal actions on both manifolds and noncommutative spaces. As examples, we present unbounded representatives of Kasparov's γ-element for the real and complex Lorentz groups and display the conformal $SL_q(2)$-equivariance of the standard spectral triple of the Podleś sphere. In pursuing descent for conformally equivariant cycles, we are led to a new framework for representing Kasparov classes. Our new representatives are unbounded, possess a dynamical quality, and also include known twisted spectral triples. We define an equivalence relation on these new representatives whose classes form an abelian group surjecting onto KK-theory. The technical innovation which underpins these results is a novel multiplicative perturbation theory.

III.1 Conformal transformations from a multiplicative perturbation theory

We begin by recalling a few facts about ternary rings of operators. Ring- or algebra-like objects with ternary product operations are known also as triple systems, and come in Lie, Jordan, and associative varieties, the latter in two kinds. In the context of abstract operator algebras there are C*- and W*-ternary rings, due to [Zet83].

Definition III.1.1. A *ternary ring of operators* on a Hilbert B-module E is a collection $\mathscr{X} \subseteq \mathrm{End}^*(E)$ which is closed under the operation

$$(x, y, z) \mapsto xy^*z.$$

We will not by default assume that a ternary ring of operators is norm-closed.

In the sense of [RW98, Lemma 2.16], a ternary ring of operators $\mathscr{X}$ is a right pre-Hilbert $\mathrm{span}(\mathscr{X}^*\mathscr{X})$-module. Its completion $\overline{\mathscr{X}}$ is then a right Hilbert $\overline{\mathrm{span}}(\mathscr{X}^*\mathscr{X})$-module. By similar considerations on the left, $\overline{\mathscr{X}}$ is a Morita equivalence $\overline{\mathrm{span}}(\mathscr{X}\mathscr{X}^*)$-$\overline{\mathrm{span}}(\mathscr{X}^*\mathscr{X})$-bimodule. We remark that, for instance, $\overline{\mathrm{span}}(\mathscr{X}\mathscr{X}^*\mathscr{X}) = \overline{\mathscr{X}}$.

In particular, every norm-closed ternary ring of operators is a Morita equivalence bimodule in a natural way. By [Zet83, Theorem 2.6], any Hilbert C*-module can be represented as a norm-closed ternary ring of operators on some Hilbert space H.

The implicit presence of ternary rings of operators will be a feature of many of our definitions. This occurs because, just as the Leibniz rule makes the domain of a commutator with a self-adjoint operator D a $*$-algebra, the domain of a mixed commutator $a \mapsto D_1 a - aD_2$ is naturally closed under the ternary product. Indeed, if, for $a, b, c \in \mathrm{End}^*(E)$,

$$D_1 a - aD_2 \qquad D_1 b - bD_2 \qquad D_1 c - cD_2$$

are bounded, then $[D_1, ab^*]$, $[D_2, a^*b]$, and $D_1 ab^*c - ab^*cD_2$ (and all other like permutations) are bounded. This can also be seen by writing D_1 and D_2 as diagonal entries of a two-by-two matrix and placing a, b, c in the upper-right corner.

We will formulate our definition of conformal transformation for higher order cycles.

Definition III.1.2. A *conformal transformation* (U, μ) from one order-$\frac{1}{1-\alpha}$ cycle, (A, E_B, D_1), to another, (A, E'_B, D_2), is a unitary map $U : E \to E'$, intertwining the representations of A, and an (even) invertible operator $\mu \in \mathrm{End}^*(E)$ which is even if the module is graded, satisfying the following. We require that $A \subseteq \overline{\mathrm{span}}(A\mathscr{M}) \cap \overline{\mathrm{span}}(\mathscr{M}A)$, where $\mathscr{M}$ is the set of $a \in \mathrm{End}^*(E)$ such that the operators

$$(U^*D_2Ua - a\mu D_1\mu^*)\mu^{-1*}\langle D_1\rangle^{-\alpha} \qquad \langle D_2\rangle^{-\alpha}U(U^*D_2Ua - a\mu D_1\mu^*)$$

are bounded and $a, a\mu, a\mu^{-1*} \in \mathrm{Lip}^*_\alpha(D_1)$.

Remarks III.1.3.

1. The easiest way for the closure condition to be satisfied is if $1 \in \mathscr{M}$; for nonunital A an approximate unit might be found to lie in $\mathscr{M}$.
2. We have $\mathscr{M}\mathscr{M}^*\mathscr{M} \subseteq \mathscr{M}$ and so $\mathscr{M}$ is a ternary ring of operators, in general not norm-closed.
3. Conformal transformations are generally neither reversible nor composable. This latter occurs very easily for two noncommuting conformal factors μ and ν. We ultimately address this issue with the *conformisms* of §III.4.2.

In §III.1.3, on p. 106, we will prove the following Theorem.

Theorem III.1.4. *Let (U, μ) be a conformal transformation from the order-$\frac{1}{1-\alpha}$ cycle (A, E_B, D_1) to the order-$\frac{1}{1-\alpha}$ cycle (A, E'_B, D_2). Then the bounded transforms (A, E_B, F_{D_1}) and (A, E'_B, F_{D_2}) are unitarily equivalent up to locally compact perturbation via the unitary U. That is*

$$(U^*F_{D_2}U - F_{D_1})a \in \mathrm{End}^0(E)$$

for all $a \in A$. Hence $[(A, E_B, F_{D_1})] = [(A, E'_B, F_{D_2})] \in KK(A, B)$ (and also $[(A, E_B, D_1)] = [(A, E'_B, D_2)]$).

III.1.1 Motivating examples

Example III.1.5. cf. [Dun20, Lemma 2.8] The simplest nontrivial example of a conformal transformation between unbounded cycles can be contructed from an unbounded cycle (A, E_B, D) and a positive number κ. The pair $(\mathrm{id}, \kappa^{1/2})$ is a conformal transformation from (A, E_B, D) to $(A, E_B, \kappa D)$.

On a geodesically complete Riemannian manifold X, there are two standard spectral triples. One relies on a spin^c structure and takes the form $(C_0(X), L^2(X, \not{S}), \not{D})$, where $\not{S}$ is a spinor bundle and $\not{D}$ is the Atiyah–Singer Dirac operator. The other depends on only the orientation and Riemannian metric, taking the form $(C_0(X), L^2(\Omega^* X), d + \delta)$ where d is the exterior derivative on differential forms $\Omega^* X$ and δ is its adjoint, the codifferential, their sum being the Hodge–de Rham Dirac operator. We consider the effect of a conformal change of metric on both these spectral triples.

Example III.1.6. The behaviour of the Atiyah–Singer Dirac operator under conformal transformations was first recorded in [Hit74, Proof of Proposition 1.3]. In the context of noncommutative geometry, see also [Bär07, Proof of Theorem 3.1]. Let $(X, \mathbf{g})$ and $(X, \mathbf{h})$ be Riemannian spin^c manifolds such that $\mathbf{h} = k^2 \mathbf{g}$. Let $\not{S}_{\mathbf{g}}$ and $\not{S}_{\mathbf{h}}$ be their associated spinor bundles. There is a canonical fibrewise isometry

$$\psi : \not{S}_{\mathbf{g}} \to \not{S}_{\mathbf{h}}.$$

Let $\not{D}_{\mathbf{g}} : \Gamma^\infty(\not{S}_{\mathbf{g}}) \to \Gamma^\infty(\not{S}_{\mathbf{g}})$ and $\not{D}_{\mathbf{h}} : \Gamma^\infty(\not{S}_{\mathbf{h}}) \to \Gamma^\infty(\not{S}_{\mathbf{h}})$ be the corresponding Dirac operators. Then, by e.g. [Hij86, Proposition 4.3.1],

$$\not{D}_{\mathbf{h}} = k^{(-n-1)/2} \circ \psi \circ \not{D}_{\mathbf{g}} \circ \psi^{-1} \circ k^{(n-1)/2}.$$

Although ψ is a fibrewise isometry, the induced map $V : L^2(X, \not{S}_{\mathbf{g}}) \to L^2(X, \not{S}_{\mathbf{h}})$ is not unitary, as the volume form changes. With the relation $\mathrm{vol}_{\mathbf{h}} = k^n \mathrm{vol}_{\mathbf{g}}$, we find that $V^* = k^n V^{-1}$. The polar decomposition is

$$U = V(V^*V)^{-1/2} = k^{-n/2} V$$

and we find that

$$\not{D}_{\mathbf{h}} = k^{-1/2} U \not{D}_{\mathbf{g}} U^* k^{-1/2}$$

or, in other words,

$$U^* \not{D}_{\mathbf{h}} U = k^{-1/2} \not{D}_{\mathbf{g}} k^{-1/2}.$$

In terms of Definition III.1.2, if $(X, \mathbf{g})$ is complete and the conformal factor k and its inverse are bounded (which is automatic if X is compact), then $(U, k^{-1/2})$ is a conformal transformation from $(C_0(X), L^2(X, \not{S}_{\mathbf{g}}), \not{D}_{\mathbf{g}})$ to $(C_0(X), L^2(X, \not{S}_{\mathbf{h}}), \not{D}_{\mathbf{h}})$. (Note that there is need for the derivative of the conformal factor to be globally bounded.)

Example III.1.7. Next, we consider the Hodge–de Rham Dirac operator. As before, let $(X, \mathbf{g})$ and $(X, \mathbf{h})$ be Riemannian manifolds such that $\mathbf{h} = k^2 \mathbf{g}$. Consider the two inner products on $\Omega^* X$ given by $\mathbf{g}$ and $\mathbf{h}$, which we will label $\langle \cdot, \cdot \rangle_{\mathbf{g}}$ and $\langle \cdot, \cdot \rangle_{\mathbf{h}}$. We will call the resulting Hilbert spaces $L^2(\Omega^* X, \mathbf{g})$ and $L^2(\Omega^* X, \mathbf{h})$. There is an obvious map

$$V : L^2(\Omega^* X, \mathbf{g}) \to L^2(\Omega^* X, \mathbf{h})$$

given by the identity on $\Omega^* X$, in other words, for $\omega \in \Omega^* X \subseteq L^2(\Omega^* X, \mathbf{g})$, $V : \omega \mapsto \omega$. Its adjoint is given on homogenous forms ω by $V^* : \omega \mapsto k^{n-2|\omega|} \omega$. Observe that if n is even the restriction of V to the middle degree forms is unitary. We make the (rather trivial) observation that

$$VV^* : \omega \mapsto k^{n-2|\omega|} \omega \qquad V^*V : \omega \mapsto k^{n-2|\omega|} \omega. \tag{III.1.8}$$

The unitary in the polar decomposition $U = V(V^*V)^{-1/2} = (VV^*)^{-1/2}V$ is given by

$$U : \omega \mapsto k^{(-n+2|\omega|)/2}\omega \qquad U^* : \omega \mapsto k^{(n-2|\omega|)/2}\omega.$$

The exterior derivative d does not depend on the metric, but its adjoint the codifferential does, so we use the notation $\delta_{\mathbf{g}}$ and $\delta_{\mathbf{h}}$ to distinguish the two codifferentials acting on Ω^*X. The invariance of the exterior derivative means that $dV = Vd$. With care over which inner product is being used, $(Vd)^* = \delta_{\mathbf{g}}V^*$ and $(dV)^* = V^*\delta_{\mathbf{h}}$. So, $\delta_{\mathbf{g}}V^* = V^*\delta_{\mathbf{h}}$ and we obtain the relations

$$V(d+\delta_{\mathbf{g}})V^* = d(VV^*) + (VV^*)\delta_{\mathbf{h}}$$

and

$$U(d+\delta_{\mathbf{g}})U^* = (VV^*)^{-1/2}V(d+\delta_{\mathbf{g}})V^*(VV^*)^{-1/2} = (VV^*)^{-1/2}d(VV^*)^{1/2} + (VV^*)^{1/2}\delta_{\mathbf{h}}(VV^*)^{-1/2}.$$

On a differential form ω of degree $|\omega|$,

$$\begin{aligned} U(d+\delta_{\mathbf{g}})U^*\omega &= k^{-(n-2(|\omega|+1))/2}d(k^{(n-2|\omega|)/2}\omega) + k^{(n-2(|\omega|-1))/2}\delta_{\mathbf{h}}(k^{-(n-2|\omega|)/2}\omega) \\ &= k\left(k^{-(n-2|\omega|)/2}d(k^{(n-2|\omega|)/2}\omega) + k^{(n-2|\omega|)/2}\delta_{\mathbf{h}}(k^{-(n-2|\omega|)/2}\omega)\right). \end{aligned}$$

For any function $f \in C^\infty(X)$,

$$\begin{aligned} f^{-1}df\omega + f\delta_{\mathbf{h}}f^{-1}\omega &= (d+\delta_{\mathbf{h}})\omega + f^{-1}[d,f]\omega + [f,\delta_{\mathbf{h}}]f^{-1}\omega \\ &= (d+\delta_{\mathbf{h}})\omega + f^{-1}[d,f]\omega - [\delta_{\mathbf{h}},f]f^{-1}\omega \\ &= (d+\delta_{\mathbf{h}})\omega + f^{-1}[d-\delta_{\mathbf{h}},f]\omega. \end{aligned}$$

Hence

$$\begin{aligned} &\left(U(d+\delta_{\mathbf{g}})U^* - k^{1/2}(d+\delta_{\mathbf{h}})k^{1/2}\right)\omega \\ &\quad= \left(k(d+\delta_{\mathbf{h}}) + k^{-(n-2|\omega|-2)/2}[d-\delta_{\mathbf{h}}, k^{(n-2|\omega|)/2}] - k^{1/2}(d+\delta_{\mathbf{h}})k^{1/2}\right)\omega \\ &\quad= \left(-k^{1/2}[d+\delta_{\mathbf{h}}, k^{1/2}] + k^{-(n-2|\omega|-2)/2}[d-\delta_{\mathbf{h}}, k^{(n-2|\omega|)/2}]\right)\omega. \end{aligned}$$

In terms of Definition III.1.2, if $(X,\mathbf{g})$ is complete and the conformal factor k and its inverse are bounded (which is automatic if X is compact), the data $(U, k^{-1/2})$ define a conformal transformation from $(C_0(X), L^2(\Omega^*X,\mathbf{g}), d+\delta_{\mathbf{g}})$ to $(C_0(X), L^2(\Omega^*X,\mathbf{h}), d+\delta_{\mathbf{h}})$.

Remark III.1.9. The extension of the Hodge-de Rham spectral triple to a spectral triple for the $\mathbb{Z}_2$-graded Clifford algebra bundle is important for Poincaré duality [Kas88, §4]. In the case of a manifold, where the functions and conformal factors are in the centre of the Clifford algebra, it is not difficult to show that our definition of conformal transformation can be modified to handle the graded commutators. We leave a discussion of the general $\mathbb{Z}_2$-graded case to another place.

In the framework of the spectral action principle, Chamseddine and Connes [CC06] calculate the effect of rescaling the Spectral Standard Model Dirac operator $D \rightsquigarrow e^{-\phi/2}De^{-\phi/2}$, where the *dilaton* ϕ is interpreted as a scalar field. Apart from the Higgs mass term, the entire Lagrangian of the Standard Model of particle physics is conformally invariant, which was a motivation for the work in Chapter III.

Example III.1.10. Suppose that we have the data of a continuous family of compact Riemannian spin^c manifolds $(M_x, \mathbf{g}_x)_{x\in X}$ parameterised by a locally compact Hausdorff space X, as in the families index theorem [LM89, §III.15]. Integration over the fibres of the total space $\mathcal{M} \to X$ along with the Dirac operators D_x on the fibre spinor bundles $\$_x$ yields an unbounded Kasparov module

$$\left(C_0(\mathcal{M}), L^2(\mathcal{M}, \$_\bullet, \mathbf{g}_\bullet)_{C_0(X)}, D_\bullet\right). \qquad \text{(III.1.11)}$$

Let $k : \mathcal{M} \to [0, \infty)$ be a family of conformal factors parameterised by X. The commutation of the conformal factors with the algebra means we obtain a new unbounded Kasparov module

$$\left(C_0(\mathcal{M}), L^2(\mathcal{M}, \$_\bullet, k^2\mathbf{g}_\bullet)_{C_0(X)}, k^{-1/2}D_\bullet k^{-1/2}\right).$$

We observe that the integration over the fibres changes, but the compactness of the fibres means we get equivalent measures. That we obtain a new unbounded Kasparov module is straightforward but of more consequence is that the classes defined by F_D and $F_{k^{-1/2}Dk^{-1/2}}$ in $KK(C_0(\mathcal{M}), C_0(X))$ coincide.

Suppose that we have another family of metrics $\mathbf{h}_\bullet$, for the same family of manifolds, giving an unbounded Kasparov module

$$\left(C_0(\mathcal{M}), L^2(\mathcal{M}, S_\bullet, \mathbf{h}_\bullet)_{C_0(X)}, D_\bullet\right). \tag{III.1.12}$$

Suppose that $\mathbf{h}_x = k_x^2\mathbf{g}$ for a (pointwise) continuous family $k_\bullet \in C^\infty(M_\bullet)$ of smooth functions and that $\sup_{x\in X}\{\|k_x\|_\infty, \|k_x^{-1}\|_\infty\} < \infty$. Then $(\mathrm{id}, k_\bullet^{-1/2})$ is a conformal transformation from (III.1.11) to (III.1.12).

The first appearance of conformal transformations in noncommutative geometry was with the preprint [CC92] on the noncommutative torus, followed up by the same authors in [CT11]; see also [CM14]. The next example is not, however, to be confused with the twisted spectral triples of [CM08], which will be examined in §III.4.

Example III.1.13. Fix a real number α. Let $C(\mathbb{T}^2_\alpha)$ be the universal C*-algebra generated by unitaries U and V subject to the relation

$$VU = e^{2\pi i\alpha}UV.$$

There are two self-adjoint (unbounded) derivations δ_1 and δ_2 on $C(\mathbb{T}^2_\alpha)$, given on generators by

$$\delta_1(U) = U \qquad \delta_1(V) = 0 \qquad \delta_2(U) = 0 \qquad \delta_2(V) = V.$$

When $\alpha = 0$, these are the derivatives $-i\partial_{\theta_1}$ and $-i\partial_{\theta_2}$ on the classical torus. There is a trace on $C(\mathbb{T}^2_\alpha)$ given by

$$\varphi(U^mV^n) = \delta_{m,0}\delta_{n,0}.$$

The completion of $C(\mathbb{T}^2_\alpha)$ in the inner product given by φ is $L^2(\mathbb{T}^2_\alpha)$. Fix a complex number τ with $\Im(\tau) > 0$. Then

$$\left(C(\mathbb{T}^2_\alpha), L^2(\mathbb{T}^2_\alpha)\otimes\mathbb{C}^2, D := \begin{pmatrix} & \delta_1 + \tau\delta_2 \\ \delta_1 + \bar{\tau}\delta_2 & \end{pmatrix}\right)$$

is a spectral triple. Now choose a positive invertible element $k \in C(\mathbb{T}^2_\alpha)$ in the domains of δ_1 and δ_2. Let $k^\circ \in B(L^2(\mathbb{T}^2_\alpha))$ be the operator of right multiplication. Then

$$\left(C(\mathbb{T}^2_\alpha), L^2(\mathbb{T}^2_\alpha)\otimes\mathbb{C}^2, D_{k^2} := \begin{pmatrix} & (k^\circ)^2(\delta_1 + \tau\delta_2) \\ (\delta_1 + \bar{\tau}\delta_2)(k^\circ)^2 & \end{pmatrix}\right)$$

is still a spectral triple. We have that

$$D_{k^2} - k^\circ Dk^\circ = \begin{pmatrix} & -k^\circ\left[\delta_1 + \tau\delta_2, k^\circ\right] \\ \left[\delta_1 + \bar{\tau}\delta_2, k^\circ\right]k^\circ & \end{pmatrix}$$

is bounded. Hence $1 \in \mathscr{M}$ and (id, k°) is a conformal transformation from the spectral triple $(C(\mathbb{T}^2_\alpha), L^2(\mathbb{T}^2_\alpha)\otimes\mathbb{C}^2, D)$ to $(C(\mathbb{T}^2_\alpha), L^2(\mathbb{T}^2_\alpha)\otimes\mathbb{C}^2, D_{k^2})$.

Let $\Phi : C(\mathbb{T}^2_\alpha) \to C(\mathbb{T})$ be the expectation coming from averaging over the circle action $U \mapsto zU$, $z \in \mathbb{T}$. Then $(C(\mathbb{T}^2_\alpha), L^2(C(\mathbb{T}^2_\alpha), \Phi)_{C(\mathbb{T})}, \delta_2)$ is an unbounded Kasparov module by [BCR15, Proposition

2.9]. Now choose a positive invertible element $k \in C(\mathbb{T}^2_\alpha)$ in the domain of δ_2. Then (id, k°) is a conformal transformation from $(C(\mathbb{T}^2_\alpha), L^2(C(\mathbb{T}^2_\alpha), \Phi)_{C(\mathbb{T})}, \delta_2)$ to the spectral triple

$$\left(C(\mathbb{T}^2_\alpha), L^2(C(\mathbb{T}^2_\alpha), \Phi)_{C(\mathbb{T})}, k^\circ \delta_2 k^\circ\right).$$

Example III.1.13 can be generalised along the lines of [Sit15], using a real spectral triple satisfying the order zero condition. Theorem III.1.4 gives a refinement of [Sit15, Lemma 14] which shows that the class in KK-theory of the conformally perturbed spectral triple is unchanged.

III.1.2 Technical preliminaries

Throughout this section we fix a countably generated right Hilbert B-module E for some C*-algebra B. The chief subtlety in using the integral formula (I.0.5) to study the bounded transform for an unbounded Kasparov module (A, E_B, D) is the commutator $(\lambda + 1 + D^2)^{-1}a - a(\lambda + 1 + D^2)^{-1}$ for $a \in A$, [CP98, Lemma 2.3]. For us, the analogous computation is still the heart of the matter, see Lemma III.1.22, but our techniques are different and described next.

Lemma III.1.14. *Let A and B be regular operators on E. If B is a symmetric operator, then so is ABA^*, provided that the domain*

$$\mathrm{dom}(ABA^*) = \{x \in \mathrm{dom}\, A^* \mid A^*x \in \mathrm{dom}\, B, BA^*x \in \mathrm{dom}\, A\}$$

is dense. If A is bounded and invertible then ABA^* is regular. If moreover B is self-adjoint then ABA^* is self-adjoint.

Proof. Given $x, y \in \mathrm{dom}(ABA^*)$, $x, y \in \mathrm{dom}(A^*)$ and $A^*y \in \mathrm{dom}(B)$, the symmetry of B gives

$$\langle ABA^*x \mid y\rangle = \langle BA^*x \mid A^*y\rangle = \langle A^*x \mid BA^*y\rangle = \langle x \mid ABA^*y\rangle$$

so ABA^* is symmetric. If A is bounded and invertible, [Wor91, §2, Example 2] shows that AB is regular and, by [Wor91, §2, Example 3], ABA^* is regular. Applying the definition of the domain of the adjoint, one readily sees that $\mathrm{dom}((ABA^*)^*) = \mathrm{dom}(ABA^*) = A^{-1*}\,\mathrm{dom}(B)$. □

In the second statement of Lemma III.1.14, the invertibility of A can be relaxed given additional assumptions [Kaa17, §6]. We will consider perturbations of the form $D \rightsquigarrow \mu D \mu^*$ for a self-adjoint regular operator D and an invertible, adjointable operator μ. The following bound is the result of a relation between the domains of fractional powers of $\langle D\rangle$ and $\langle \mu D \mu^*\rangle$, using Theorem A.3.4 of §A.3.

Lemma III.1.15. *Let D be a self-adjoint regular operator and μ an invertible adjointable operator. For all $0 < \alpha \leq 1$ we have*

$$\mathrm{dom}(\mu\langle D\rangle^\alpha\mu^*) = \mathrm{dom}((\mu\langle D\rangle\mu^*)^\alpha) = \mathrm{dom}\langle \mu D\mu^*\rangle^\alpha$$

and the inequalities

$$\left\|\langle D\rangle^\alpha\mu^*(\mu\langle D\rangle\mu^*)^{-\alpha}\right\| \leq \|\mu^{-1}\|^\alpha\|\mu\|^{1-\alpha} \qquad \left\|(\mu\langle D\rangle\mu^*)^\alpha\mu^{-1*}\langle D\rangle^{-\alpha}\right\| \leq \|\mu\|^\alpha\|\mu^{-1}\|^{1-\alpha}.$$

Proof. The domain statement follows from Theorem A.3.4. For the first inequality, in the context of Theorem A.3.4, let $A = \langle D\rangle$ and $B = \mu\langle D\rangle\mu^*$ so that $\mu^*\,\mathrm{dom}\, B = \mathrm{dom}\, A$. We have

$$\begin{aligned}\left\|\langle D\rangle^\alpha\mu^*(\mu\langle D\rangle\mu^*)^{-\alpha}\right\| &= \|A^\alpha\mu^*B^{-\alpha}\| \leq \|A\mu^*B^{-1}\|^\alpha\|\mu^*\|^{1-\alpha}\\ &= \left\|\langle D\rangle\mu^*(\mu\langle D\rangle\mu^*)^{-1}\right\|^\alpha\|\mu^*\|^{1-\alpha}\\ &= \|\mu^{-1}\|^\alpha\|\mu\|^{1-\alpha}.\end{aligned}$$

For the second, in the context of Theorem A.3.4, let $A = \mu\langle D\rangle\mu^*$ and $B = \langle D\rangle$, so that $\mu^{-1*}\operatorname{dom} B = \operatorname{dom} A$. We obtain that

$$\begin{aligned}\left\|(\mu\langle D\rangle\mu^*)^\alpha\mu^{-1*}\langle D\rangle^{-\alpha}\right\| &= \|A^\alpha\mu^{-1*}B^{-\alpha}\| \leq \|A\mu^{-1*}B^{-1}\|^\alpha\|\mu^{-1*}\|^{1-\alpha}\\ &= \left\|(\mu\langle D\rangle\mu^*)\mu^{-1*}\langle D\rangle^{-1}\right\|^\alpha \|\mu^{-1}\|^{1-\alpha}\\ &= \|\mu\|^\alpha\|\mu^{-1}\|^{1-\alpha}\end{aligned}$$

as required. □

We recall tools ensuring convergence of regular self-adjoint operators on a Hilbert module E_B.

Theorem III.1.16. *[WN92, §1] Let T be a normal regular operator on E and $f \in C_b(\sigma(T))$. Let $(f_n)_{n\in\mathbb{N}} \subseteq C_b(\sigma(T))$ be a sequence of functions with common bound which converge to f uniformly on compact subsets. Then $f_n(T)$ converges to $f(T)$ as $n \to \infty$ in the strict topology on $M(\operatorname{End}^0(E))$, and hence in the $*$-strong topology on $\operatorname{End}^*(E)$.*

For the final statement, recall that the strict topology on $M(\operatorname{End}^0(E)) = \operatorname{End}^*(E)$ agrees with the $*$-strong topology on norm-bounded subsets [RW98, Proposition C.7].

The proofs of the following two Theorems are essentially unchanged from the Hilbert space case.

Theorem III.1.17. *cf. [RS80, Theorem VIII.25(a)], [Oli09, Proposition 10.1.18] Let $\mathscr{C} \subseteq E$ be a core for a self-adjoint regular operator T on E. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of self-adjoint regular operators such that, for all $n \in \mathbb{N}$, $\mathscr{C} \subseteq \operatorname{dom} T_n$ and, for all $\xi \in \mathscr{C}$, $T_n\xi$ converges to $T\xi$ as $n \to \infty$. Then T_n converges to T in the strong resolvent sense as $n \to \infty$.*

Theorem III.1.18. *cf. [RS80, Theorem VIII.20(b)], [Oli09, Proposition 10.1.9] A sequence $(T_n)_{n\in\mathbb{N}}$ of self-adjoint regular operators on E converges to a self-adjoint regular operator T in the strong resolvent sense if and only if, for all $f \in C_b(\mathbb{R})$, $f(T_n)$ converges strongly to $f(T)$ as $n \to \infty$.*

Let $(\varphi_n)_{n\in\mathbb{N}} \subset C_c(\mathbb{R})$ be a sequence of positive functions, bounded by 1 and converging uniformly on compact subsets to the constant function 1. Let D be a self-adjoint regular operator. By Theorem III.1.16, the bounded operators $(\varphi_n(D))_{n\in\mathbb{N}}$ converge $*$-strongly to 1. We will consider the bounded operators $d_n = D\varphi_n(D)$. On an element $\xi \in \operatorname{dom} D$,

$$d_n\xi = D\varphi_n(D)\xi = \varphi_n(D)(D\xi) \to D\xi.$$

In particular, by Theorem III.1.17, $d_n \to D$ in the strong resolvent sense. By Theorem III.1.18, F_{d_n} converges strongly to F_D as $n \to \infty$.

Proposition III.1.19. *Let D be a self-adjoint regular operator and μ an invertible adjointable operator. Then $\mu d_n\mu^*$ converges to $\mu D\mu^*$ in the strong resolvent sense as $n \to \infty$. Furthermore, $\mu\langle d_n\rangle\mu^*$ converges to $\mu\langle D\rangle\mu^*$ in the strong resolvent sense.*

Let a be a bounded operator such that $a\operatorname{dom} D \subseteq \operatorname{dom} D$. With $a_n = \varphi_n(D)a\varphi_n(D)$, we find that $d_na_n\langle d_n\rangle^{-1}$ converges strongly to $Da\langle D\rangle^{-1}$ as $n \to \infty$. In consequence, $[d_n, a_n]\langle d_n\rangle^{-1}$ converges strongly to $[D,a]\langle D\rangle^{-1}$.

Proof. First, apply Theorem III.1.17 to the self-adjoint regular operator $\mu D\mu^*$ and the sequence $(\mu d_n\mu^*)_{n\in\mathbb{N}}$ of bounded operators. Noting that $\operatorname{dom}(\mu D\mu^*) = \mu^{-1*}\operatorname{dom} D$, on an element $\mu^{-1*}\xi \in \operatorname{dom}(\mu D\mu^*)$,

$$(\mu d_n\mu^*)\mu^{-1*}\xi = \mu d_n\xi \to \mu D\xi$$

as $n \to \infty$. Hence, $\mu d_n\mu^*$ converges to $\mu D\mu^*$ in the strong resolvent sense. On an element $\xi \in \operatorname{dom} D$,

$$\langle d_n\rangle\xi = (1 + (D\varphi_n(D))^2)^{1/2}\xi = (1 + (D\varphi_n(D))^2)^{1/2}\langle D\rangle^{-1}(\langle D\rangle\xi).$$

The function

$$x \mapsto \frac{(1+(x\varphi_n(x))^2)^{1/2}}{(1+x^2)^{1/2}} = \left(1 - \frac{1-\varphi_n(x)^2}{1+x^{-2}}\right)^{1/2}$$

is bounded above by 1 and below by φ_n and so converges to 1 on compact subsets. Applying Theorem III.1.16,

$$\langle d_n\rangle\xi = (1+(D\varphi_n(D))^2)^{1/2}\langle D\rangle^{-1}(\langle D\rangle\xi) \to \langle D\rangle\xi$$

and we proceed as before. For the second statement we have

$$d_n a_n \langle d_n\rangle^{-1} = D\varphi_n(D)^2 a\varphi_n(D)\langle D\varphi_n(D)\rangle^{-1} = \varphi_n(D)^2\left(Da\langle D\rangle^{-1}\right)\langle D\rangle\varphi_n(D)\langle D\varphi_n(D)\rangle^{-1}.$$

The function

$$x \mapsto \frac{(1+x^2)^{1/2}\varphi_n(x)}{(1+(x\varphi_n(x))^2)^{1/2}} = \left(1 - \frac{1-\varphi_n(x)^2}{1+x^2\varphi_n(x)^2}\right)^{1/2}$$

is bounded above by 1 and below by φ_n and so converges to 1 on compact subsets. Applying Theorem III.1.16,

$$d_n a_n\langle d_n\rangle^{-1} = \varphi_n(D)^2\left(Da\langle D\rangle^{-1}\right)\langle D\rangle\varphi_n(D)\langle D\varphi_n(D)\rangle^{-1} \to Da\langle D\rangle^{-1}$$

strongly, as $n \to \infty$. For the second part,

$$[d_n, a_n]\langle d_n\rangle^{-1} = d_n a_n\langle d_n\rangle^{-1} - a_n F_{d_n} \to Da\langle D\rangle^{-1} - aF_D$$

strongly, as required. □

As an application, we prove an operator inequality needed for applications involving summability.

Proposition III.1.20. *Let D be a self-adjoint regular operator on a Hilbert B-module E and μ an invertible adjointable operator on E. Then*

$$C^{-1}\mu^{-1*}(1+D^2)^{-1}\mu^{-1} \le (1+(\mu D\mu^*)^2)^{-1} \le C\mu^{-1*}(1+D^2)^{-1}\mu^{-1}$$

where $C = \max\{\|\mu\|^2, \|\mu^{-1}\|^2\}$.

Hence if J is a hereditary ideal of $\mathrm{End}^*(B)$, not necessarily closed, then $(1+(\mu D\mu^*)^2)^{-1} \in J$ if and only if $(1+D^2)^{-1} \in J$. In particular, this applies if $B = \mathbb{C}$, so that E is a Hilbert space and J is any two-sided ideal of $B(E)$, not necessarily closed [Bla06, §II.5.2], such as Schatten ideals.

Proof. If $\mu^*\mu\,\mathrm{dom}\,D \subseteq \mathrm{dom}\,D$, we could proceed more straightforwardly. As we do not assume this, we will use the (bounded) operators $d_n = D\varphi_n(D)$ and Proposition III.1.19 to write

$$\begin{aligned}1+(\mu d_n\mu^*)^2 &= 1+\mu d_n\mu^*\mu d_n\mu^* \le 1+\|\mu\|^2\mu d_n^2\mu^* = \mu(\mu^{-1}\mu^{-1*}+\|\mu\|^2 d_n^2)\mu^* \\ &\le \mu(\|\mu^{-1}\|^2+\|\mu\|^2 d_n^2)\mu^* = \|\mu\|^2\mu(\|\mu\|^{-2}\|\mu^{-1}\|^2+d_n^2)\mu^* \\ &\le \|\mu\|^2\max\{1, \|\mu\|^{-2}\|\mu^{-1}\|^2\}\mu(1+d_n^2)\mu^* = \max\{\|\mu\|^2, \|\mu^{-1}\|^2\}\mu(1+d_n^2)\mu^*.\end{aligned}$$

Hence, $(1+(\mu d_n\mu^*))^{-1} \ge C^{-1}\mu^{-1*}(1+d_n^2)^{-1}\mu^{-1}$, and by Theorem III.1.18 and Proposition III.1.19, $(1+(\mu d_n\mu^*)^2)^{-1}$ converges strongly to $(1+(\mu D\mu^*)^2)^{-1}$ and $(1+d_n^2)^{-1}$ converges strongly to $(1+D^2)^{-1}$ as $n \to \infty$. Thus,

$$C^{-1}\mu^{-1*}(1+D^2)^{-1}\mu^{-1} \le (1+(\mu D\mu^*)^2)^{-1},$$

and similarly,

$$1+(\mu d_n\mu^*)^2 \ge \min\{\|\mu\|^{-2}, \|\mu^{-1}\|^{-2}\}\mu(1+d_n^2)\mu^*$$

and

$$(1+(\mu D\mu^*)^2)^{-1} \le C\mu^{-1*}(1+D^2)^{-1}\mu^{-1}$$

as required. □

We use the notation $\mathfrak{T}_{a,b}(x) = ax - xb$ for $a, b, x \in \operatorname{End}^*(E)$. The following inequality controlling $\mathfrak{T}_{a,b}(x)$ is based on Stampfli [Sta70, Theorem 8]; see also Archbold [Arc78].

Lemma III.1.21. *Let a and b be elements of a C^*-algebra A. Define the bounded linear operator*

$$\mathfrak{T}_{a,b} : A \to A \qquad x \mapsto ax - xb.$$

If a and b are positive, then

$$\|\mathfrak{T}_{a,b}\| \leq \max\{\|a\| - \|b^{-1}\|^{-1}, \|b\| - \|a^{-1}\|^{-1}\}$$

where $\|a^{-1}\|^{-1}$ is considered to be zero if a is not invertible, and likewise for b.

Proof. Firstly, $\|\mathfrak{T}_{a,b}\| \leq \|a\| + \|b\|$. For any $\lambda \in \mathbb{C}$, $\mathfrak{T}_{a-\lambda,b-\lambda} = \mathfrak{T}_{a,b}$, so $\|\mathfrak{T}_{a,b}\| \leq \|a - \lambda\| + \|b - \lambda\|$. For any $\lambda_1, \lambda_2 \in \mathbb{C}$,

$$\|\mathfrak{T}_{a,b}\| \leq \|a - \lambda_1\| + \|b - \lambda_2\| + |\lambda_1 - \lambda_2|.$$

To obtain the required bound, let

$$\lambda_1 = \frac{1}{2}(\|a\| + \|a^{-1}\|^{-1}) \qquad \lambda_2 = \frac{1}{2}(\|b\| + \|b^{-1}\|^{-1})$$

so that, because a and b are positive,

$$\|a - \lambda_1\| = \frac{1}{2}(\|a\| - \|a^{-1}\|^{-1}) \qquad \|b - \lambda_2\| = \frac{1}{2}(\|b\| - \|b^{-1}\|^{-1}).$$

Then

$$\begin{aligned}
\|\mathfrak{T}_{a,b}\| &\leq \frac{1}{2}(\|a\| - \|a^{-1}\|^{-1}) + \frac{1}{2}(\|b\| - \|b^{-1}\|^{-1}) + \left|\frac{1}{2}(\|a\| + \|a^{-1}\|^{-1}) - \frac{1}{2}(\|b\| + \|b^{-1}\|^{-1})\right| \\
&= \frac{1}{2}\left((\|a\| - \|b^{-1}\|^{-1}) + (\|b\| - \|a^{-1}\|^{-1}) + \left|(\|a\| - \|b^{-1}\|^{-1}) - (\|b\| - \|a^{-1}\|^{-1})\right|\right) \\
&= \max\{\|a\| - \|b^{-1}\|^{-1}, \|b\| - \|a^{-1}\|^{-1}\}
\end{aligned}$$

as required. □

It is proved in [Sta70, Theorem 8, Corollary 2] that, if A has a faithful irreducible representation, then there is an equality

$$\|\mathfrak{T}_{a,b}\| = \inf_{\lambda \in \mathbb{C}} (\|a - \lambda\| + \|b - \lambda\|)$$

for any $a, b \in A$.

III.1.3 A multiplicative perturbation theory

The technical tool which allows us to extend the definitions of conformality and equivariance to unbounded Kasparov cycles is a multiplicative perturbation theory. This perturbation theory allows us to relate properties of an unbounded self-adjoint regular operator D and its bounded transform $F_D := D(1 + D^2)^{-1/2} = D\langle D\rangle^{-1}$ to conformally rescaled versions $D_1 = \mu D \mu^*$ and F_{D_1}.

Lemma III.1.22. *Let D be a self-adjoint regular operator and μ an invertible adjointable operator on E. Let a be an adjointable operator such that $a\mu^{-1*} \operatorname{dom} D \subseteq \mu^{-1*} \operatorname{dom} D$. Then, with $D_1 = \mu D \mu^*$ and $D_2 = \mu\langle D\rangle\mu^*$, and for all $\lambda \geq 0$*

$$\begin{aligned}
-(\lambda + \langle D_1\rangle^2)^{-1} a + a(\lambda + D_2^2)^{-1} &= (\lambda + \langle D_1\rangle^2)^{-1} a\mu \mathfrak{T}_{(\mu^*\mu)^{-1},\mu^*\mu}(\langle D\rangle^{-1})\mu^{-1} D_2(\lambda + D_2^2)^{-1} \\
&\quad + D_1(\lambda + \langle D_1\rangle^2)^{-1}\left([D_1, a]D_2^{-1} - \mu^{-1*}[F_D, \mu^* a\mu]\mu^{-1}\right) D_2(\lambda + D_2^2)^{-1} \\
&\quad + (\lambda + \langle D_1\rangle^2)^{-1}\left(\mu F_D \mu^{-1}[D_1, a]D_2^{-1} + \mu[F_D, \mu^{-1} a\mu]F_D\mu^{-1}\right) D_2^2(\lambda + D_2^2)^{-1}
\end{aligned}$$

as everywhere-defined operators.

Proof. If $\mu^*\mu \operatorname{dom} D \subseteq \operatorname{dom} D$, we could proceed more straightforwardly. As we do not make this assumption, we use the approximation arguments of §III.1.2. Let $(\varphi_n)_{n\in\mathbb{N}} \subset C_c(\mathbb{R})$ be a sequence of positive functions, bounded by 1 and converging uniformly on compact subsets to the constant function 1. Let $d_n = D\varphi_n(D)$ and set

$$a_n = \mu^{-1*}\varphi_n(D)\mu^* a\mu^{-1*}\varphi_n(D)\mu^*.$$

Note for future reference that we may use the bounded transform $F_{d_n} = d_n\langle d_n\rangle^{-1}$ to write

$$\begin{aligned}
&[\mu d_n\mu^*, a_n](\mu\langle d_n\rangle\mu^*)^{-1} \\
&\qquad = \mu d_n\varphi_n(D)\mu^* a\mu^{-1*}\varphi_n(D)\langle d_n\rangle^{-1}\mu^{-1} - \mu^{-1*}\varphi_n(D)\mu^* a\mu^{-1*}\varphi_n(D)\mu^*\mu F_{d_n}\mu^{-1} \\
&\qquad = \mu[d_n, \varphi_n(D)\mu^* a\mu^{-1*}\varphi_n(D)]\langle d_n\rangle^{-1}\mu^{-1} + \mu\varphi_n(D)\mu^* a\mu^{-1*}\varphi_n(D)F_{d_n}\mu^{-1} \\
&\qquad - \mu^{-1*}\varphi_n(D)\mu^* a\mu^{-1*}\varphi_n(D)\mu^*\mu F_{d_n}\mu^{-1}
\end{aligned}$$

so that we will be in a position to apply Proposition III.1.19 to the first term, while the other two are uniformly bounded in n. Because d_n is bounded, we may write

$$\begin{aligned}
&-(\lambda + \langle\mu d_n\mu^*\rangle^2)^{-1}a_n + a_n(\lambda + (\mu\langle d_n\rangle\mu^*)^2)^{-1} \\
&\quad = (\lambda + \langle\mu d_n\mu^*\rangle^2)^{-1}\left(-a_n(\lambda + (\mu\langle d_n\rangle\mu^*)^2) + (\lambda + \langle\mu d_n\mu^*\rangle^2)a_n\right)(\lambda + (\mu\langle d_n\rangle\mu^*)^2)^{-1} \\
&\quad = (\lambda + \langle\mu d_n\mu^*\rangle^2)^{-1}\left(-a_n(\mu\langle d_n\rangle\mu^*)^2 + \langle\mu d_n\mu^*\rangle^2 a_n\right)(\lambda + (\mu\langle d_n\rangle\mu^*)^2)^{-1}.
\end{aligned} \tag{III.1.23}$$

Expanding the middle factor and using the identity $F_{d_n}d_n - \langle d_n\rangle = -\langle d_n\rangle^{-1}$ yields

$$\begin{aligned}
&\langle\mu d_n\mu^*\rangle^2 a_n - a_n(\mu\langle d_n\rangle\mu^*)^2 \\
&\quad = a_n + \mu d_n\mu^*\mu d_n\mu^* a_n - a_n\mu\langle d_n\rangle\mu^*\mu\langle d_n\rangle\mu^* \\
&\quad = a_n + \mu d_n\mu^*[\mu d_n\mu^*, a_n] + \mu d_n\mu^* a_n\mu d_n\mu^* - a_n\mu\langle d_n\rangle\mu^*\mu\langle d_n\rangle\mu^* \\
&\quad = a_n + \mu d_n\mu^*[\mu d_n\mu^*, a_n] - \mu d_n[F_{d_n}, \mu^* a_n\mu]\langle d_n\rangle\mu^* + \mu d_n F_{d_n}\mu^* a_n\mu\langle d_n\rangle\mu^* \\
&\qquad - a_n\mu\langle d_n\rangle\mu^*\mu\langle d_n\rangle\mu^* \\
&\quad = a_n + \mu d_n\mu^*[\mu d_n\mu^*, a_n] - \mu d_n[F_{d_n}, \mu^* a_n\mu]\langle d_n\rangle\mu^* + \mu F_{d_n}\mu^{-1}\mu d_n\mu^* a_n\mu\langle d_n\rangle\mu^* \\
&\qquad - a_n\mu\langle d_n\rangle\mu^*\mu\langle d_n\rangle\mu^* \\
&\quad = a_n + \mu d_n\mu^*[\mu d_n\mu^*, a_n] - \mu d_n[F_{d_n}, \mu^* a_n\mu]\langle d_n\rangle\mu^* + \mu F_{d_n}\mu^{-1}[\mu d_n\mu^*, a_n]\mu\langle d_n\rangle\mu^* \\
&\qquad + \mu F_{d_n}\mu^{-1}a_n\mu d_n\mu^*\mu\langle d_n\rangle\mu^* - a_n\mu\langle d_n\rangle\mu^*\mu\langle d_n\rangle\mu^* \\
&\quad = a_n + \mu d_n\mu^*[\mu d_n\mu^*, a_n] - \mu d_n[F_{d_n}, \mu^* a_n\mu]\langle d_n\rangle\mu^* + \mu F_{d_n}\mu^{-1}[\mu d_n\mu^*, a_n]\mu\langle d_n\rangle\mu^* \\
&\qquad + \mu[F_{d_n}, \mu^{-1}a_n\mu]d_n\mu^*\mu\langle d_n\rangle\mu^* + a_n\mu(F_{d_n}d_n - \langle d_n\rangle)\mu^*\mu\langle d_n\rangle\mu^* \\
&\quad = a_n + \mu d_n\mu^*[\mu d_n\mu^*, a_n] - \mu d_n[F_{d_n}, \mu^* a_n\mu]\langle d_n\rangle\mu^* + \mu F_{d_n}\mu^{-1}[\mu d_n\mu^*, a_n]\mu\langle d_n\rangle\mu^* \\
&\qquad + \mu[F_{d_n}, \mu^{-1}a_n\mu]d_n\mu^*\mu\langle d_n\rangle\mu^* - a_n\mu\langle d_n\rangle^{-1}\mu^*\mu\langle d_n\rangle\mu^* \\
&\quad = a_n\mu\mathfrak{T}_{(\mu^*\mu)^{-1},\mu^*\mu}(\langle d_n\rangle^{-1})\mu^{-1}(\mu\langle d_n\rangle\mu^*) \\
&\qquad + (\mu d_n\mu^*)\left([\mu d_n\mu^*, a_n](\mu\langle d_n\rangle\mu^*)^{-1} - \mu^{-1*}[F_{d_n}, \mu^* a_n\mu]\mu^{-1}\right)(\mu\langle d_n\rangle\mu^*) \\
&\qquad + \left(\mu F_{d_n}\mu^{-1}[\mu d_n\mu^*, a_n](\mu\langle d_n\rangle\mu^*)^{-1} + \mu[F_{d_n}, \mu^{-1}a_n\mu]F_{d_n}\mu^{-1}\right)(\mu\langle d_n\rangle\mu^*)^2
\end{aligned}$$

since $\mathfrak{T}_{\mu^{-1}\mu^{-1*},\mu^*\mu}(\langle d_n\rangle^{-1}) = \mu^{-1}\mu^{-1*}\langle d_n\rangle^{-1} - \langle d_n\rangle^{-1}\mu^*\mu$. Substituting into (Ⅲ.1.23) yields

$$\begin{aligned}
&-(\lambda + \langle \mu d_n\mu^*\rangle^2)^{-1}a_n + a_n(\lambda + (\mu\langle d_n\rangle\mu^*)^2)^{-1} \\
&\quad= (\lambda + \langle \mu d_n\mu^*\rangle^2)^{-1}a_n\mu\mathfrak{T}_{(\mu^*\mu)^{-1},\mu^*\mu}(\langle d_n\rangle^{-1})\mu^{-1}(\mu\langle d_n\rangle\mu^*)(\lambda + (\mu\langle d_n\rangle\mu^*)^2)^{-1} \\
&\qquad+ (\mu d_n\mu^*)(\lambda + \langle \mu d_n\mu^*\rangle^2)^{-1}\left([\mu d_n\mu^*, a_n](\mu\langle d_n\rangle\mu^*)^{-1} - \mu^{-1*}[F_{d_n}, \mu^*a_n\mu]\mu^{-1}\right) \\
&\qquad\quad\times (\mu\langle d_n\rangle\mu^*)(\lambda + (\mu\langle d_n\rangle\mu^*)^2)^{-1} \\
&\qquad+ (\lambda + \langle \mu d_n\mu^*\rangle^2)^{-1}\left(\mu F_{d_n}\mu^{-1}[\mu d_n\mu^*, a_n](\mu\langle d_n\rangle\mu^*)^{-1} + \mu[F_{d_n}, \mu^{-1}a_n\mu]F_{d_n}\mu^{-1}\right) \\
&\qquad\quad\times (\mu\langle d_n\rangle\mu^*)^2(\lambda + (\mu\langle d_n\rangle\mu^*)^2)^{-1}.
\end{aligned} \tag{Ⅲ.1.24}$$

By Proposition Ⅲ.1.19, the right-hand side of (Ⅲ.1.24) converges strongly to

$$\begin{aligned}
&(\lambda + \langle \mu D\mu^*\rangle^2)^{-1}a\mu\mathfrak{T}_{(\mu^*\mu)^{-1},\mu^*\mu}(\langle D\rangle^{-1})\mu^{-1}(\mu\langle D\rangle\mu^*)(\lambda + (\mu\langle D\rangle\mu^*)^2)^{-1} \\
&\quad+ (\mu D\mu^*)(\lambda + \langle \mu D\mu^*\rangle^2)^{-1}\left([\mu D\mu^*, a](\mu\langle D\rangle\mu^*)^{-1} - \mu^{-1*}[F_D, \mu^*a\mu]\mu^{-1}\right) \\
&\qquad\times (\mu\langle D\rangle\mu^*)(\lambda + (\mu\langle D\rangle\mu^*)^2)^{-1} \\
&\quad+ (\lambda + \langle \mu D\mu^*\rangle^2)^{-1}\left(\mu F_D\mu^{-1}[\mu D\mu^*, a](\mu\langle D\rangle\mu^*)^{-1} + \mu[F_D, \mu^{-1}a\mu]F_D\mu^{-1}\right) \\
&\qquad\times (\mu\langle D\rangle\mu^*)^2(\lambda + (\mu\langle D\rangle\mu^*)^2)^{-1}
\end{aligned}$$

and we obtain the required equality of everywhere-defined operators. □

Lemma Ⅲ.1.25. *Let D be a self-adjoint regular operator and μ an invertible,adjointable operator on E. Let a be an adjointable operator such that $a\mu^{-1*}\operatorname{dom} D \subseteq \mu^{-1*}\operatorname{dom} D$. Suppose further that, for some $0 \le \alpha < 1$,*

$$[F_D, \mu^*a\mu]\langle D\rangle^{1-\alpha} \qquad [F_D, \mu^{-1}a\mu]\langle D\rangle^{1-\alpha} \qquad [\mu D\mu^*, a]\mu^{-1*}\langle D\rangle^{-\alpha}$$

are bounded. Then, with $D_1 = \mu D\mu^*$ and $D_2 = \mu\langle D\rangle\mu^*$, for $\lambda \ge 0$ and $\beta \le 1-\alpha$

$$\left\| D_1\left((\lambda + \langle D_1\rangle^2)^{-1}a - a(\lambda + D_2^2)^{-1}\right)\mu\langle D\rangle^\beta \right\| \le c_1(\lambda + c_0)^{-1+(\alpha+\beta)/2}$$

where $c_0 = \min\{1, \|\mu^{-1}\|^{-4}\}$ and $c_1 \ge 0$ is independent of λ.

Proof. First, by Lemma Ⅲ.1.15, $\|D_2^{-\beta}\mu\langle D\rangle^\beta\| = \|\langle D\rangle^\beta\mu^*D_2^{-\beta}\| \le \|\mu^{-1}\|^\beta\|\mu\|^{1-\beta}$ so

$$\begin{aligned}
&\left\| D_1\left((\lambda + \langle D_1\rangle^2)^{-1}a - a(\lambda + D_2^2)^{-1}\right)\mu\langle D\rangle^\beta \right\| \\
&\qquad\le \left\| D_1\left((\lambda + \langle D_1\rangle^2)^{-1}a - a(\lambda + D_2^2)^{-1}\right)D_2^\beta \right\| \|\mu^{-1}\|^\beta\|\mu\|^{1-\beta},
\end{aligned}$$

By Lemma Ⅲ.1.21, $\|\mathfrak{T}_{(\mu^*\mu)^{-1},(\mu^*\mu)}\| \le \max\{\|\mu^{-1}\|^2 - \|\mu^{-1}\|^{-2}, \|\mu\|^2 - \|\mu\|^{-2}\}$. We compute that

$$\begin{aligned}
&\left\| D_1\left((\lambda + \langle D_1\rangle^2)^{-1}a - a(\lambda + D_2^2)^{-1}\right)D_2^\beta \right\| \\
&\quad\le \left\| D_1(\lambda + \langle D_1\rangle^2)^{-1}a\mu\mathfrak{T}_{(\mu^*\mu)^{-1},\mu^*\mu}(\langle D\rangle^{-1})\mu^{-1}D_2^{1+\beta}(\lambda + D_2^2)^{-1} \right\| \\
&\qquad+ \left\| D_1^2(\lambda + \langle D_1\rangle^2)^{-1}\left([D_1, a]\mu^{-1*}\langle D\rangle^{-\alpha} - \mu^{-1*}[F_D, \mu^*a\mu]\langle D\rangle^{1-\alpha}\right) \times \right. \\
&\qquad\quad\left. \times \langle D\rangle^\alpha\mu^*D_2^{-\alpha}D_2^{\alpha+\beta}(\lambda + D_2^2)^{-1} \right\| \\
&\qquad+ \left\| D_1(\lambda + \langle D_1\rangle^2)^{-1}\left(\mu F_D\mu^{-1}[D_1, a]\mu^{-1*}\langle D\rangle^{-\alpha} + \mu[F_D, \mu^{-1}a\mu]\langle D\rangle^{1-\alpha}F_D^2\right) \right. \\
&\qquad\quad\left. \times \langle D\rangle^\alpha\mu^*D_2^{-\alpha}D_2^{1+\alpha+\beta}(\lambda + D_2^2)^{-1} \right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\| D_1(\lambda + \langle D_1\rangle^2)^{-1} \right\| \|a\| \|\mu\| \left\| \mathfrak{T}_{(\mu^*\mu)^{-1},\mu^*\mu}(\langle D\rangle^{-1}) \right\| \|\mu^{-1}\| \left\| D_2^{1+\beta}(\lambda + D_2^2)^{-1} \right\| \\
&\quad + \left\| D_1^2(\lambda + \langle D_1\rangle^2)^{-1} \right\| \left(\left\| [D_1, a]\mu^{-1*}\langle D\rangle^{-\alpha} \right\| - \left\| \mu^{-1*}[F_D, \mu^* a\mu]\langle D\rangle^{1-\alpha} \right\| \right) \\
&\qquad \times \left\| \langle D\rangle^{\alpha} \mu^* D_2^{-\alpha} \right\| \left\| D_2^{\alpha+\beta}(\lambda + D_2^2)^{-1} \right\| \\
&\quad + \left\| D_1(\lambda + \langle D_1\rangle^2)^{-1} \right\| \left(\|\mu\| \|\mu^{-1}\| \left\| [D_1, a]\mu^{-1*}\langle D\rangle^{-\alpha} \right\| + \|\mu\| \left\| [F_D, \mu^{-1} a\mu]\langle D\rangle^{1-\alpha} \right\| \right) \\
&\qquad \times \left\| \langle D\rangle^{\alpha} \mu^* D_2^{-\alpha} \right\| \left\| D_2^{1+\alpha+\beta}(\lambda + D_2^2)^{-1} \right\| \\
&\leq (\lambda + 1)^{-1/2} \|a\| \|\mu\| \max\{\|\mu^{-1}\|^2 - \|\mu^{-1}\|^{-2}, \|\mu\|^2 - \|\mu\|^{-2}\} \|\mu^{-1}\| (\lambda + \|\mu^{-1}\|^{-4})^{(-1+\beta)/2} \\
&\quad + \left(\left\| [D_1, a]\mu^{-1*}\langle D\rangle^{-\alpha} \right\| - \|\mu^{-1}\| \left\| [F_D, \mu^* a\mu]\langle D\rangle^{1-\alpha} \right\| \right) \\
&\qquad \times \|\mu^{-1}\|^{\alpha} \|\mu\|^{1-\alpha} (\lambda + \|\mu^{-1}\|^{-4})^{(-2+\alpha+\beta)/2} \\
&\quad + (\lambda + 1)^{-1/2} \left(\|\mu\| \|\mu^{-1}\| \left\| [D_1, a]\mu^{-1*}\langle D\rangle^{-\alpha} \right\| + \|\mu\| \left\| [F_D, \mu^{-1} a\mu]\langle D\rangle^{1-\alpha} \right\| \right) \\
&\qquad \times \|\mu^{-1}\|^{\alpha} \|\mu\|^{1-\alpha} (\lambda + \|\mu^{-1}\|^{-4})^{(-1+\alpha+\beta)/2} \\
&\leq c_1'(\lambda + c_0)^{-1+(\alpha+\beta)/2}
\end{aligned}
$$

where $c_0 = \min\{1, \|\mu^{-1}\|^{-4}\}$ and $c_1' \geq 0$ is a constant independent of λ. Hence,

$$
\left\| D_1 \left((\lambda + \langle D_1\rangle^2)^{-1} a - a(\lambda + D_2^2)^{-1} \right) \mu\langle D\rangle^{\beta} \right\| \leq c_1(\lambda + c_0)^{-1+(\alpha+\beta)/2}
$$

for $c_1 = c_1' \|\mu^{-1}\|^{\beta} \|\mu\|^{1-\beta}$. □

Lemma III.1.26. *Let D be a self-adjoint regular operator and μ an invertible adjointable operator on E. Let a be an adjointable operator such that $a\mu^{-1*}\operatorname{dom} D \subseteq \mu^{-1*}\operatorname{dom} D$. Suppose further that, for some $0 \leq \alpha < 1$,*

$$
[F_D, \mu^* a\mu]\langle D\rangle^{1-\alpha} \qquad [F_D, \mu^{-1} a\mu]\langle D\rangle^{1-\alpha} \qquad [\mu D\mu^*, a]\mu^{-1*}\langle D\rangle^{-\alpha}
$$

are bounded. Then, with $D_1 = \mu D\mu^*$ and $D_2 = \mu\langle D\rangle\mu^*$,

$$
D_1 \left(\langle D_1\rangle^{-1} a - a D_2^{-1} \right) \mu\langle D\rangle^{\beta}
$$

is bounded for $\beta < 1 - \alpha$.

Proof. Using the integral formula (I.0.5),

$$
D_1 \left(\langle D_1\rangle^{-1} a - a D_2^{-1} \right) \mu\langle D\rangle^{\beta} = \frac{1}{\pi} \int_0^{\infty} \lambda^{-1/2} D_1 \left((\lambda + \langle D_1\rangle^2)^{-1} a - a(\lambda + D_2^2)^{-1} \right) \mu\langle D\rangle^{\beta} d\lambda.
$$

By Proposition III.1.25, the integrand is bounded and the integral is norm convergent when

$$
\int_0^{\infty} \lambda^{-1/2} (\lambda + c_0)^{-1+(\alpha+\beta)/2} d\lambda
$$

is convergent, that is, when $\beta < 1 - \alpha$. □

Theorem III.1.27. *Let D_0 be a self-adjoint regular operator and μ an invertible adjointable operator on E. Let a be an adjointable operator such that $a\mu^{-1*}\operatorname{dom} D_0 \subseteq \mu^{-1*}\operatorname{dom} D_0$. Suppose further that, for some $0 \leq \alpha < 1$,*

$$
[F_{D_0}, \mu^* a\mu]\langle D_0\rangle^{1-\alpha} \qquad [F_{D_0}, \mu^{-1} a\mu]\langle D_0\rangle^{1-\alpha} \qquad [F_{D_0}, a\mu]\langle D_0\rangle^{1-\alpha} \qquad [\mu D_0\mu^*, a]\mu^{-1*}\langle D_0\rangle^{-\alpha}
$$

are bounded. Then, with $D_1 = \mu D_0\mu^*$, the operator

$$
(F_{D_1} - F_{D_0}) a\mu\langle D_0\rangle^{\beta}
$$

is bounded for $\beta < 1 - \alpha$.

Proof. We have

$$\begin{aligned}(F_{D_1} - F_{D_0})a\mu &= F_{D_1}a\mu - a\mu F_{D_0} - [F_{D_0}, a\mu] \\ &= F_{D_1}a\mu - aD_1\mu^{-1*}\langle D_0\rangle^{-1} - [F_{D_0}, a\mu] \\ &= F_{D_1}a\mu - D_1 a\mu^{-1*}\langle D_0\rangle^{-1} + [D_1, a]\mu^{-1*}\langle D_0\rangle^{-1} - [F_{D_0}, a\mu] \\ &= D_1\left(\langle D_1\rangle^{-1}a - a(\mu\langle D_0\rangle\mu^*)^{-1}\right)\mu + [D_1, a]\mu^{-1*}\langle D_0\rangle^{-1} - [F_{D_0}, a\mu].\end{aligned}$$

Multiplying on the right by $\langle D\rangle^{\beta}$, the first term remains bounded by Lemma III.1.26. The remaining two terms are bounded owing to the last two of our displayed assumptions. □

Theorem III.1.28. *Let D be a self-adjoint regular operator and μ an invertible adjointable operator on E. Let a be an adjointable operator such that $\{\mu^*a\mu, \mu^{-1}a\mu, a\mu, \mu^*a\mu^{-1*}\}\operatorname{dom} D \subseteq \mu^{-1*}\operatorname{dom} D$. Suppose further that, for some $0 \leq \alpha < 1$,*

$$[D, \mu^*a\mu]\langle D\rangle^{-\alpha} \qquad [D, \mu^{-1}a\mu]\langle D\rangle^{-\alpha} \qquad [D, a\mu]\langle D\rangle^{-\alpha} \qquad [\mu D\mu^*, a]\mu^{-1*}\langle D\rangle^{-\alpha}$$

are bounded. Then, with $D_1 = \mu D\mu^*$,

$$(F_{D_1} - F_D)a\mu\langle D\rangle^{\beta}$$

is bounded for $\beta < 1 - \alpha$. If b is an adjointable operator such that $b^*\mu^{-1*}\operatorname{dom} D \subseteq \operatorname{dom} D$, then $(F_{D_1} - F_D)ab\langle D\rangle^{\beta}$ is bounded. If c is a bounded operator such that $(1 + D^2)^{-1}c$ is compact, then $(F_{D_1} - F_D)abc$ is compact.

Proof. Applying Theorem I.0.6, we find that

$$[F_D, \mu^*a\mu]\langle D\rangle^{1-\gamma} \qquad [F_D, \mu^{-1}a\mu]\langle D\rangle^{1-\gamma} \qquad [F_{D_0}, a\mu]\langle D\rangle^{1-\gamma}$$

are bounded for $\gamma > \alpha$. Then, by Theorem III.1.27, $(F_{D_1} - F_D)a\mu\langle D\rangle^{\beta}$ is bounded for all $\beta < 1 - \gamma$, and so for all $\beta < 1 - \alpha$. The remaining statements follow immediately. □

Remark III.1.29. In Theorem III.1.28, that $[\mu D\mu^*, a]\mu^{-1*}\langle D\rangle^{-\alpha}$ is bounded is equivalent to

$$\begin{aligned}D\mu^{-1}[\mu\mu^*, a]\mu^{-1*}\langle D\rangle^{-\alpha} &= D(\mu^*a\mu^{-1*} - \mu^{-1}a\mu)\langle D\rangle^{-\alpha} \\ &= \mu^{-1}[\mu D\mu^*, a]\mu^{-1*}\langle D\rangle^{-\alpha} - [D, \mu^{-1}a\mu]\langle D\rangle^{-\alpha}\end{aligned}$$

being bounded, using the assumption that $[D, \mu^{-1}a\mu]\langle D\rangle^{-\alpha}$ is bounded. In other words, that $\mu\mu^*$ and a almost commute.

Corollary III.1.30. *Let D be a self-adjoint regular operator and μ an invertible adjointable operator on E. Suppose that, for some $0 \leq \alpha < 1$,*

$$[F_D, \mu]\langle D\rangle^{1-\alpha} \qquad [F_D, \mu^*\mu]\langle D\rangle^{1-\alpha}$$

are bounded. Then, with $D_1 = \mu D\mu^*$,

$$(F_{D_1} - F_D)\mu\langle D\rangle^{\beta}$$

is bounded for $\beta < 1 - \alpha$. If $\mu^*\operatorname{dom} D \subseteq \operatorname{dom} D$, then $(F_{D_1} - F_D)\langle D\rangle^{\beta}$ is bounded.

Corollary III.1.31. *Let D be a self-adjoint regular operator and μ an invertible adjointable operator on E. Suppose that $\mu\operatorname{dom} D \subseteq \operatorname{dom} D$ and, for some $0 \leq \alpha < 1$,*

$$[D, \mu]\langle D\rangle^{-\alpha} \qquad \langle D\rangle^{-\alpha}[D, \mu]$$

are bounded. Then, with $D_1 = \mu D\mu^*$, the operator

$$(F_{D_1} - F_D)\langle D\rangle^{\beta}$$

is bounded for $\beta < 1 - \alpha$.

Corollary III.1.32. *Let D_0 and D_1 be self-adjoint regular operators and μ an invertible adjointable operator on E. Suppose that $\mu \operatorname{dom} D_0 \subseteq \operatorname{dom} D_0$ and, for some $0 \leq \alpha < 1$,*

$$(\mu^{-1} D_1 \mu^{-1*} - D_0)\langle D_0 \rangle^{-\alpha} \qquad [D_0, \mu]\langle D_0 \rangle^{-\alpha} \qquad \langle D_0 \rangle^{-\alpha}[D_0, \mu]$$

are bounded. Then the operator

$$(F_{D_1} - F_{D_0})\langle D_0 \rangle^{\beta}$$

is bounded for $\beta < 1 - \alpha$.

Theorem III.1.33. *Let D be a self-adjoint regular operator and μ an invertible adjointable operator on E. Let a and b be adjointable operators such that $\{\mu^* a, \mu^{-1} a, a, b\mu, b\mu^{-1*}\} \operatorname{dom} D \subseteq \operatorname{dom} D$. Suppose further that, for some $0 \leq \alpha < 1$,*

$$\langle D \rangle^{-\alpha}[D, a\mu] \qquad \langle D \rangle^{-\alpha}[D, a\mu^{-1*}] \qquad \langle D \rangle^{-\alpha}[D, a] \qquad [D, b\mu]\langle D \rangle^{-\alpha} \qquad [\mu D \mu^*, a^* b]\mu^{-1*}\langle D \rangle^{-\alpha}$$

are bounded. Then, with $D_1 = \mu D \mu^*$, the operator

$$(F_{D_1} - F_D) a^* b \mu \langle D \rangle^{\beta}$$

is bounded for $\beta < 1 - \alpha$. If c is an adjointable operator such that $c\mu^{-1*} \operatorname{dom} D \subseteq \operatorname{dom} D$, then $(F_{D_1} - F_D) a^* b c^* \langle D \rangle^{\beta}$ is bounded. If d is an adjointable operator such that $(1 + D^2)^{-1} d$ is compact, then $(F_{D_1} - F_D) a^* b c^* d$ is compact.

Proof. This follows from Theorem III.1.28, using [GM15, Proposition A.5] for the appropriate Leibniz rule to relate the differing commutator conditions. □

Now, returning to the concept of conformal transformation, we have:

Proof of Theorem III.1.4. Let (U, μ) be a conformal transformation from (A, E_B, D_1) to (A, E'_B, D_2). By Proposition I.1.1 and Lemma III.1.15,

$$(U^* F_{D_2} U a - a F_{\mu D_1 \mu^*}) \mu \langle D_0 \rangle^{\beta}$$

is bounded for $a \in \mathscr{M}$. Let $b, c \in \mathscr{M}$ and consider the operators

$$D = \begin{pmatrix} U^* D_2 U & \\ & \mu D_1 \mu^* \end{pmatrix} \qquad B = \begin{pmatrix} & b \\ 0 & \end{pmatrix} \qquad C = \begin{pmatrix} & c \\ 0 & \end{pmatrix}$$

on $E \oplus E'$. By assumption and using Lemma III.1.15,

$$[D, B]\langle D \rangle^{-\alpha} = \begin{pmatrix} & (U^* D_2 U b - b \mu D_1 \mu^*)\langle \mu D_1 \mu^* \rangle^{-\alpha} \\ 0 & \end{pmatrix} \quad \text{and} \quad [D, C]\langle D \rangle^{-\alpha}$$

are bounded. By [GM15, Proposition A.5],

$$[D, B^* C]\langle D \rangle^{-\alpha} = \begin{pmatrix} 0 & \\ & [\mu D_1 \mu^*, b^* c]\langle \mu D_1 \mu^* \rangle^{-\alpha} \end{pmatrix}$$

extends to an adjointable operator. Again using Lemma III.1.15, $[\mu D_1 \mu^*, b^* c]\mu^{-1*}\langle D_1 \rangle^{-\alpha}$ is bounded and we may apply Theorem III.1.33 to obtain that

$$(F_{\mu D_1 \mu^*} - F_{D_1}) b^* c \mu \langle D_1 \rangle^{\beta}$$

is bounded for $\beta < 1 - \alpha$. Then

$$(U^* F_{D_2} U - F_{D_1}) a b^* c = (U^* F_{D_2} U a - a F_{\mu D_1 \mu^*}) b^* c - [F_{D_1}, a] b^* c + a (F_{\mu D_1 \mu^*} - F_{D_1}) b^* c$$

so that $(U^* F_{D_2} U - F_{D_1}) a b^* c \mu \langle D_0 \rangle^{\beta}$ is bounded. For $d \in \mathscr{M}$ and $e \in A$ we find

$$(U^* F_{D_2} U - F_{D_1}) a b^* c d^* e = (U^* F_{D_2} U - F_{D_1}) a^* b c \mu \langle D_1 \rangle^{\beta} (\langle D_1 \rangle^{-\beta} \mu^{-1} d^* \langle D_1 \rangle^{\beta}) \langle D_1 \rangle^{-\beta} e$$

is compact. By the inclusion $A \subseteq \overline{\operatorname{span}}((\mathscr{M}^* \mathscr{M})^2 A)$, we are done. □

III.1.3.1 A partial converse

A partial converse result is possible, in the sense that these kinds of estimates on bounded transforms always arise from an additive and a multiplicative perturbation of the unbounded operator. This is not quite precise due to differences in the differentiability assumptions. The following is nearly a converse to Corollary III.1.30.

Theorem III.1.34. *Let D_1 and D_2 be self-adjoint regular operators with equal domains such that, for some $0 < \alpha \leq 1$,*

$$(F_{D_1} - F_{D_2})\langle D_1\rangle^\alpha$$

is bounded on $\operatorname{dom}\langle D_1\rangle^\alpha$. Then there exist a bounded invertible operator μ and a self-adjoint regular operator T such that

$$D_2 = \mu D_1 \mu^* + T$$

and both

$$\langle D_1\rangle^{-1/2} T \langle D_1\rangle^{-1/2+\alpha} \qquad \left([F_{D_1}, \mu] - T\langle D_2\rangle^{-1}\right)\langle D_1\rangle^\alpha$$

are bounded. Furthermore, if $1/2 \leq \alpha$,

$$T\langle D_1\rangle^{-1+\alpha} \qquad [F_{D_1}, \mu]\langle D_1\rangle^\alpha$$

are bounded.

Proof. Let $\mu = \langle D_2\rangle^{1/2}\langle D_1\rangle^{-1/2}$ and $T = \langle D_2\rangle^{1/2}(F_{D_2} - F_{D_1})\langle D_2\rangle^{1/2}$, defined on $\operatorname{dom} D_1$, so that

$$\begin{aligned}
\mu D_1 \mu^* + T &= \langle D_2\rangle^{1/2}\langle D_1\rangle^{-1/2} D_1 \langle D_1\rangle^{-1/2}\langle D_2\rangle^{1/2} + \langle D_2\rangle^{1/2}(F_{D_2} - F_{D_1})\langle D_2\rangle^{1/2} \\
&= \langle D_2\rangle^{1/2}\left(F_{D_1} + (F_{D_2} - F_{D_1})\right)\langle D_2\rangle^{1/2} \\
&= D_2.
\end{aligned}$$

We have

$$\begin{aligned}
[F_{D_1}, \mu] &= \left(F_{D_1}\langle D_2\rangle^{1/2} - \langle D_2\rangle^{1/2} F_{D_1}\right)\langle D_1\rangle^{-1/2} \\
&= \left(\langle D_2\rangle^{1/2}(F_{D_2} - F_{D_1}) + (F_{D_2} - F_{D_1})\langle D_2\rangle^{1/2}\right)\langle D_1\rangle^{-1/2} \\
&= \left(T\langle D_2\rangle^{-1/2} + \langle D_2\rangle^{-1/2} T\right)\langle D_1\rangle^{-1/2} \\
&= T\langle D_2\rangle^{-1} + (F_{D_2} - F_{D_1}).
\end{aligned}$$

Because the domains of D_1 and D_2 are equal, $(F_{D_2} - F_{D_1})\langle D_2\rangle^\alpha$ is bounded and the statement follows from the boundedness of

$$\langle D_2\rangle^{-1/2} T \langle D_2\rangle^{-1/2+\alpha} = (F_{D_2} - F_{D_1})\langle D_2\rangle^\alpha \qquad \left([F_{D_1}, \mu] - T\langle D_2\rangle^{-1}\right)\langle D_2\rangle^\alpha = (F_{D_2} - F_{D_1})\langle D_2\rangle^\alpha.$$

Suppose that $1/2 \leq \alpha$. It is sufficient to prove that

$$T\langle D_2\rangle^{-1+\alpha} = \langle D_2\rangle^{1/2}(F_{D_2} - F_{D_1})\langle D_2\rangle^{-1/2+\alpha}$$

is bounded. If $\alpha = 1/2$,

$$T\langle D_2\rangle^{-1/2} = \langle D_2\rangle^{1/2}(F_{D_2} - F_{D_1})$$

and we are done. If $1/2 < \alpha \leq 1$, both $1/2$ and $-1/2 + \alpha$ are positive, and we can interpolate between

$$(F_{D_1} - F_{D_2})\langle D_2\rangle^\alpha \quad \text{and} \quad \langle D_2\rangle^\alpha (F_{D_1} - F_{D_2})$$

as in [Les05, Proposition A.1], adjusted for Hilbert modules in [LM19, Lemma 7.7] (see also §A.3). □

III.1.4 The logarithmic transform: multiplicative to additive

Conformal transformations of unbounded Kasparov modules are not preserved by the exterior product. This is exemplified by the fact that the Cartesian product of two conformally perturbed Riemannian manifolds $(X_1, k_1^2\mathbf{g}_1)$ and $(X_2, k_2^2\mathbf{g}_2)$ is not a conformal perturbation of the Cartesian product $(X_1 \times X_2, \mathbf{g}_1 \oplus \mathbf{g}_2)$, unless $k_1(x) = k_2(y)$ for all $x \in X_1$ and $y \in X_2$, i.e. $k_1 = k_2$ is a constant. The *logarithmic dampening* of [GMR19] provides a way of turning conformal transformations into locally bounded perturbations, at the expense of much of the geometrical information encoded by the Dirac operator.

Proposition III.1.35. *Let D be a self-adjoint regular operator on a right Hilbert B-module E and let $a \in \operatorname{End}_B^*(E)$ preserve $\operatorname{dom} D$. Suppose also that $[F_D, a]\log\langle D\rangle$ is bounded. Then, with*

$$L_D = F_D \log\langle D\rangle = D\log((1+D^2)^{1/2})(1+D^2)^{-1/2},$$

the commutator $[L_D, a]$ is bounded.

Proof. By [GMR19, Lemma 1.15], the condition $a \operatorname{dom} D \subseteq \operatorname{dom} D$ implies that $a \operatorname{dom}\log\langle D\rangle \subseteq \operatorname{dom}\log\langle D\rangle$ and that $[\log\langle D\rangle, a]$ is bounded. Using also the condition on $[F_D, a]$,

$$[L_D, a] = F_D[\log\langle D\rangle, a] + [F_D, a]\log\langle D\rangle$$

is bounded. □

Corollary III.1.36. *Let D_0 and D_1 be self-adjoint regular operators on right Hilbert B-modules E_0 and E_1. Suppose that there is an operator $a \in \operatorname{Hom}_B^*(E_0, E_1)$ such that $a \operatorname{dom} D_0 \subseteq \operatorname{dom} D_1$ and*

$$(F_{D_1}a - aF_{D_0})\log\langle D_0\rangle$$

extends to an adjointable operator. Then $L_{D_1}a - aL_{D_0}$ is bounded.

Theorem III.1.37. *Let (U, μ) be a conformal transformation from the order-$\frac{1}{1-\alpha}$ cycle (A, E_B, D_1) to the order-$\frac{1}{1-\alpha}$ cycle (A, E'_B, D_2). Then the logarithmic transforms (A, E_B, L_{D_1}) and (A, E'_B, L_{D_2}) are related by the unitary U, up to locally bounded perturbation; in particular, A is contained in the closure of the set of $a \in \operatorname{End}^*(E)$ such that*

$$(U^*L_{D_2}U - L_{D_1})a \qquad [L_{D_1}, a] \qquad [L_{D_2}, UaU^*]$$

are bounded.

Proof. Let $a, b, c \in \mathscr{M}$ so that $(U^*F_{D_2}U - F_{D_1})ab^*c\mu\langle D_0\rangle^\beta$

$$\begin{aligned}(U^*L_{D_2}U - L_{D_1})ab^*c\mu &= U^*L_{D_2}Uab^*c\mu - ab^*c\mu L_{D_1} - [L_{D_1}, ab^*c\mu] \\ &= U^*F_{D_2}U(U^*\log\langle D_2\rangle Uab^*c\mu - ab^*c\mu\log\langle D_1\rangle) \\ &\qquad + (U^*F_{D_2}U - F_{D_1})ab^*c\mu\log\langle D_1\rangle - F_{D_1}[\log\langle D_1\rangle, ab^*c\mu]\end{aligned}$$

is bounded, by the proof of Theorem III.1.4. Let $d \in \operatorname{Lip}_\alpha^*(D)$ and multiply on the right by $\mu^{-1}d$. Then $(U^*L_{D_2}U - L_{D_1})a^*bcd$ is bounded and, by the inclusions $A \subseteq \overline{\operatorname{span}}(\mathscr{M}A) \subseteq \overline{\operatorname{span}}(\mathscr{M}\mathscr{M}^*\mathscr{M}\operatorname{Lip}_\alpha^*(D))$, we are done. □

III.1.5 The singular case

Conformal factors on noncompact manifolds need not be bounded nor have bounded inverse. In that setting, we can take a suitable open cover and assemble local estimates. This idea motivates the next definition. In the following we stress that $\overline{\operatorname{span}}$ means the norm completion of finite linear combinations.

Definition III.1.38. A *singular conformal transformation* $(U, (\mu_i)_{i\in I})$ from one order-$\frac{1}{1-\alpha}$ cycle, (A, E_B, D_1), to another, (A, E'_B, D_2), is a unitary map $U : E \to E'$, intertwining the representations of A, and a family $(\mu_i)_{i\in I} \subseteq \mathrm{End}^*(E)$ of (even) invertible operators such that

$$A \subseteq \overline{\mathrm{span}}_{i\in I} A\mathscr{M}_i \cap \overline{\mathrm{span}}_{i\in I} \mathscr{M}_i A$$

where $\mathscr{M}_i$ is the set of $a \in \mathrm{End}^*(E)$ such that

$$(U^* D_2 U a - a\mu_i D_1 \mu_i^*)\mu_i^{-1*}\langle D_1\rangle^{-\alpha} \qquad \langle D_2\rangle^{-\alpha} U(U^* D_2 U a - a\mu_i D_1 \mu_i^*)$$

are bounded, $a, a\mu_i, a\mu_i^{-1*} \in \mathrm{Lip}^*_\alpha(D_1)$, and $UaU^* \in \mathrm{Lip}^*_\alpha(D_2)$.

Remark III.1.39. As in the non-singular case, $\mathscr{M}_i$ is a ternary ring of operators, generally not closed. In particular, $\overline{\mathrm{span}}(\mathscr{M}_i\mathscr{M}_i^*\mathscr{M}_i) = \overline{\mathscr{M}_i}$.

Theorem III.1.40. *Let $(U, (\mu_n)_{n\in\mathbb{N}})$ be a singular conformal transformation from (A, E_B, D_1) to (A, E'_B, D_2). Then the bounded transforms (A, E_B, F_{D_1}) and (A, E'_B, F_{D_2}) are related by the unitary U, up to locally compact perturbation, i.e.*

$$(U^* F_{D_2} U - F_{D_1})a \in \mathrm{End}^0(E)$$

for all $a \in A$.

Proof. As in the Proof of Theorem III.1.4, $(U^* F_{D_2} U - F_{D_1})ab^*c\mu_i\langle D_0\rangle^\beta$ is bounded for all $a, b, c \in \mathscr{M}_i$. For $d, e \in \mathscr{M}_i$ and $f \in A$ we find

$$(U^* F_{D_2} U - F_{D_1})ab^*cd^*ef = (U^* F_{D_2} U - F_{D_1})a^*bc\mu_i\langle D_1\rangle^\beta(\langle D_1\rangle^{-\beta}\mu_i^{-1}d^*e\langle D_1\rangle^\beta)\langle D_1\rangle^{-\beta}f$$

is compact. The inclusion of $A \subseteq \overline{\mathrm{span}}_{i\in I}(\mathscr{M}_i A) = \overline{\mathrm{span}}_{i\in I}((\mathscr{M}_i\mathscr{M}_i^*)^2\mathscr{M}_i A)$ proves the statement. □

Example III.1.41. Let us reprise Example III.1.6, in which we considered Riemannian spin^c manifolds $(X, \mathbf{g})$ and $(X, \mathbf{h})$ such that $\mathbf{h} = k^2\mathbf{g}$. Suppose that $(X, \mathbf{g})$ is geodesically complete, so that $\not{D}_\mathbf{g}$ is self-adjoint. It may or may not be the case that $(X, \mathbf{h})$ is complete and $\not{D}_\mathbf{h}$ is self-adjoint, depending on the properties of k, although that is guaranteed if k is bounded with bounded inverse. Let $(O_i)_{i\in I}$ be an open cover of X such that k is bounded and invertible when restricted to any O_i. (This can be ensured by choosing a relatively compact cover.) Choose a family $(k_i)_{i\in I}$ of positive smooth functions which are bounded and invertible and agree with k on the corresponding O_i. Let $f \in C_c^\infty(O_i)$, so that

$$\begin{aligned} U^*\not{D}_\mathbf{h} Uf - fk_i^{-1/2}\not{D}_\mathbf{g} k_i^{-1/2} &= k^{-1/2}\not{D}_\mathbf{g} k^{-1/2}f - fk_i^{-1/2}\not{D}_\mathbf{g} k_i^{-1/2} \\ &= k^{-1/2}[\not{D}_\mathbf{g}, f]k_i^{-1/2} \\ &= k_i^{-1/2}[\not{D}_\mathbf{g}, f]k_i^{-1/2} \end{aligned}$$

is bounded. Then $(U, (k_i^{-1/2})_{i\in I})$ is a singular conformal transformation from the spectral triple $(C_0(X), L^2(X, S_\mathbf{g}), \not{D}_\mathbf{g})$ to $(C_0(X), L^2(X, S_\mathbf{h}), \not{D}_\mathbf{h})$, provided that $(X, \mathbf{h})$ is complete so that the latter is a spectral triple. In the context of Example III.1.7, $(U, (k_i^{-1/2})_{i\in I})$ is a singular conformal transformation from $(C_0(X), L^2(\Omega^*X, \mathbf{g}), d + \delta_\mathbf{g})$ to $(C_0(X), L^2(\Omega^*X, \mathbf{h}), d + \delta_\mathbf{h})$.

If either or both of $(X, \mathbf{g})$ and $(X, \mathbf{h})$ fails to be complete, the failure of self-adjointness of the Dirac operator(s) means that one requires the technology of half-closed chains and relative spectral triples. We do not pursue this here; for more details, see [Hil10, DGM18, FGMR19].

An abstract treatment of open covers, for the purposes of unbounded KK-theory, can be found in [Dun22]; see, in particular, [Dun22, Lemma 4.3].

In the following example, inspired by the modular cycles of [Kaa21], one should think of $\Delta_-\Delta_+^{-1}$ as the conformal factor, which can be both unbounded and noninvertible. Later, in Proposition III.4.7, we directly generalise the results of [Kaa21].

Proposition III.1.42. *Let (A, E_B, D_1) and (A, E_B, D_2) be unbounded Kasparov modules. Let Δ_+ and Δ_- be commuting positive adjointable operators such that*

- *For all $a \in A$, $(a(\Delta_+ + \Delta_-)(\Delta_+ + \Delta_- + \frac{1}{n})^{-1})_{n=1}^\infty$ converges in operator norm to a.*
- *$\Delta_+, \Delta_- \in \operatorname{Lip}^*_\alpha(D_1)$; and*
- *$A \subseteq \overline{\operatorname{span}}(A\mathscr{N}) \cap \overline{\operatorname{span}}(\mathscr{N}A)$, where $\mathscr{N}$ is the set of $a \in \operatorname{Lip}^*_\alpha(D_1) \cap \operatorname{Lip}^*_\alpha(D_2)$ such that $a \operatorname{dom} D_1 \subseteq \operatorname{dom} D_2$, $a^* \operatorname{dom} D_2 \subseteq \operatorname{dom} D_1$, and*

$$(D_2 a\Delta_+ - aD_1\Delta_-)\langle D_1\rangle^{-\alpha} \qquad \langle D_2\rangle^{-\alpha}(D_2 a\Delta_+ - aD_1\Delta_-)$$

extend to adjointable operators.

Let $(h_n)_{n\in\mathbb{N}_{\geq 1}} \subseteq C_b^\infty(\mathbb{R}_+^\times)$ be any sequence of positive functions with bounded reciprocals which agree with the function $x \mapsto x^{-1/2}$ on the interval $[\frac{1}{n}, n]$. Then $(1, (h_n(\Delta_+)h_n(\Delta_-)^{-1})_{n\in\mathbb{N}_{\geq 1}})$ is a singular conformal transformation from (A, E_B, D_1) to (A, E_B, D_2).

For the proof, we shall make use of the smooth functional calculus of §A.4.2.

Lemma III.1.43. *Let A be a C^*-algebra represented by π on a Hilbert module E. Let $h \in C \subseteq \operatorname{End}^*(E)$ be a strictly positive element of a C^*-algebra C such that, for a dense subset of $a \in A$, the sequence*

$$(\pi(a)h(h + 1/n)^{-1})_{n=1}^\infty$$

converges to $\pi(a)$. Then $\pi(A)$ is contained in the closure of $\pi(A)C$.

Proof. First, note that $(h(h+1/n)^{-1})_{n=1}^\infty$ is an approximate unit for C. For every $a \in A$ such that the sequence $(\pi(a)h(h + 1/n)^{-1})_{n=1}^\infty \subseteq \pi(a)C$ converges in norm to $\pi(a)$, $\pi(a) \in \overline{\pi(a)C}$. □

Proof of Proposition III.1.42. First, the smooth functional calculus of Theorem A.4.18 shows that the $h_n(\Delta_+)h_n(\Delta_-)^{-1} \in \operatorname{Lip}^*_\alpha(D_1)$ is bounded. Second, let $f_1, f_2 \in C_c^\infty((\frac{1}{n}, n))$ and $a \in \mathscr{N}$, and define $b \in \operatorname{End}^*(E)$ to be the product

$$af_1(\Delta_+)f_2(\Delta_-) \in \mathscr{N}C_0((\tfrac{1}{n}, n))(\Delta_+)C_0((\tfrac{1}{n}, n))(\Delta_-).$$

Then $bh_n(\Delta_+)h_n(\Delta_-)^{-1} = b\Delta_+^{-1/2}\Delta_-^{1/2}$. Again using the smooth functional calculus,

$$\begin{aligned}
&\big(D_2 b - bh_n(\Delta_+)h_n(\Delta_-)^{-1}D_1 h_n(\Delta_+)h_n(\Delta_-)^{-1}\big)(h_n(\Delta_+)h_n(\Delta_-)^{-1})^{-1}\langle D_1\rangle^{-\alpha} \\
&\quad = (D_2 a\Delta_+ - aD_1\Delta_-)\langle D_1\rangle^{-\alpha}\big(\langle D_1\rangle^{\alpha}\Delta_+^{-1/2}\Delta_-^{-1/2}f_1(\Delta_+)f_2(\Delta_-)\langle D_1\rangle^{-\alpha}\big) \\
&\qquad + a\left[D_1, \Delta_-^{1/2}\Delta_+^{-1/2}f_1(\Delta_+)f_2(\Delta_-)\right]\langle D_1\rangle^{-\alpha}
\end{aligned}$$

and

$$\begin{aligned}
&\langle D_2\rangle^{-\alpha}\big(D_2 b - bh_n(\Delta_+)h_n(\Delta_-)^{-1}D_1 h_n(\Delta_+)h_n(\Delta_-)^{-1}\big) \\
&\quad = \langle D_2\rangle^{-\alpha}(D_2 a\Delta_+ - aD_1\Delta_-)\Delta_+^{-1}f_1(\Delta_+)f_2(\Delta_-) \\
&\qquad + \big(\langle D_1\rangle^{\alpha}a^*\langle D_2\rangle^{-\alpha}\big)^*\langle D_1\rangle^{-\alpha}\left[D_1, \Delta_-^{1/2}\Delta_+^{-1/2}f_1(\Delta_+)f_2(\Delta_-)\right]h_n(\Delta_+)h_n(\Delta_-)^{-1}
\end{aligned}$$

extend to adjointable operators. Hence $b \in \mathscr{M}_n$. The closure of $C_0((\frac{1}{n}, n))(\Delta_+)C_0((\frac{1}{n}, n))(\Delta_-)$ is $C^*(\Delta_+, \Delta_-)$. By Lemma III.1.43, we have $A \subseteq \overline{AC^*(\Delta_+, \Delta_-)}$ and

$$\overline{\operatorname{span}}_{i\in I}A\mathscr{M}_i \cap \overline{\operatorname{span}}_{i\in I}\mathscr{M}_i A \supseteq \overline{\operatorname{span}}(A\mathscr{N}C^*(\Delta_+, \Delta_-)) \cap \overline{\operatorname{span}}(\mathscr{N}C^*(\Delta_+, \Delta_-)A) \supseteq A,$$

as required. □

III.2 Conformal group equivariance

It is not clear that Definition I.2.7 is the correct generalisation of equivariance to unbounded KK-theory. Definition I.2.7 is natural in the sense that the exterior product and descent map are well-defined and Kucerovsky's conditions [Kuc97, Theorem 13] for the Kasparov product still suffice [Kuc94, Theorem 8.12]. On the other hand, let us examine 'patient zero' of noncommutative geometry: a complete Riemannian spin^c manifold $(X, \mathbf{g})$ with spinor bundle S and Dirac operator $\not{D}$, forming the spectral triple $\big(C(X), L^2(X,S), \not{D}\big)$. The largest group for which this is uniformly equivariant, in the sense of Definition I.2.7, is the isometry group $\mathrm{Iso}(X, \mathbf{g})$. What is the largest group for which the Fredholm module

$$\Big(C(X), L^2(X,S), F_{\not{D}}\Big)$$

given by the bounded transform is equivariant, and can a geometric interpretation be put upon it? The answer to this question is that the Fredholm module above is equivariant under the conformal group $\mathrm{Conf}(X, \mathbf{g})$ of X. That this is maximal is confirmed by [Bär07, Theorem 3.1].

Example III.2.1. The simplest example exhibiting this discrepancy is the real line and its Dirac spectral triple $(C_0(\mathbb{R}), L^2(\mathbb{R}), i\partial_x)$. We will compare two group actions on $\mathbb{R}$: translations by $\mathbb{R}$ and dilation by $\mathbb{R}_+^\times$, i.e. addition and multiplication, respectively. The affine group $\mathbb{R} \rtimes \mathbb{R}_+^\times$ acts on $\mathbb{R}$ by $\varphi_{(a,b)} : x \mapsto ax + b$, for $(a,b) \in \mathbb{R} \rtimes \mathbb{R}_+^\times$. Let $V_{(a,b)}$ be the pullback by $\varphi_{(a,b)}^{-1} = \varphi_{(a^{-1},-a^{-1}b)}$ on $L^2(\mathbb{R})$. For $\xi, \eta \in L^2(\mathbb{R})$, we have

$$\int_0^\infty \overline{(V_{(a,b)}\xi)(x)}\eta(x)dx = \int_0^\infty \overline{\xi(a^{-1}(x-b))}\eta(x)dx = \int_0^\infty \overline{\xi(y)}\eta(ay+b)ady$$

so $V_{(a,b)}^* = aV_{(a,b)}^{-1} = aV_{(a^{-1},-a^{-1}b)}$. The unitary part of the polar decomposition of $V_{(a,b)}$ is, therefore, $U_{(a,b)} = a^{-1/2}V_{(a,b)}$. By the chain rule, for $\xi \in C_c^\infty(\mathbb{R})$,

$$(U_{(a,b)}\partial_x U_{(a,b)}^*\xi)(x) = a^{-1/2}(\partial_x U_{(a,b)}^*\xi)(a^{-1}(x-b)) = a^{-3/2}(U_{(a,b)}^*\xi)'(a^{-1}(x-b)) = a^{-1}\xi'(x)$$

so that $U_{(a,b)} i\partial_x U_{(a,b)}^* = a^{-1} i\partial_x$. For the subgroup $\mathbb{R}$ ($a = 1$), the spectral triple $(C_0(\mathbb{R}), L^2(\mathbb{R}), i\partial_x)$ is isometrically equivariant in the sense of Definition I.2.7. On the other hand, when $a \neq 1$, for $f \in C_c^\infty(\mathbb{R})$,

$$U_{(a,b)} i\partial_x U_{(a,b)}^* f - f i\partial_x = (a^{-1} - 1) i\partial_x f + [i\partial_x, f]$$

is as unbounded as $i\partial_x$, so condition 4 of Definition I.2.7 is not satisfied. On the other hand,

$$(U_{(a,b)} F_{i\partial_x} U_{(a,b)}^* - F_{i\partial_x})f = (F_{a^{-1}i\partial_x} - F_{i\partial_x})f = i\partial_x \left((a^2 + (i\partial_x)^2)^{-1/2} - (1 + (i\partial_x)^2)^{-1/2}\right) f$$

is compact, as $y \mapsto y\left((a^2+y^2)^{-1/2} - (1+y^2)^{-1/2}\right)$ is in $C_0(\mathbb{R})$. Hence $(C_0(\mathbb{R}), L^2(\mathbb{R}), F_{i\partial_x})$ is equivariant for all of $\mathbb{R} \rtimes \mathbb{R}_+^\times$. In this section, we will make a definition of equivariance in unbounded KK-theory which can cope with this and similar examples. (We remark that multiplication by -1, although an isometry, is not orientation-preserving and has the effect of multiplying by -1 in $KK_1(C_0(\mathbb{R}), \mathbb{C})$, rather than preserving the class.)

Definition III.2.2. An order-$\frac{1}{1-\alpha}$ A-B-cycle (A, E_B, D) is *conformally equivariant* if E is a G-equivariant A-B-correspondence and there exists a $*$-strongly continuous family $(\mu_g)_{g\in G} \subseteq \mathrm{End}^*(E)$ of (even) invertible operators satisfying the following. We require that $A \subseteq \overline{\mathrm{span}}(A\mathcal{Q}) \cap \overline{\mathrm{span}}(\mathcal{Q}A)$, where $\mathcal{Q}$ is the set of $a \in \mathrm{Lip}_\alpha^*(E)$ such that for all $g \in G$ we have $\{a\mu_g, a\mu_g^{-1*}\}\,\mathrm{dom}\,D \subseteq \mathrm{dom}\,D \cap U_g\,\mathrm{dom}\,D$, and the maps

$$\begin{array}{lll} g \mapsto (U_g D U_g^* a - a\mu_g D \mu_g^*)\mu_g^{-1*}\langle D\rangle^{-\alpha} & g \mapsto [D, a\mu_g]\langle D\rangle^{-\alpha} & g \mapsto [D, a\mu_g^{-1*}]\langle D\rangle^{-\alpha} \\ g \mapsto U_g\langle D\rangle^{-\alpha}U_g^*(U_g D U_g^* a - a\mu_g D\mu_g^*) & g \mapsto \langle D\rangle^{-\alpha}[D, a\mu_g] & g \mapsto \langle D\rangle^{-\alpha}[D, a\mu_g^{-1*}] \end{array}$$

are $*$-strongly continuous from G into bounded operators (but need not be globally bounded). We call $\mu = (\mu_g)_{g\in G}$ the conformal factor.

Remarks III.2.3.

1. When $\mu_g = 1$ for all $g \in G$, this Definition reduces to Definition I.2.7 of uniformly equivariant G-cycles.
2. Also, if $\mu_e = 1$, for elements $a \in \operatorname{End}^*(E)$ satisfying that
$$[D, a\mu_g]\langle D\rangle^{-\alpha}$$
is bounded, a is automatically in $\operatorname{Lip}^*_\alpha(D)$.
3. Note also that it is sufficient that $1 \in \mathcal{Q}$ for the closure conditions to be satisfied; in the nonunital case, an approximate unit might be used.

Theorem III.2.4. *Let (A, E_B, D) be a conformally G-equivariant order-$\frac{1}{1-\alpha}$ cycle. Then (A, E_B, F_D) is a G-equivariant bounded Kasparov module.*

Proof. The only difference from the non-equivariant case is the need to show that, for every $a \in A$, $g \mapsto (F_D - U_g F_D U_g^*)a$ is norm-continuous as a map from G into $\operatorname{End}^0(E)$.

By definition, for every $a \in \mathcal{Q}$, the maps $f_0 : g \mapsto \mu_g^{-1}$ and

$$f_{1,a} : g \mapsto (U_g D U_g^* a - a\mu_g D\mu_g^*)\mu_g^{-1*}\langle D\rangle^{-\alpha} \qquad f_{2,a} : g \mapsto \langle D\rangle^{-\alpha} U_g^*(U_g D U_g^* a - a\mu_g D\mu_g^*)$$
$$f_{3,a} : g \mapsto [D, a\mu_g]\langle D\rangle^{-\alpha} \qquad f_{4,a} : g \mapsto \langle D\rangle^{-\alpha}[D, a\mu_g]$$
$$f_{5,a} : g \mapsto [D, a\mu_g^{-1*}]\langle D\rangle^{-\alpha} \qquad f_{6,a} : g \mapsto \langle D\rangle^{-\alpha}[D, a\mu_g^{-1*}]$$

are $*$-strongly continuous as a map from G into $\operatorname{End}^*(E)$. By Lemma A.1.12, this is equivalent to $f_{i,a}|_K$ residing in $\operatorname{End}^*(C(K, E))$ for every compact subset $K \subseteq G$.

Fix a compact subset $K \subseteq G$ and let $\tilde{E} = C(K, E)$. Define $\tilde{D}$ to be the self-adjoint regular operator on $\tilde{E}$ given by D at each point of K. Let U denote the $\mathbb{C}$-linear map from $\tilde{E}$ to itself given by $g \mapsto U_g$. Let $\tilde{\mu} \in \operatorname{End}^*(\tilde{E})$ be given by $g \mapsto \mu_g$. For every $a \in \operatorname{End}^*(E)$, let $\tilde{a}$ be given by a at each point of G. Then, for every $a \in \mathcal{Q}$,

$$(U\tilde{D}U^*\tilde{a} - \tilde{a}\tilde{\mu}\tilde{D}\tilde{\mu}^*)\tilde{\mu}^{-1*}\langle\tilde{D}\rangle^{-\alpha} \qquad \langle\tilde{D}\rangle^{-\alpha}U^*(U\tilde{D}U_g^*\tilde{a} - \tilde{a}\tilde{\mu}\tilde{D}\tilde{\mu}^*)$$
$$[\tilde{D}, \tilde{a}\tilde{\mu}]\langle\tilde{D}\rangle^{-\alpha} \qquad \langle\tilde{D}\rangle^{-\alpha}[\tilde{D}, \tilde{a}\tilde{\mu}] \qquad [\tilde{D}, \tilde{a}\tilde{\mu}^{-1*}]\langle\tilde{D}\rangle^{-\alpha} \qquad \langle\tilde{D}\rangle^{-\alpha}[\tilde{D}, \tilde{a}\tilde{\mu}^{-1*}] \qquad [\tilde{D}, \tilde{a}]$$

are adjointable endomorphisms of $\tilde{E}$. Let $a, b, c, d \in \mathcal{Q}$. As in the Proof of Theorem III.1.4,

$$[\tilde{\mu}\tilde{D}\tilde{\mu}^*, \tilde{b}^*\tilde{c}]\tilde{\mu}^{-1*}\langle\tilde{D}\rangle^{-\alpha}$$

is bounded. We apply Theorem III.1.33 to obtain that $(F_{\tilde{\mu}\tilde{D}\tilde{\mu}^*} - F_{\tilde{D}})\tilde{b}^*\tilde{c}\tilde{d}^*\langle\tilde{D}\rangle^{\beta}$ is bounded for $\beta < 1-\alpha$. Furthermore, as

$$(U\tilde{D}U^*\tilde{a} - \tilde{a}\tilde{\mu}\tilde{D}\tilde{\mu}^*)\tilde{\mu}^{-1*}\langle\tilde{D}\rangle^{-\alpha}$$

is bounded, Proposition I.1.1, shows that

$$(UF_{\tilde{D}}U^*\tilde{a} - \tilde{a}F_{\tilde{\mu}\tilde{D}\tilde{\mu}^*})\tilde{\mu}\langle\tilde{D}\rangle^{\beta}$$

is too. Taking care because U is only $\mathbb{C}$-linear, we have

$$\begin{aligned}(UF_{\tilde{D}}U^* - F_{\tilde{D}})\tilde{a}\tilde{b}^*\tilde{c}\tilde{d}^* &= U[F_{\tilde{D}}, U^*]\tilde{a}\tilde{b}^*\tilde{c}\tilde{d}^* = U[F_{\tilde{D}}, U^*\tilde{a}\tilde{b}^*\tilde{c}]\tilde{d}^* - [F_{\tilde{D}}, \tilde{a}\tilde{b}^*\tilde{c}]\tilde{d}^* \\ &= U(F_{\tilde{D}}U^*\tilde{a} - U^*\tilde{a}F_{\tilde{\mu}\tilde{D}\tilde{\mu}^*})\tilde{b}^*\tilde{c}\tilde{d}^* + \tilde{a}(F_{\tilde{\mu}\tilde{D}\tilde{\mu}^*}\tilde{b}^*\tilde{c}\tilde{d}^* - \tilde{b}^*\tilde{c}F_{\tilde{D}}) - [F_{\tilde{D}}, \tilde{a}\tilde{b}^*\tilde{c}]\tilde{d}^* \\ &= U(F_{\tilde{D}}U^*\tilde{a} - U^*\tilde{a}F_{\tilde{\mu}\tilde{D}\tilde{\mu}^*})\tilde{b}^*\tilde{c}\tilde{d}^* + \tilde{a}(F_{\tilde{\mu}\tilde{D}\tilde{\mu}^*} - F_{\tilde{D}})\tilde{b}^*\tilde{c}\tilde{d}^* - [F_{\tilde{D}}, \tilde{a}]\tilde{b}^*\tilde{c}\tilde{d}^*\end{aligned}$$

so that $(UF_{\tilde{D}}U^* - F_{\tilde{D}})\tilde{a}\tilde{b}^*\tilde{c}\tilde{d}^*\langle\tilde{D}\rangle^{\beta}$ is bounded. Letting $e \in A$ we have

$$(UF_{\tilde{D}}U^* - F_{\tilde{D}})\tilde{a}\tilde{b}^*\tilde{c}\tilde{d}^*\tilde{e} \tag{III.2.5}$$

is in $\mathrm{End}^0(\tilde{E}) = \mathrm{End}^0(C(K,E))$.

Define the map $f' : g \mapsto (F_D - U_g F_D U_g^*)ab^*cd^*e$ from G into bounded operators on E. By Lemma A.1.8, the norm-continuity of f' is equivalent to the condition that $f'|_K$ be in $\mathrm{End}^0(C(K,E))$ for every compact subset $K \subseteq G$. By the inclusion of $A \subseteq \overline{\mathcal{Q}\mathcal{Q}^*\mathcal{Q}\mathcal{Q}^*A}$, we are done. □

Example III.2.6. Let $(X, \mathbf{g})$ be a complete Riemannian spin^c manifold with spinor bundle $\not{S}$ and Atiyah–Singer Dirac operator $\not{D}$. Let G be a locally compact group with a spin^c-preserving conformal action φ on X, so that $\varphi_g^*(\mathbf{g}) = k_g^2 \mathbf{g}$ for $g \in G$. If the conformal factors $(k_g)_{g\in G}$ are each bounded and invertible (for instance, if X is compact), then $(C_0(X), L^2(X, \not{S}_{\mathbf{g}}), \not{D})$ is a conformally G-equivariant spectral triple with conformal factors $(k_{g^{-1}}^{-1/2})_{g\in G}$.

Example III.2.7. Let $(X, \mathbf{g})$ be a complete oriented Riemannian manifold with Hodge–de Rham operator $d+\delta$. Let G be a locally compact group with a conformal action φ on X, so that $\varphi_g^*(\mathbf{g}) = k_g^2 \mathbf{g}$ for $g \in G$. If the conformal factors $(k_g)_{g\in G}$ are each bounded and invertible (for instance, if X is compact), then $(C_0(X), L^2(\Omega^* X), d+\delta)$ is a conformally G-equivariant spectral triple with conformal factors $(k_{g^{-1}}^{-1/2})_{g\in G}$.

Example III.2.8. Let P be a principal circle bundle over a compact Hausdorff space X. Let $\Phi : C(P) \to C(X)$ be the conditional expectation given by averaging over the circle action. By [CNNR11, Proposition 2.9],

$$(C(P), L^2(P, \Phi)_{C(X)}, N = -i\partial_\theta) \tag{III.2.9}$$

is an unbounded Kasparov module, where N is the number operator on the spectral subspaces, equivalent to the vertical Dirac operator $-i\partial_\theta$ acting on each fibre. Let G be a group acting on P and X, compatibly with the surjection $P \to X$. Suppose that φ acts differentiably between the fibres. Since the circle is one-dimensional, $\varphi_g^*(d\theta^2) = k_g^2 d\theta^2$ for a family of functions $(k_g)_{g\in G} \in C(P)$. We obtain that (III.2.9) is conformally G-equivariant with conformal factors $(k_{g^{-1}}^{-1/2})_{g\in G}$.

In the following Example, we give a truly noncommutative example of conformal equivariance, showing that the order-2 spectral triple for the C*-algebra of the Heisenberg group built in §II.4.2 is conformally equivariant.

Example III.2.10. Recall the order-2 spectral triple

$$(C^*(\mathsf{H}_3), L^2(\mathsf{H}_3, \mathbb{C}^2), M_\ell)$$

of §II.4.2, where $\ell : \mathsf{H}_3 \to \mathscr{C}\ell_3$ is the weight given by

$$\ell : \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \mapsto (a\gamma_1 + b\gamma_2)(a^2+b^2)^{1/2} + c\gamma_3 \,.$$

There is an action of $\mathbb{R}_+^\times$ on H_3 by automorphisms, given for $t \in \mathbb{R}_+^\times$ by

$$\begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & ta & t^2c \\ & 1 & tb \\ & & 1 \end{pmatrix}.$$

Let $V_t \in B(L^2(\mathsf{H}_3))$ be given by the pullback

$$V_t\xi(a,b,c) = \xi(t^{-1}a, t^{-1}b, t^{-2}c)$$

on $\xi \in L^2(\mathsf{H}_3)$. Then

$$\langle V_t^*\xi \mid \eta\rangle = \int \xi(t^{-1}a, t^{-1}b, t^{-2}c)\eta(a,b,c)dadbdc = \int \xi(x,y,z)\eta(tx,ty,t^2z)t^4dxdydz = t^4\langle \xi \mid V_{t^{-1}}\eta\rangle$$

so that $V_t^* = t^4 V_{t^{-1}}$. The unitary in the polar decomposition is given by $U_t = t^{-2}V_t$. Noting that

$$\ell(ta, tb, t^2c) = t^2\ell(a,b,c)$$

we see that the operator M_ℓ transforms as

$$\begin{aligned}(U_tM_\ell U_t^*\xi)(a,b,c) &= t^{-2}(M_\ell U_t^*\xi)(t^{-1}a, t^{-1}b, t^{-2}c)\\ &= t^{-2}\ell(t^{-1}a, t^{-1}b, t^{-2}c)(U_t^*\xi)(t^{-1}a, t^{-1}b, t^{-2}c)\\ &= t^{-2}\ell(a,b,c)\xi(a,b,c)\\ &= t^{-2}(M_\ell\xi)(a,b,c)\end{aligned}$$

on a vector $\xi \in L^2(\mathsf{H}_3, \mathbb{C}^2)$. In summary, the data $(C^*(\mathsf{H}_3), L^2(\mathsf{H}_3,\mathbb{C}^2), M_\ell)$, together with the action $(U_t)_{t\in\mathbb{R}}$ of the group $\mathbb{R}_+^\times$ and conformal factors given by $\mu_t = t^{-1}$, constitute a conformally $\mathbb{R}_+^\times$-equivariant 2nd-order spectral triple.

The C*-algebra of the Heisenberg group can be identified with a continuous field of Moyal planes (with one classical plane) over $\mathbb{R}$ [ENN93, §4]. In this picture, the group action is dilation on $\mathbb{R}$ and a corresponding scaling of the parameters of the Moyal planes.

We generalise Example III.2.10 to all Carnot groups and their dilation actions in §IV.3.1.

One limitation of conformal equivariance is that the exterior product becomes ill-defined. This is exemplified by the fact that the conformal group of the Cartesian product of Riemannian manifolds is generically smaller than the product of the conformal groups. However, at the bounded level of KK-theory, the exterior product is known to exist by Kasparov's Technical Theorem. The logarithmic transform of §III.1.4 will provide a way of turning conformal equivariance into uniform equivariance, making the exterior product constructive, at the expense of much of the geometric information encoded by the Dirac operator. In a similar way, descent and the dual Green–Julg map are not well-defined for conformally equivariant cycles. One way of resolving this is by the logarithmic transform; another will be given in §III.4.1.

Theorem III.2.11. *Let (A, E_B, D) be a conformally G-equivariant order-$\frac{1}{1-\alpha}$ cycle with conformal factor μ. Then (A, E_B, L_D) is a uniformly G-equivariant unbounded Kasparov module.*

Proof. The only difference from the non-equivariant case is the need to show that A is contained in the closure of the set of $a \in \operatorname{End}^*(E)$ such that $[L_D, a]$ extends to an adjointable operator and $g \mapsto (L_D - U_gL_DU_g^*)a$ is $*$-strongly continuous as a map from G into $\operatorname{End}^*(E)$.

Fix a compact subset $K \subseteq G$ and let $\tilde{E} = C(K, E)$. As in the Proof of Theorem III.2.4, define $\tilde{D}$ to be the self-adjoint regular operator on $\tilde{E}$ given by D at each point of K. Let U denote the $\mathbb{C}$-linear map from $\tilde{E}$ to itself given by $g \mapsto U_g$. Let $\tilde{\mu} \in \operatorname{End}^*(\tilde{E})$ be given by $g \mapsto \mu_g$. For every $a \in \operatorname{End}^*(E)$, let $\tilde{a}$ be given by a at each point of G. Let $a, b, c \in \mathcal{Q}$; then as in (III.2.5)

$$(UF_{\tilde{D}}U^* - F_{\tilde{D}})\tilde{a}\tilde{b}^*\tilde{c}\tilde{\mu}\langle\tilde{D}\rangle^\beta$$

is bounded for $\beta < 1-\alpha$. Hence,

$$\begin{aligned}(UL_{\tilde{D}}U^* - L_{\tilde{D}})\tilde{a}\tilde{b}^*\tilde{c}\tilde{\mu} &= UL_{\tilde{D}}U^*\tilde{a}\tilde{b}^*\tilde{c}\tilde{\mu} - \tilde{a}^*b\tilde{\mu}L_{\tilde{D}} - [L_{\tilde{D}}, \tilde{a}\tilde{b}^*\tilde{c}\tilde{\mu}]\\ &= UF_{\tilde{D}}U^*(U\log\langle\tilde{D}\rangle U^*\tilde{a}\tilde{b}^*\tilde{c}\tilde{\mu} - \tilde{a}\tilde{b}^*\tilde{c}\tilde{\mu}\log\langle\tilde{D}\rangle)\\ &\quad + (UF_{\tilde{D}}U^* - F_{\tilde{D}})\tilde{a}\tilde{b}^*\tilde{c}\tilde{\mu}\log\langle\tilde{D}\rangle - F_D[\log\langle\tilde{D}\rangle, \tilde{a}\tilde{b}^*\tilde{c}\tilde{\mu}]\end{aligned}$$

is bounded. By the invertibility of $\tilde{\mu}$, $(UL_{\tilde{D}}U^* - L_{\tilde{D}})\tilde{a}\tilde{b}^*\tilde{c} \in \operatorname{End}^*(C(K,E))$.

Let $d \in A$ and define the map $f' : g \mapsto (L_D - U_gL_DU_g^*)ab^*cd^*$ from G into bounded operators on E. By Lemma A.1.12, the $*$-strong-continuity of f' is equivalent to the condition that $f'|_K$ be in $\operatorname{End}^*(C(K,E))$ for every compact subset $K \subseteq G$, which it is. By the inclusion of $A \in \overline{\operatorname{span}}(\mathcal{Q}\mathcal{Q}^*\mathcal{Q}A)$, we are done. $\square$

Remark III.2.12. Let A be a unital C*-algebra and $\mathscr{A}$ a dense unital $*$-subalgebra of A. Let $(\mathscr{A}, H, D)$ be a conformally G-equivariant p-summable order-m spectral triple. Because $(U_g F_D U_g^* - F_D)(1+D^2)^{\beta/2}$ is bounded for $\beta < m^{-1}$, the G-equivariant Fredholm module (A, H, F_D) is q-summable over $\mathscr{A}$ for any $q > mp$ (see Definition I.2.6).

We note in particular that there are obstructions to finite summability which persist also in the setting above. Connes' obstruction [Con89] (see also [GRU19]) shows that there are no finitely summable G-equivariant higher order spectral triples over A if $A \rtimes G$ is purely infinite. Puschnigg's generalization [Pus11] of the rigidity results of Bader–Furman–Gelander–Monod [BFGM07] goes even further when G is a higher rank lattice and implies essentially that there are no conformally G-equivariant finitely summable higher order spectral triples over a unital A.

III.2.1 The γ-element for the real and complex Lorentz groups

In this section, we lift to unbounded KK-theory the γ-elements constructed for $SO(2n+1, 1)$, $SO(2n, 1)$, and $SU(n, 1)$ by Kasparov [Kas84], Chen [Che96], and Julg and Kasparov [JK95], respectively. We have opted to present them with notation close to the original sources. For a unified treatment, see [AJV19, §5.3].

In each case, the Bernstein–Gelfand–Gelfand (BGG) complex [ČS09] for a sphere, considered as a symmetric space, is cleft in twain. For the real Lorentz groups, the BGG complex is the de Rham complex and, for the complex Lorentz groups, it is the Rumin complex [Rum94]. In the case of $SO(2n+1, 1)$, the symmetric space is $\mathbf{S}^{2n}$. The sphere being even dimensional, the middle-degree forms are split into the two eigenspaces of the Hodge star operator, which division is conformally invariant and, indeed, appears in the BGG complex. In the cases of $SO(2n, 1)$ and $SU(n, 1)$, the symmetric space is $\mathbf{S}^{2n-1}$. The sphere being odd-dimensional necessitates the addition of the L^2 harmonic forms on a real or complex hyperbolic space to be added to the half-complex, along with an operator related to the Poisson transform. The sphere $\mathbf{S}^{2n-1}$ is considered as the boundary of $\mathbb{R}\mathbf{H}^{2n} = SO(2n, 1)/S(O(n) \times O(1))$ or $\mathbb{C}\mathbf{H}^n = SU(n, 1)/S(U(n) \times U(1))$.

It is possible that the framework of conformally equivariant unbounded KK-theory could be used to treat the other rank-one groups, $Sp(n, 1)$ and the real form $F_{4(-20)}$, lifting the construction in [Jul19]; however, there, the resulting complex contains differential operators of different orders. In rank two, there is a construction by Yuncken [Yun11] of the γ-element in bounded KK-theory of $SL(3, \mathbb{C})$, using the BGG complex of the flag manifold. A similar construction is proposed for the other rank-two complex semisimple groups [Yun18]. The BGG complex, in full generality, has been put on a sound analytical footing in [DH22] and subsequently fitted into bounded KK-theory in [Gof24], although with limitations on equivariance. The lifting of these constructions to the unbounded picture remains a difficult task, likely to require a substantial renovation of the axioms of an unbounded Kasparov module, beyond what is done here. A step in this direction is the treatment of 'mixed-order' situations in noncommutative geometry in Chapter IV.

III.2.1.1 The case of $SO(2n+1, 1)$

Following [Kas84, §4], we begin with the sphere $\mathbf{S}^{2n}$ on which $SO(2n+1, 1)$ acts conformally and its Hodge–de Rham Dirac operator. As we have seen, we can build a conformally $SO(2n+1, 1)$-equivariant spectral triple

$$(C(\mathbf{S}^{2n}), L^2(\Omega^* \mathbf{S}^n), d + \delta).$$

In order to obtain the KK-class of the γ-element, we split the complexified exterior algebra into two subspaces, each preserved by the Dirac operator. On a $2n$-dimensional manifold, the codifferential is equal to $\delta = d^* = - \star d \star$ and the Hodge star satisfies that

$$\star^2 : \alpha \mapsto (-1)^{|\alpha|} \alpha \qquad \star^* : \alpha \mapsto (-1)^{|\alpha|} \star \alpha$$

for homogeneous $\alpha \in \Omega^*\mathbf{S}^n$. The Hodge star and Hodge-de Rham operator are related by

$$(d+\delta)\star\alpha = (d\star - (-1)^{|\alpha|}\star d)\alpha = \star((-1)^{|\alpha|+1}\star d\star - (-1)^{|\alpha|}d)\alpha = (-1)^{|\alpha|+1}\star(d-\delta)\alpha.$$

Define the map $\epsilon : \alpha \mapsto i^{|\alpha|(|\alpha|+1)-n}\alpha = (-1)^{|\alpha|(|\alpha|+1)/2}i^{-n}\alpha$, so that

$$(\star\epsilon)^2\alpha = i^{|\alpha|(|\alpha|+1)-n}\star\epsilon\star\alpha = i^{|\alpha|(|\alpha|+1)-n}i^{(2n-|\alpha|)((2n-|\alpha|)+1)-n}(-1)^{|\alpha|}\alpha = \alpha$$

and

$$(\star\epsilon)^*\alpha = (-1)^{(2n-|\alpha|)(2n-|\alpha|+1)/2}i^n(-1)^{|\alpha|}\star\alpha = (-1)^{|\alpha|(|\alpha|+1)/2}i^{-n}\star\alpha = \star\epsilon\alpha,$$

meaning that $\star\epsilon$ is a self-adjoint unitary. We have

$$(d+\delta)\star\epsilon\alpha = i^{|\alpha|(|\alpha|+1)-n}(d+\delta)\star\alpha = i^{2|\alpha|+2+|\alpha|(|\alpha|+1)-n}\star(d-\delta)\alpha$$

and

$$\epsilon d\alpha = i^{2|\alpha|+2+|\alpha|(|\alpha|+1)-n}d\alpha \qquad \epsilon\delta\alpha = -i^{2|\alpha|+2+|\alpha|(|\alpha|+1)-n}\delta\alpha.$$

Hence $\star\epsilon$ commutes with $d+\delta$ and we can decompose the exterior algebra into

$$\Omega^*\mathbf{S}^{2n} = \Omega_1^* \oplus \Omega_2^* := \operatorname{im}\left(\frac{1}{2}(1+\star\epsilon)\right) \oplus \operatorname{im}\left(\frac{1}{2}(1-\star\epsilon)\right).$$

We thus have a spectral triple

$$(C(\mathbf{S}^{2n}), L^2(\Omega_1^*), d+\delta)$$

which is still conformally $SO(2n+1,1)$-equivariant and isometrically $SO(2n+1)$-equivariant. By forgetting the action of the algebra, we obtain a representative $(\mathbb{C}, L^2(\Omega_1^*), d+\delta)$ of a class $\gamma \in KK^{SO(2n+1,1)}(\mathbb{C},\mathbb{C})$. The only harmonic forms on $\mathbf{S}^{2n}$ are scalar multiples of $1 \in \Omega^0\mathbf{S}^{2n}$ and the volume form $\mathrm{vol} \in \Omega^{2n}\mathbf{S}^{2n}$. One can check that

$$\star\epsilon 1 = i^{-n}\mathrm{vol} \qquad \star\epsilon\mathrm{vol} = i^n 1 \qquad \frac{1}{2}(1+\star\epsilon)(1+i^{-n}\mathrm{vol}) = 1 + i^{-n}\mathrm{vol}.$$

Hence the only harmonic forms in Ω_1^* are scalar multiples of $(1+i^{-n}\mathrm{vol})$. The form $(1+i^{-n}\mathrm{vol})$ being $SO(2n+1)$-invariant, the restriction $r^{SO(2n+1,1),SO(2n+1)}(\gamma)$ represents $1 \in KK^{SO(2n+1)}(\mathbb{C},\mathbb{C})$. By [AJV19, Proposition 5.9], because γ is the image of an element of $KK^{SO(2n+1,1)}(C(\mathbf{S}^{2n}),\mathbb{C})$ and restricts to $1 \in KK^{SO(2n+1)}(\mathbb{C},\mathbb{C})$, γ is really the γ-element of $SO(2n+1,1)$.

III.2.1.2 The case of $SO(2n,1)$

Following [Che96, §3.1], we begin with the sphere $\mathbf{S}^{2n-1}$, on which $SO(2n,1)$ acts conformally, and its Hodge–de Rham operator. As in the even-dimensional case, we can build a conformally $SO(2n,1)$-equivariant spectral triple

$$(C(\mathbf{S}^{2n-1}), L^2(\Omega^*\mathbf{S}^{2n-1}), d+\delta).$$

To obtain the correct class in $KK_0^{SO(2n,1)}(\mathbb{C},\mathbb{C})$ for the γ-element, we will cut the differential forms in two, as we did for $SO(2n+1,1)$, and add an additional operator.

Let $\mathbf{D}^{2n}$ be the open unit ball with Euclidean metric. The Poincaré disc model is a conformal identification of the hyperbolic space $\mathbb{R}\mathbf{H}^{2n}$ with $\mathbf{D}^{2n}$. As we saw in Example III.1.7 (in particular (III.1.8)) the pullback map $L^2(\Omega^n\mathbb{R}\mathbf{H}^{2n}) \to L^2(\Omega^n\mathbf{D}^{2n})$ is automatically unitary because the forms are of middle degree. Let $I : \mathrm{dom}(I) \subset L^2(\Omega^n\mathbb{R}\mathbf{H}^{2n}) \to L^2(\Omega^n\mathbf{S}^{2n-1})$ be the restriction to the boundary $\mathbf{S}^{2n-1}$ of the ball. Let $\mathscr{H} \subseteq L^2(\Omega^n\mathbb{R}\mathbf{H}^{2n})$ be the L^2 harmonic forms on the real hyperbolic $2n$-space and let $\mathscr{H}_\infty \subset \mathscr{H}$ be those forms in the domain of I. We have a complex

$$0 \longrightarrow \mathscr{H}_\infty \xrightarrow{I} \Omega^n\mathbf{S}^{2n-1} \xrightarrow{d} \Omega^{n+1}\mathbf{S}^{2n-1} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{2n-1}\mathbf{S}^{2n-1} \longrightarrow 0$$

which is invariant under the pullback by the action φ of $SO(2n,1)$. When we complete the spaces of the complex to Hilbert spaces, pullback by the action of $SO(2n,1)$ is not unitary. On $L^2(\Omega^n\mathbf{S}^{2n-1})$ the unitaries $(U_g)_{g\in G}$ implementing the group action φ act by

$$U_g : \xi \mapsto k_{g^{-1}}^{-(-(2n-1)+2n)/2}\varphi^*_{g^{-1}}(\xi) = k_{g^{-1}}^{-1/2}\varphi^*_{g^{-1}}(\xi).$$

As in Example III.1.7,

$$U_g d U_g^* - k_{g^{-1}}^{-1/2} d k_{g^{-1}}^{-1/2}$$

is bounded. However, on the hyperbolic space $\mathbb{R}\mathbf{H}^{2n}$, the group $SO(2n,1)$ acts by isometries. Because the map I commutes with pullback by the group action, $U_g I U_g^* = k_{g^{-1}}^{-1/2} I$, which is not the same behaviour as the rest of the complex displays, the overall exponent of the conformal factor being $-1/2$ rather than -1. On all of $L^2(\Omega^*\mathbf{S}^{2n-1})$ the Laplacian $\Delta = d\delta + \delta d$ transforms so that

$$U_g \Delta^{1/4} U_g^* - k_{g^{-1}}^{-1/2}\Delta^{1/4}$$

is of order $-1/2$. We will replace the operator I in the complex with $\Delta^{1/4} I$, in the hope of obtaining the right conformal scaling.

We need also an operator on $\mathscr{H}$ to act as the conformal factor, because neither functions on $\mathbf{S}^{2n-1}$ nor on $\overline{\mathbf{D}}^{2n}$ are represented naturally on $\mathscr{H}$. By [Che96, Proposition 3.2], there is a polar decomposition $I = \Delta^{1/4} B$, where $B : \mathscr{H} \to L^2(\Omega^n\mathbf{S}^{2n-1})$ is an isometry with range $\Omega^n\mathbf{S}^{2n-1} \cap \ker d$. The operator $B^* k_{g^{-1}}^{-1/2} B$ is positive and invertible on $\mathscr{H}$ because

$$B^* k_{g^{-1}}^{-1/2} B \geq B^* \|k_{g^{-1}}^{1/2}\|^{-1} B = \|k_{g^{-1}}^{1/2}\|^{-1} 1_{\mathscr{H}}.$$

We compute that both

$$\Delta^{1/4} I (B^* k_{g^{-1}}^{-1/2} B) - k_{g^{-1}}^{-1/2}\Delta^{1/4} I = [\Delta^{1/2} P_{\ker d}, k_{g^{-1}}^{-1/2}] B$$

and

$$\begin{aligned} &U_g \Delta^{1/4} I U_g^* - k_{g^{-1}}^{-1/2} \Delta^{1/4} I (B^* k_{g^{-1}}^{-1/2} B) \\ &\qquad = \left(U_g \Delta^{1/4} U_g^* - k_{g^{-1}}^{-1/2}\Delta^{1/4}\right) k_{g^{-1}}^{-1/2}\Delta^{1/4} B - k_{g^{-1}}^{-1/2}\Delta^{1/4}\left[\Delta^{1/4} P_{\ker d}, k_{g^{-1}}^{-1/2}\right] B \end{aligned}$$

are bounded. With $D = \Delta^{1/4} I + I^* \Delta^{1/4} + d + \delta$, the Hodge decomposition theorem $\Omega^n\mathbf{S}^{2n-1} = \ker(\Delta) \oplus Im(d) \oplus Im(\delta)$ shows that $D^2|_{\mathscr{H}_\infty} = B^* d\delta B$ has at most a finite dimensional kernel, while $D^2|_{\Omega^n\mathbf{S}^{2n-1}} = \Delta^{1/2} P_{\ker d}\Delta^{1/2} + \delta d = d\delta + \delta d = \Delta$. On the rest of the complex, D^2 agrees with Δ and so D has compact resolvent. Therefore,

$$(\mathbb{C}, \mathscr{H} \oplus L^2(\Omega^{\geq n}\mathbf{S}^{2n}), \Delta^{1/4} I + I^*\Delta^{1/4} + d + \delta)$$

is a conformally $SO(2n,1)$-equivariant spectral triple with conformal factors $\mu_g = B^* k_{g^{-1}}^{-1/2} B \oplus k_{g^{-1}}^{-1/2}$. Its bounded transform (more exactly its phase) is the γ-element constructed by Chen [Che96, §3.1].

To show that we have obtained the γ-element, independent of the bounded transform, we would need a representation of $C(\overline{\mathbf{D}}^{2n})$ so as to apply [AJV19, Proposition 5.10]. For this purpose, Chen shows that the phase of the larger complex

$$\begin{array}{ccccccccccccc} & & & & 0 & \longrightarrow & \mathscr{H}_\infty & \xrightarrow{I} & \Omega^n\mathbf{S}^{2n-1} & \xrightarrow{d} \cdots \xrightarrow{d} & \Omega^{2n-1}\mathbf{S}^{2n-1} & \longrightarrow & 0 \\ & & & & & & \oplus & & \oplus & & \oplus & & \oplus \\ 0 & \longrightarrow & \Omega^0\mathbb{R}\mathbf{H}^{2n} & \xrightarrow{d} \cdots \xrightarrow{d} & \Omega^n\mathbb{R}\mathbf{H}^{2n}/\mathscr{H}_\infty & & \xrightarrow{d} & & \Omega^{n+1}\mathbb{R}\mathbf{H}^{2n} & \xrightarrow{d} \cdots \xrightarrow{d} & \Omega^{2n}\mathbb{R}\mathbf{H}^{2n} & \longrightarrow & 0 \end{array}$$

gives a Fredholm module for $C(\overline{\mathbf{D}}^{2n})$. Unfortunately, at the level of unbounded Kasparov modules, the construction cannot be carried through because the Hodge–de Rham operator on $\mathbb{R}\mathbf{H}^{2n}$ does not have compact resolvent. Although we do not pursue it here, this defect can be remedied by appealing to the framework of relative spectral triples [FGMR19, Fri25]. The larger complex can be assembled into a relative spectral triple for $C_0(\mathbb{R}\mathbf{H}^{2n}) \lhd C(\overline{\mathbf{D}}^{2n})$ in the sense of [Fri25, Definition 2.8] cf. [Fri25, Example 2.15]. We can show that the K-homology class of the relative spectral triple extends to a class for $C(\overline{\mathbf{D}}^{2n})$ by showing that the boundary map applied to the class of the relative spectral triple is zero. To compute the boundary map as in [HR00, §8.5], one uses the phase rather than the bounded transform. Since the phase already gives a Fredholm module for all of $C(\overline{\mathbf{D}}^{2n})$ the boundary map is zero and we conclude that we do obtain a K-homology class for $C(\overline{\mathbf{D}}^{2n})$.

III.2.1.3 The case of $SU(n,1)$

Following [JK95], we consider the sphere $\mathbf{S}^{2n-1}$, on which $SU(n,1)$ acts by CR-automorphisms. This is not a conformal group action. We replace the de Rham complex with the Rumin complex [Rum94], a refinement depending on a contact structure. A treatment of the Rumin complex in the context of spectral noncommutative geometry and unbounded KK-theory can be found in §IV.2.4. The analytical underpinnings of the Rumin complex, and the much more general class of Rockland complexes, have recently been examined in [DH22]. For the time being, we limit ourselves to outlining those points which we require.

Let X be a $(2n-1)$-dimensional contact manifold with contact structure $H \subseteq TX$. By this, it is meant that there exists a one-form θ such that $H = \ker\theta$ and $d\theta|_H$ is nondegenerate. The nondegeneracy of $d\theta|_H$ is equivalent to $\theta \wedge (d\theta)^{n-1}$ being a volume form. Such a one-form θ is a *contact form* and is not unique. However, if τ is another contact form, then the equality $\ker\tau = \ker\theta$ implies that $\tau = f\theta$ for a nonvanishing smooth function f on X. Conversely, $f\theta$ will be a contact form for any nonvanishing smooth function f on X.

The *Rumin complex* associated to a contact manifold X is a refinement of the de Rham complex of X, depending only on the contact structure (and not on the choice of contact form). For the construction of the Rumin complex on X, we do require a choice of θ, to define two differential ideals of $\Omega^* X$,

- $\mathcal{I}$, the ideal generated by θ and $d\theta$, and
- $\mathcal{J}$, the ideal of forms $\omega \in \Omega^* X$ such that $\theta \wedge \omega$ and $d\theta \wedge \omega$ are zero.

The Rumin complex is built by combining the quotient complex $\Omega^* X/\mathcal{I}^*$ and the subcomplex $\mathcal{J}^*$. These complexes are spliced together using a map $D_H : \Omega^{n-1}X/\mathcal{I}^{n-1} \to \mathcal{J}^n$. The *Rumin differential* D_H is given by $\omega \mapsto d\tilde{\omega}$ where $\tilde{\omega}$ is the unique lift of ω such that $\theta \wedge d\tilde{\omega} = 0$. Surprisingly, D_H is well-defined, is a second-order differential operator, and completes the Rumin complex

$$0 \longrightarrow \Omega^0 X \xrightarrow{d_H} \Omega^1 X/\mathcal{I}^1 \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^{n-1}X/\mathcal{I}^{n-1} \xrightarrow{D_H} \mathcal{J}^n \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{J}^{2n-1} \longrightarrow 0\ ,$$

whose cohomology coincides with the de Rham cohomology. Here, we have denoted the exterior differential on the quotient complex and subcomplex by d_H. The mixture of first- and second-order operators means that the construction of a spectral triple from the Rumin complex requires careful thought; see §IV.2.4. For the construction of the γ-element of $SU(n,1)$, however, this issue will not arise, as we shall see.

Let us fix a contact form θ and choose a Riemannian metric $\mathbf{g}$ on X. We require that these be compatible, in the sense that H is orthogonal to the *Reeb field*, the (unique) vector field Z such that $\theta(Z) = 1$ and $\iota_Z(d\theta) = 0$. Using the metric on $\Omega^k X$ induced by $\mathbf{g}$, we obtain a version

$$\star_H : \Omega^k X/\mathcal{I}^k \to \mathcal{J}^{2n-1-k} \qquad \star_H : \mathcal{J}^k \to \Omega^{2n-1-k}X/\mathcal{I}^{2n-1-k}$$

of the Hodge star operator by the relation $\overline{\alpha} \wedge \star_H \beta = (\alpha, \beta)\theta \wedge (d\theta)^{n-1}$. We thereby obtain formal adjoints of the operators in the Rumin complex, viz. $d_H^* = (-1)^k \star_H d_H \star_H$ and $D^* = (-1)^n \star_H D_H \star_H$. We also obtain the *Rumin Laplacian*, given by

$$\Delta_H = \begin{cases} (n-1-k)d_H d_H^* + (n-k)d_H^* d_H & \text{on } \Omega^k X/\mathcal{J}^k, 0 \leq k \leq n-2 \\ (d_H d_H^*)^2 + D_H^* D_H & \text{on } \Omega^{n-1} X/\mathcal{J}^{n-1} \\ D_H D_H^* + (d_H^* d_H)^2 & \text{on } \mathcal{J}^n \\ (n-k)d_H d_H^* + (n-1-k)d_H^* d_H & \text{on } \mathcal{J}^k, n+1 \leq k \leq 2n-1 \end{cases}.$$

The Rumin Laplacian is hypoelliptic, fourth-order on $\Omega^{n-1}X/\mathcal{J}^{n-1}$ and $\mathcal{J}^n$ and second-order elsewhere.

The contact form θ determines a symplectic form $d\theta$ on H. A CR-structure on X is the additional datum of a complex structure J on H such that $d\theta(X, JY) = \mathbf{g}(X, Y)$ for all $X, Y \in H$. A CR-automorphism of X is a diffeomorphism φ such that the Jacobian φ' preserves and acts complex-linearly on $H \subseteq TX$. Because the Rumin complex depends only on the contact structure, the operators d_H and D_H are unchanged. Again, because the contact structure is preserved, the pullback $\varphi^*(\theta)$ of the contact form must be $f\theta$ for some nonvanishing smooth function on X. Hence

$$\varphi^*(\mathbf{g})(X, Y) = (fd\theta + df \wedge \theta)(X, JY) = fd\theta(X, JY) = f\mathbf{g}(X, Y)$$

for all $X, Y \in H$. On the other hand, the induced metric on TX/H is multiplied by f^2. One can check that the induced metric on the Rumin complex is multiplied by f^{-k} on $\Omega^k X/\mathcal{J}^k$ and f^{-k-1} on $\mathcal{J}^k$. In this sense, CR-automorphisms behave in a similar way to conformal diffeomorphisms.

To construct the γ-element for $SU(n, 1)$, following [JK95, §6(b)], we begin with the Rumin complex on the contact sphere $\mathbf{S}^{2n-1}$ with the round metric, on which the group acts by CR-automorphisms. To obtain the correct class in $KK^{SU(n,1)}(\mathbb{C}, \mathbb{C})$ for the γ-element, we will cut the Rumin complex in two, as we did for $SO(2n+1, 1)$ and $SO(2n, 1)$, and add an additional operator, as we did for the latter. The extra map is the Szegö map S constructed in [JK95, Theorem 2.12] from $\Omega^{n-1}\mathbf{S}^{2n-1}/\mathcal{J}^{n-1}$ to the L^2 harmonic n-forms $\mathscr{H}^n \subseteq \Omega^n \mathbb{C}\mathbf{H}^{2n}$ on the complex hyperbolic space. The sphere $\mathbf{S}^{2n-1}$ can be attached to $\mathbb{C}\mathbf{H}^{2n}$ as its boundary, forming the closed disc $\overline{\mathbf{D}}^{2n}$. The Szegö map takes $\omega \in \Omega^{n-1}\mathbf{S}^{2n-1}/\mathcal{J}^{n-1}$, lifts it uniquely to $\tilde{\omega}$ such that $\theta \wedge d\tilde{\omega} = 0$ (as in the construction of D_H), extends $\tilde{\omega}$ to $\eta \in \Omega^{n-1}\overline{\mathbf{D}}^{2n}$ so that $d\eta \in L^2(\Omega^n \mathbb{C}\mathbf{H}^{2n})$, and then projects η down to $S\omega \in \mathscr{H}^n$. It turns out that such a process gives a well-defined map S, whose kernel is $\ker D_H$, invariant under pullback by the action of $SU(n, 1)$. We dissect the Rumin complex and graft in the Szegö map S, obtaining

$$0 \longrightarrow \Omega^0\mathbf{S}^{2n-1} \xrightarrow{d_H} \Omega^1\mathbf{S}^{2n-1}/\mathcal{J}^1 \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^{n-1}\mathbf{S}^{2n-1}/\mathcal{J}^{n-1} \xrightarrow{S} \mathscr{H}^n_\infty \longrightarrow 0$$

where $\mathscr{H}^n_\infty$ is the image of S, dense in $\mathscr{H}^n$. This new complex is still invariant under pullback by the action φ of $SU(n, 1)$. When we complete the spaces of the complex to Hilbert spaces, pullback by the action of $SU(n, 1)$ is not unitary. The unitary action is, for $\omega \in L^2(\Omega^k/\mathcal{J}^k)$ and $\xi \in \mathscr{H}^n$,

$$U_g\omega = f_{g^{-1}}^{\frac{n-k}{2}} \varphi_{g^{-1}}^* \omega \qquad U_g\xi = \varphi_{g^{-1}}^*\xi,$$

where $(f_g)_{g \in SU(n,1)}$ is a family of nonvanishing, positive, smooth functions on $\mathbf{S}^{2n-1}$. By similar computations to those for Example III.1.7, for the unitary implementors U_g we have that

$$U_g d_H U_g^* \omega = f_{g^{-1}}^{\frac{n-(k+1)}{2}} d_H f_{g^{-1}}^{-\frac{n-k}{2}} \omega = f_{g^{-1}}^{-\frac{1}{2}} d_H \omega + f_{g^{-1}}^{\frac{n-(k+1)}{2}} \left[d_H, f_{g^{-1}}^{-\frac{n-k}{2}}\right]\omega$$

so that $U_g d_H U_g^* - f_{g^{-1}}^{-1/4} d_H f_{g^{-1}}^{-1/4}$ is bounded. On the hyperbolic space $\mathbb{C}\mathbf{H}^n$, the group $SU(n, 1)$ acts by isometries. Because the map S commutes with pullback by the group action, $U_g S U_g^* = S f_{g^{-1}}^{-1/2}$. Unlike in the case of $SO(2n, 1)$, there is no discrepancy between the conformal behaviours of d_H and

S. It remains to construct a conformal factor on $\mathscr{H}^n$. By [JK95, Proof of Theorem 6.6(ii)], there is a polar decomposition $S = \Phi(S)\Delta_H^{1/4}$, where $\Phi(S) : L^2(\Omega^{n-1}\mathbf{S}^{2n-1}/\mathcal{J}^{n-1}) \to \mathscr{H}^n$ is a coisometry with kernel $\ker D_H$. The operator $\Phi(S)f_{g^{-1}}^{-1/4}\Phi(S)^*$ is positive and invertible on $\mathscr{H}^n$ because

$$\Phi(S)f_{g^{-1}}^{-1/4}\Phi(S)^* \geq \Phi(S)\|f_{g^{-1}}^{1/4}\|^{-1}\Phi(S)^* = \|f_{g^{-1}}^{1/4}\|^{-1}.$$

We compute that both

$$Sf_{g^{-1}}^{-1/4} - \left(\Phi(S)f_{g^{-1}}^{-1/4}\Phi(S)^*\right)S = \Phi(S)\left[(1-\ker D)\Delta_H^{-1/4}, f_{g^{-1}}^{-1/4}\right]$$

and

$$U_gSU_g^* - \left(\Phi(S)f_{g^{-1}}^{-1/4}\Phi(S)^*\right)Sf_{g^{-1}}^{-1/4} = \Phi(S)\left[(1-\ker D)\Delta_H^{-1/4}, f_{g^{-1}}^{-1/4}\right]f_{g^{-1}}^{-1/4}$$

are bounded. The operator $d_H + d_H^* + S + S^*$ has compact resolvent by an argument very similar to the case of $SO(2n,1)$, using this time the compactness of the resolvent of the Rumin Laplacian [JK95, Corollary 5.20]. For example, on $\Omega^{n-1}\mathbf{S}^{2n-1}/\mathcal{J}^{n-1}$ one can check that

$$\begin{aligned}(d_H + d_H^* + S + S^*)^2|_{\Omega^{n-1}\mathbf{S}^{2n-1}/\mathcal{J}^{n-1}} &= \Delta_H^{1/4}(1-\ker D)\Delta_H^{1/4} + d_Hd_H^* \\ &= (D_H^*D_H)^{1/2} + d_Hd_H^* \\ &= \Delta_H^{1/2},\end{aligned}$$

and the other cases are similar. In summary, we have constructed a conformally $SU(n,1)$-equivariant spectral triple

$$(\mathbb{C}, L^2(\Omega^{\leq n-1}\mathbf{S}^{2n-1}/\mathcal{J}^{\leq n-1}) \oplus \mathscr{H}^n, d_H + d_H^* + S + S^*)$$

with conformal factors $\mu_g = f_{g^{-1}}^{-1/4} \oplus \Phi(S)f_{g^{-1}}^{-1/4}\Phi(S)^*$. The phase of this spectral triple is exactly the Fredholm module of [JK95, Corollary 6.10] whose class is $\gamma \in KK^{SU(n,1)}(\mathbb{C},\mathbb{C})$.

To show that we have obtained the γ-element without directly using the result of Julg and Kasparov, it would be necessary, as in the case of $SO(2n,1)$ to expand the complex to accommodate a representation of $\overline{\mathbf{D}}^{2n}$. However, as before, the resolvent would not be compact. Furthermore, it is unclear whether sufficient analytical tools are available to obtain bounded commutators.

III.3 Conformal quantum group equivariance

Conformal group actions of a nontrivial kind are already rare in the classical setup of Riemannian manifolds, as the Ferrand–Obata theorem [Fer96, Theorem A] shows. The conformal group of a Riemannian metric must be the isometry group of a conformally equivalent metric, unless the manifold is conformally equivalent to a round sphere $\mathbf{S}^n$ or Euclidean space $\mathbb{R}^n$. It seems that the rarity of large conformal groups carries over to the noncommutative setting. A possible example of a noncommutative geometry with interesting conformal group is the Podleś sphere. As we shall see in §III.3.1, this hope is realised; however the conformal geometry of the Podleś sphere is not governed by a group but rather by a quantum group.

To generalise Definition I.3.8 to conformal (co)actions, we will consider a conformal factor μ which is an unbounded operator on $E \otimes S$, where E is a Hilbert B-module and S is a C*-bialgebra. It is necessary to allow μ to be unbounded in the 'S direction', as can be seen from classical group equivariance. To apply the multiplicative perturbation theory of §III.1.3, we will require μ to be *S-matched*, in the sense of §A.1.2, meaning roughly that μ is locally bounded in the S-direction. We denote by K_S the Pedersen ideal of S.

Definition III.3.1. Let A and B be C*-algebras equipped with coactions of a C*-bialgebra S. An order-$\frac{1}{1-\alpha}$ A-B-cycle (A, E_B, D) is *conformally S-equivariant* if E is an S-equivariant A-B-correspondence and there exists an (even) S-matched operator μ on $(E \otimes S)_{B \otimes S}$ whose inverse is also S-matched, satisfying the following. For $a \in \mathrm{Lip}^*_\alpha(D)$, let $\mathcal{S}_a$ be the set of $s \in M(S)$ such that

$$\{(a \otimes s)\mu, (a \otimes s)\mu^{-1*}\} \operatorname{dom}(D \otimes 1)(1 \otimes K_S) \subseteq \operatorname{dom}(D \otimes 1) \cap V_E \operatorname{dom}(D \otimes_{\delta_B} 1)$$

and

$$\begin{gathered} \Big(V_E(D \otimes_{\delta_B} 1)V_E^*(a \otimes s) - (a \otimes s)\mu(D \otimes 1)\mu^*\Big)\mu^{-1*}\langle D \otimes 1\rangle^{-\alpha}, \\ V_E\langle D \otimes_{\delta_B} 1\rangle^{-\alpha} V_E^*\Big(V_E(D \otimes_{\delta_B} 1)V_E^*(a \otimes s) - (a \otimes s)\mu(D \otimes 1)\mu^*\Big), \\ [D \otimes 1, (a \otimes s)\mu]\langle D \otimes 1\rangle^{-\alpha}, \qquad [D \otimes 1, (a \otimes s)\mu^{-1*}]\langle D \otimes 1\rangle^{-\alpha}, \\ \langle D \otimes 1\rangle^{-\alpha}[D \otimes 1, (a \otimes s)\mu], \qquad \text{and } \langle D \otimes 1\rangle^{-\alpha}[D \otimes 1, (a \otimes s)\mu^{-1*}] \end{gathered}$$

extend to S-matched operators. Let $\mathcal{Q}$ be the set of $a \in \mathrm{Lip}^*_\alpha(D)$ such that $S \subseteq \overline{\operatorname{span}}(S\mathcal{S}_a) \cap \overline{\operatorname{span}}(\mathcal{S}_a S)$. Then we require that $A \subseteq \overline{\operatorname{span}}(A\mathcal{Q}) \cap \overline{\operatorname{span}}(\mathcal{Q}A)$.

If A and B are C*-algebras with $\mathbb{G}$-actions, an order-$\frac{1}{1-\alpha}$ cycle (A, E_B, D) is *conformally $\mathbb{G}$-equivariant* if it is conformally $C_0^r(\mathbb{G})$-equivariant.

Remarks III.3.2.

1. When $\mu = 1$, Definition III.3.1 reduces to Definition I.3.8 of uniformly S-equivariant cycles.
2. For a discrete quantum group $\mathbb{G}$, when $C_0(\mathbb{G})$ is isomorphic as an algebra to the C*-algebraic direct sum
$$\bigoplus_{\lambda \in \Lambda} M_{n_\lambda}(\mathbb{C})$$
of finite-dimensional matrix algebras, the Pedersen ideal $K_{C_0(\mathbb{G})}$ is the algebraic direct sum. In this case, the conformal factor and the admissible unitary would be labelled by the index set $\lambda \in \Lambda$, so that
$$V_E^\lambda \in \mathrm{Hom}^*_B(E \otimes_{\delta_B} (B \otimes \mathbb{C}^{n_\lambda}), E \otimes \mathbb{C}^{n_\lambda}) \qquad \mu^\lambda \in \mathrm{End}^*_B(E \otimes \mathbb{C}^{n_\lambda})$$
and the equivariance conditions on $a \in \mathcal{Q}$ become that
$$\begin{gathered} \Big(V_E^\lambda(D \otimes_{\delta_B} 1)V_E^{\lambda*}(a \otimes s) - (a \otimes s)\mu^\lambda(D \otimes 1)\mu^{\lambda*}\Big)(\mu^\lambda)^{-1*}\langle D \otimes 1\rangle^{-\alpha}, \\ V_E^\lambda\langle D \otimes_{\delta_B} 1\rangle^{-\alpha} V_E^{\lambda*}\Big(V_E^\lambda(D \otimes_{\delta_B} 1)V_E^{\lambda*}(a \otimes s) - (a \otimes s)\mu^\lambda(D \otimes 1)\mu^{\lambda*}\Big), \\ [D \otimes 1, (a \otimes s)\mu^\lambda]\langle D \otimes 1\rangle^{-\alpha}, \qquad [D \otimes 1, (a \otimes s)(\mu^\lambda)^{-1*}]\langle D \otimes 1\rangle^{-\alpha}, \\ \langle D \otimes 1\rangle^{-\alpha}[D \otimes 1, (a \otimes s)\mu^\lambda], \qquad \text{and } \langle D \otimes 1\rangle^{-\alpha}[D \otimes 1, (a \otimes s)(\mu^\lambda)^{-1*}] \end{gathered}$$
be bounded for all $\lambda \in \Lambda$.

Theorem III.3.3. *A conformally S-equivariant order-$\frac{1}{1-\alpha}$ cycle (A, E_B, D), with conformal factor μ, gives rise to an S-equivariant bounded Kasparov module (A, E_B, F_D).*

Proof. The only point of difference from the non-equivariant case is the need to prove that, for every $a \in A$ and $s \in S$, $(F_D \otimes 1 - V_E(F_D \otimes_{\delta_B} 1)V_E^*)a \otimes s$ is compact. Let c be a positive element of K_S, so that, by Proposition A.1.20, the restriction of μ to the $B \otimes \overline{\operatorname{span}}(ScS)$-module $E \otimes \overline{\operatorname{span}}(ScS)$ is bounded. For the time being, we work on the module $E \otimes \overline{\operatorname{span}}(ScS)$. Let $a_1, a_2, a_3, a_4 \in \mathcal{Q}$ and $s_1, s_2, s_3, s_4 \in \mathcal{S}_{a_1}, \mathcal{S}_{a_2}, \mathcal{S}_{a_3}, \mathcal{S}_{a_4}$. As in the Proof of Theorem III.1.4,

$$[\mu(D \otimes 1)\mu^*, a_2^* a_3 \otimes s_2^* s_3]\mu^{-1*}\langle D\rangle^{-\alpha}$$

is bounded. We apply Theorem III.1.33 to obtain that

$$(F_{\mu(D\otimes 1)\mu^*} - F_D \otimes 1)a_2^* a_3 a_4^* \langle D\rangle^\beta \otimes s_2^* s_3 s_4^*$$

is bounded for $\beta < 1-\alpha$. Furthermore,

$$((D \otimes_{\delta_B} 1)V_E^*(a_1 \otimes s_1) - V_E^*(a_1 \otimes s_1)\mu(D \otimes 1)\mu^*)\mu^{-1*}(\langle D\rangle^{-\alpha} \otimes 1)$$

is bounded and, by Proposition I.1.1,

$$((F_D \otimes_{\delta_B} 1)V_E^*(a_1 \otimes s_1) - V_E^*(a_1 \otimes s_1)(F_{\mu D\mu^*} \otimes 1))\mu(\langle D\rangle^\beta \otimes 1)$$

is too. Now we have

$$\begin{aligned}
&(V_E(F_D \otimes_{\delta_B} 1)V_E^* - F_D \otimes 1)a_1 a_2^* a_3 a_4^* \otimes s_1 s_2^* s_3 s_4^* \\
&\quad = V_E((F_D \otimes_{\delta_B} 1)V_E^* - V_E^*(F_D \otimes 1))a_1 a_2^* a_3 a_4^* \otimes s_1 s_2^* s_3 s_4^* \\
&\quad = V_E\left((F_D \otimes_{\delta_B} 1)V_E^*(a_1 a_2^* a_3 \otimes s_1 s_2^* s_3) - V_E^*(a_1 a_2^* a_3 \otimes s_1 s_2^* s_3)(F_D \otimes 1)\right)(a_4^* \otimes s_4^*) \\
&\qquad - [F_D, a_1 a_2^* a_3]a_4^* \otimes s_1 s_2^* s_3 s_4^* \\
&\quad = V_E\left((F_D \otimes_{\delta_B} 1)V_E^*(a_1 \otimes s_1) - V_E^*(a_1 \otimes s_1)F_{\mu(D\otimes 1)\mu^*}\right)(a_2^* a_3 a_4^* \otimes s_2^* s_3 s_4^*) \\
&\qquad + (a_1 \otimes s_1)\left(F_{\mu(D\otimes 1)\mu^*}(a_2^* a_3 \otimes s_2^* s_3) - (a_2^* a_3 \otimes s_2^* s_3)(F_D \otimes 1)\right)(a_4^* \otimes s_4^*) \\
&\qquad - [F_D, a_1 a_2^* a_3]a_4^* \otimes s_1 s_2^* s_3 s_4^* \\
&\quad = V_E\left((F_D \otimes_{\delta_B} 1)V_E^*(a_1 \otimes s_1) - V_E^*(a_1 \otimes s_1)F_{\mu(D\otimes 1)\mu^*}\right)(a_2^* a_3 a_4^* \otimes s_2^* s_3 s_4^*) \\
&\qquad + (a_1 \otimes s_1)\left(F_{\mu(D\otimes 1)\mu^*} - F_D \otimes 1\right)(a_2^* a_3 a_4^* \otimes s_2^* s_3 s_4^*) \\
&\qquad - [F_D, a_1]a_2^* a_3 a_4^* \otimes s_1 s_2^* s_3 s_4^*
\end{aligned}$$

so that $(V_E(F_D \otimes_{\delta_B} 1)V_E^* - F_D \otimes 1)a_1 a_2^* a_3 a_4^* \langle D\rangle^\beta \otimes s_1 s_2^* s_3 s_4^*$ is bounded. Let $a_5 \in A$ and note that $c \in \overline{\mathrm{span}}(ScS) \trianglelefteq S$. Then

$$(V_E(F_D \otimes_{\delta_B} 1)V_E^* - F_D \otimes 1)a_1 a_2^* a_3 a_4^* a_5 \otimes s_1 s_2^* s_3 s_4^* c \tag{III.3.4}$$

is an element of $\mathrm{End}^0(E) \otimes \overline{\mathrm{span}}(ScS)$. As the compacts on $E \otimes \overline{\mathrm{span}}(ScS)_{B\otimes\overline{\mathrm{span}}(ScS)}$ are

$$\overline{\mathrm{span}}(EE^* \otimes ScSScS) = \mathrm{End}^0(E) \otimes \overline{\mathrm{span}}(ScS) \trianglelefteq \mathrm{End}^0(E) \otimes S = \mathrm{End}^0(E \otimes S)$$

for each $c \in K_S$, we see that (III.3.4) defines a compact endomorphism on $E \otimes S$. Because $S \subseteq \overline{\mathcal{S}_{a_1}\mathcal{S}_{a_2}^*\mathcal{S}_{a_3}\mathcal{S}_{a_4}^* K_S}$ and $A \subseteq \overline{\mathcal{Q}^*\mathcal{Q}\mathcal{Q}^*\mathcal{Q}A}$,

$$(V_E(F_D \otimes_{\delta_B} 1)V_E^* - F_D \otimes 1)a \otimes s$$

is compact for all $a \in A$ and $s \in S$. □

Theorem III.3.5. *A conformally S-equivariant order-$\frac{1}{1-\alpha}$ cycle (A, E_B, D) gives rise to a uniformly S-equivariant order-$\frac{1}{1-\alpha}$ cycle (A, E_B, L_D) via the logarithmic transform.*

Proof. By the Proof of Theorem III.3.3, $(V_E(F_D \otimes_{\delta_B} 1)V_E^* - F_D \otimes 1)a_1 a_2^* a_3 a_4^* \langle D\rangle^\beta \otimes s_1 s_2^* s_3 s_4^* c$ is bounded on $E \otimes S$ for $a_1, a_2, a_3, a_4 \in \mathcal{Q}$, $s_1, s_2, s_3, s_4 \in \mathcal{S}_{a_1}, \mathcal{S}_{a_2}, \mathcal{S}_{a_3}, \mathcal{S}_{a_4}$, $c \in K_S$, and $\beta < 1-\alpha$. Then

$$\left[\begin{pmatrix} V_E(F_D \otimes_{\delta_B} 1)V_E^* & \\ & F_D \otimes 1 \end{pmatrix}, \begin{pmatrix} & a_1 a_2^* a_3 a_4^* \otimes s_1 s_2^* s_3 s_4^* c \\ 0 & \end{pmatrix}\right] \left\langle \begin{pmatrix} V_E(F_D \otimes_{\delta_B} 1)V_E^* & \\ & F_D \otimes 1 \end{pmatrix} \right\rangle^\beta$$

is bounded and

$$\begin{pmatrix} & a_1a_2^*a_3a_4^* \otimes s_1s_2^*s_3s_4^*c \\ 0 & \end{pmatrix} \operatorname{dom} \begin{pmatrix} V_E(D \otimes_{\delta_B} 1)V_E^* & \\ & D \otimes 1 \end{pmatrix} \subseteq \operatorname{dom} \begin{pmatrix} V_E(D \otimes_{\delta_B} 1)V_E^* & \\ & D \otimes 1 \end{pmatrix}.$$

Applying Proposition III.1.35,

$$\left[\begin{pmatrix} V_E(L_D \otimes_{\delta_B} 1)V_E^* & \\ & L_D \otimes 1 \end{pmatrix}, \begin{pmatrix} & a_1a_2^*a_3a_4^* \otimes s_1s_2^*s_3s_4^*c \\ 0 & \end{pmatrix} \right]$$

is bounded and therefore so is $(V_E(L_D \otimes_{\delta_B} 1)V_E^* - L_D \otimes 1)a_1a_2^*a_3a_4^* \otimes s_1s_2^*s_3s_4^*c$. For any $a_5 \in A$,

$$(V_E(L_D \otimes_{\delta_B} 1)V_E^* - L_D \otimes 1)a_1a_2^*a_3a_4^*a_5 \otimes s_1s_2^*s_3s_4^*c$$

is bounded. We have $S \subseteq \overline{\mathcal{S}_{a_1}\mathcal{S}_{a_2}^*\mathcal{S}_{a_3}\mathcal{S}_{a_4}^*K_S}$ and $A \subseteq \overline{\mathcal{Q}^*\mathcal{Q}\mathcal{Q}^*\mathcal{Q}A}$, as required. □

Proposition III.3.6. *Let G be a locally compact group. An order-$\frac{1}{1-\alpha}$ cycle is conformally $C_0(G)$-equivariant if and only if it is conformally G-equivariant.*

Proof. Use Proposition A.1.24. Because $C_0(G)$ is abelian, for $a \in \mathcal{Q}$, $\mathcal{S}_a$ will always contain the Pedersen ideal $K_{C_0(G)} = C_c(G)$. □

III.3.1 The Podleś sphere

The compact quantum group $SU_q(2)$ has polynomial algebra $\mathcal{O}(SU_q(2))$ generated by a, b, c, d subject to the relations

$$ab = qba \qquad ac = qca \qquad bd = qdb \qquad cd = qdc \qquad bc = cb \qquad ad = 1 + qbc \qquad da = 1 + q^{-1}bc$$

and with adjoints $a^* = d$, $b^* = -qc$, $c^* = -q^{-1}b$, $d^* = a$. The polynomial algebra $\mathcal{O}(SU_q(2))$ is spanned by the Peter–Weyl elements t^l_{ij} with $l \in \frac{1}{2}\mathbb{N}$ and $i, j \in \{-l, -l+1, \dots, l-1, l\}$. The generators form the fundamental representation $l = \frac{1}{2}$, that is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} t^{\frac12}_{-\frac12,-\frac12} & t^{\frac12}_{-\frac12,\frac12} \\ t^{\frac12}_{\frac12,-\frac12} & t^{\frac12}_{\frac12,\frac12} \end{pmatrix}.$$

In terms of this basis, the coproduct and counit are

$$\Delta(t^l_{i,j}) = \sum_k t^l_{i,k} \otimes t^l_{k,j} \qquad \varepsilon(t^l_{i,j}) = \delta_{i,j}$$

and the adjoint is related to the antipode by ${t^l_{i,j}}^* = S(t^l_{j,i})$.

Dual to $SU_q(2)$ is the discrete quantum group $\widehat{SU_q(2)}$ [VY20, §4.2.3], whose function algebra $C_0(\widehat{SU_q(2)}) = C^*(SU_q(2))$ is the closed span of matrix elements τ^l_{ij} with $l \in \frac{1}{2}\mathbb{N}$ and $i, j \in \{-l, -l+1, \dots, l-1, l\}$, subject to

$$\tau^l_{i,j}\tau^{l'}_{i',j'} = \delta_{l,l'}\delta_{j,i'}\tau^l_{i,j'} \qquad {\tau^l_{i,j}}^* = \tau^l_{j,i}\,.$$

In particular, as C*-algebras,

$$C_0(\widehat{SU_q(2)}) = C^*(SU_q(2)) \cong \bigoplus_{l \in \frac12\mathbb{N}} M_{2l}(\mathbb{C})\,.$$

We may choose τ^l_{ij} so that the pairing between $C^*(SU_q(2))$ and $C(SU_q(2))$ is given by

$$(\tau^l_{ij}, t^{l'}_{i'j'}) = \delta_{l,l'}\delta_{i,i'}\delta_{j,j'}$$

and the multiplicative unitary $W \in M(C(SU_q(2)) \otimes C^*(SU_q(2)))$ is $W = \sum_{l,i,j} t^l_{i,j} \otimes \tau^l_{i,j}$.

The quantum universal enveloping algebra $\breve{U}_q(\mathfrak{sl}(2))$ is generated by K, K^{-1}, E, F subject to

$$KK^{-1} = K^{-1}K = 1 \qquad KEK^{-1} = qE \qquad KFK^{-1} = q^{-1}F \qquad [E,F] = \frac{K^2 - K^{-2}}{q - q^{-1}}$$

with coproduct

$$\Delta(K) = K \otimes K \qquad \Delta(E) = E \otimes K + K^{-1} \otimes E \qquad \Delta(F) = F \otimes K + K^{-1} \otimes F$$

and counit and antipode

$$\varepsilon(K) = 1 \qquad \varepsilon(E) = \varepsilon(F) = 0 \qquad S(K) = K^{-1} \qquad S(E) = -qE \qquad S(F) = -q^{-1}F.$$

Note that this is not the same as $U_q(\mathfrak{sl}(2))$, although the latter is a Hopf subalgebra of $\breve{U}_q(\mathfrak{sl}(2))$ [KS97, §3.1.2]. There is a nondegenerate pairing $(\cdot,\cdot)$ between $\breve{U}_q(\mathfrak{sl}(2))$ and $\mathcal{O}(SU_q(2))$ [KS97, Theorem 4.21]. By this pairing, $\breve{U}_q(\mathfrak{sl}(2))$ is an algebra of unbounded operators affiliated to $C^*(SU_q(2))$. We may define left and right actions of $\breve{U}_q(\mathfrak{sl}(2))$ on $\mathcal{O}(SU_q(2))$ by

$$X \rightharpoonup \alpha = \alpha_{(1)}(X, \alpha_{(2)}) \qquad \alpha \leftharpoonup X = (X, \alpha_{(1)})\alpha_{(2)}.$$

The left and right actions of K are automorphisms of $\mathcal{O}(SU_q(2))$ and have the properties

$$(K \rightharpoonup \alpha)^* = K^{-1} \rightharpoonup \alpha^* \qquad (\alpha \leftharpoonup K)^* = \alpha^* \leftharpoonup K^{-1}.$$

In terms of the Peter–Weyl basis, $K \rightharpoonup t^l_{i,j} = q^j t^l_{i,j}$ and $t^l_{i,j} \leftharpoonup K = q^i t^l_{i,j}$. We also record the relationships $S^{-1}(\alpha) = K^2 \rightharpoonup S(\alpha) \leftharpoonup K^{-2}$ and $\phi(\alpha\beta) = \phi(\beta(K^2 \rightharpoonup \alpha \leftharpoonup K^2))$ for the left Haar state ϕ on $C(SU_q(2))$. The unitary antipode R on $C(SU_q(2))$ is then given by $R(\alpha) = K \rightharpoonup S(\alpha) \leftharpoonup K^{-1}$; on the Peter–Weyl basis, $R(t^l_{ij}) = K \rightharpoonup {t^l_{ji}}^* \leftharpoonup K^{-1} = (K^{-1} \rightharpoonup t^l_{ji} \leftharpoonup K)^* = q^{-i+j}{t^l_{ji}}^*$.

The Podleś sphere $\mathbf{S}^2_q$ has polynomial algebra $\mathcal{O}(\mathbf{S}^2_q)$, the subalgebra of $\mathcal{O}(SU_q(2))$ generated by

$$\begin{aligned}
A &= -q^{-1}bc = c^*c = t^{1/2*}_{1/2,-1/2}t^{1/2}_{1/2,-1/2} = q^{-2}t^{1/2}_{-1/2,1/2}t^{1/2*}_{-1/2,1/2} = -q^{-1}[2]_q^{-1}t^1_{00} \\
B &= ac^* = -q^{-1}ab = t^{1/2}_{-1/2,-1/2}t^{1/2*}_{1/2,-1/2} = -q^{-1/2}[2]_q^{-1/2}t^1_{-10} \\
B^* &= cd = t^{1/2}_{1/2,-1/2}t^{1/2}_{1/2,1/2} = q^{-1/2}t^{1/2}_{1/2,-1/2}t^{1/2*}_{-1/2,-1/2} = [2]_q^{-1/2}q^{1/2}t^1_{10}.
\end{aligned}$$

and is spanned by t^l_{i0}. The subspaces $S_+ = \mathrm{span}\{t^l_{i,\frac{1}{2}} \mid l, i\}$ and $S_- = \mathrm{span}\{t^l_{i,-\frac{1}{2}} \mid l, i\}$ of $\mathcal{O}(SU_q(2))$ are the spinor bundles of the Podleś sphere. They can be completed under the inner product on $\mathcal{O}(SU_q(2))$ given by the left Haar state. The natural Dirac operator defining a spectral triple $(C(S^2_q), L^2(S_+ \oplus S_-), D)$ is [DS03, Theorem 8]

$$D = \begin{pmatrix} & \partial_E \\ \partial_F & \end{pmatrix}$$

where $\partial_E = E \rightharpoonup$ and $\partial_F = F \rightharpoonup$ or, in terms of the Peter–Weyl basis,

$$\partial_E t^l_{i,j} = \sqrt{[l+1/2]_q^2 - [j+1/2]_q^2}\, t^l_{i,j+1} \qquad \partial_F t^l_{i,j} = \sqrt{[l+1/2]_q^2 - [j-1/2]_q^2}\, t^l_{i,j-1}\,.$$

(Here, we use the convention $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$ for q-numbers.) We abbreviate these coefficients as $\kappa^l_k = \sqrt{[l+1/2]_q^2 - [k-1/2]_q^2}$. We have the twisted derivation property

$$\partial_E(\alpha\beta) = \partial_E(\alpha)(K \rightharpoonup \beta) + (K^{-1} \rightharpoonup \alpha)\partial_E(\beta) \qquad \partial_F(\alpha\beta) = \partial_F(\alpha)(K \rightharpoonup \beta) + (K^{-1} \rightharpoonup \alpha)\partial_F(\beta)$$

which shows that D has bounded commutators with elements of $\mathcal{O}(\mathbf{S}^2_q)$. The relationships

$$\partial_E(\alpha^*) = -q\partial_F(\alpha)^* \qquad \partial_F(\alpha^*) = -q^{-1}\partial_E(\alpha)^*$$

can be used to show that D is self-adjoint [Sen11, Lemma A.1].

There is an action of $SU_q(2)$ on $\mathbf{S}^2_q$ given by the restriction of the coaction of $C(SU_q(2))$ on itself to $C(\mathbf{S}^2_q)$. The spectral triple $(\mathcal{O}(S^2_q), L^2(S_+ \oplus S_-), D)$ is constructed to be isometric with respect to this action, cf. [DS03, §4]. We can phrase this in terms of a right coaction

$$\delta_\Delta : \alpha \mapsto \Sigma(R \otimes 1)\Delta\alpha \qquad\qquad t^l_{ij} \mapsto \sum_k t^l_{kj} \otimes q^{-i+k} {t^l_{ki}}^*$$

of $C(SU_q(2))$ on $C(\mathbf{S}^2_q)$, where R is the unitary antipode. We can write the admissible unitary as

$$V_\Delta(t^l_{ij} \otimes t^{l'}_{i'j'}) = \sum_k t^l_{kj} \otimes q^{-i+k} {t^l_{ki}}^* t^{l'}_{i'j'} .$$

We then have

$$\begin{aligned}(\partial_E \otimes 1)V_\Delta(t^l_{ij} \otimes t^{l'}_{i'j'}) &= \sum_k \kappa^l_{j+1} t^l_{k,j+1} \otimes q^{-i+k} {t^l_{ki}}^* t^{l'}_{i'j'} \\ &= \kappa^l_{j+1} V_\Delta(t^l_{i,j+1} \otimes t^{l'}_{i'j'}) = V_\Delta(\partial_E \otimes 1)(t^l_{ij} \otimes t^{l'}_{i'j'})\end{aligned}$$

and, similarly, that $\partial_F \otimes 1$ commutes with V_Δ, which means that $(C(S^2_q), L^2(S_+ \oplus S_-), D)$ is isometrically equivariant for the action of $SU_q(2)$.

In addition, there is an action of $\widehat{SU_q(2)}$ on $\mathbf{S}^2_q$ given by the restriction of the adjoint action of $C(SU_q(2))$ on itself to $C(\mathbf{S}^2_q)$ [Voi11, §4]. Together, these actions give an action of $SL_q(2) = SU_q(2) \bowtie \widehat{SU_q(2)}$, the Drinfeld double of $SU_q(2)$, which can be thought of as the quantisation of the classical Lorentz group $SL(2, \mathbb{C})$ action on the sphere $\mathbf{S}^2$. The left adjoint action of $C(SU_q(2))$ is given by

$$\mathrm{ad}(\alpha) : \beta \to \alpha_{(1)}\beta S(\alpha_{(2)}) .$$

For $z \in \mathbb{C}$, we define a slightly adjusted action

$$\omega_z(\alpha) : \beta \to \alpha_{(1)}\beta(K^{2z} \rightharpoonup S(\alpha_{(2)})) .$$

For any $\alpha \in C(SU_q(2))$, $\omega_z(\alpha)$ preserves the subalgebra $C(\mathbf{S}^2_q)$ and its spinor bundles. In terms of the Peter–Weyl basis,

$$\omega_z(t^l_{i,j})(\beta) = \sum_k q^{-2zk} t^l_{i,k} \beta {t^l_{j,k}}^* \quad \text{and} \quad \omega_z({t^l_{i,j}}^*)(\beta) = \sum_k q^{2((z-1)k+j)} {t^l_{i,k}}^* \beta t^l_{j,k} .$$

With respect to the inner product on $C(SU_q(2))$ given by the Haar state ϕ, ω_1 is self-adjoint; in general,

$$\langle \omega_z(\alpha)(\beta) \mid \gamma \rangle = \langle \beta \mid \omega_{-z+2}(\alpha^*)(\gamma) \rangle.$$

From the left action ω_1 of $C(SU_q(2))$ on itself, we obtain a right coaction of $C^*(SU_q(2))$ on $C(SU_q(2))$ by the formula

$$\beta_{(0)}(\beta_{(1)}, \alpha) = \omega_1(\alpha)(\beta),$$

using the Sweedler notation $\delta_{\omega_1}(\beta) = \beta_{(0)} \otimes \beta_{(1)}$ for the coaction. In particular, we obtain that

$$\delta_{\omega_1}(t^l_{i,j}) = \sum_{l',i',j'} \omega_1(t^{l'}_{i',j'})(t^l_{i,j}) \otimes \tau^{l'}_{i',j'} .$$

The admissible unitary V_{ω_1} on $L^2(S_+ \oplus S_-) \otimes C^*(SU_q(2))$ is given by

$$V_{\omega_1} = \sum_{l,i,j} \omega_1(t^l_{i,j}) \otimes \tau^l_{i,j} = \sum_k q^{-2k} t^l_{i,k} \cdot {t^l_{j,k}}^* \otimes \tau^l_{i,j} = \sum_{k,k'} q^{-2k'} t^l_{i,k} \cdot {t^l_{j,k'}}^* \otimes \tau^l_{i,k} \tau^l_{k',j}\,.$$

We claim that the spectral triple $(C(S^2_q), L^2(S_+ \oplus S_-), D)$ is conformally $\widehat{SU_q(2)}$-equivariant. The conformal geometry of the Podleś sphere is examined at the level of bounded KK-theory in [NV10, Voi11]. Because $\widehat{SU_q(2)}$ is discrete, the conformal factor μ will be the sum of components $\mu^l \in B(L^2(S_+ \oplus S_-)) \otimes M_{2l}(\mathbb{C})$, $l \in \frac{1}{2}\mathbb{N}_{\geq 1}$ labelling the irreducible representations of $SU_q(2)$. Noting that $C(\mathbf{S}^2_q)$ is unital, conformal equivariance will be a consequence of

$$V^l_{\omega_1}(D \otimes 1) V^{l*}_{\omega_1} - \mu^l (D \otimes 1) \mu^{l*} \qquad [D \otimes 1, \mu^l]$$

being bounded for all $l \in \frac{1}{2}\mathbb{N}_{\geq 1}$.

Note that $(K \otimes K) \rightharpoonup (1 \otimes S)\Delta(\alpha) = (1 \otimes S)\Delta(\alpha)$ because

$$\begin{aligned}
(K \otimes K) \rightharpoonup (1 \otimes S)\Delta(t^l_{i,j}) &= \sum_k K \rightharpoonup t^l_{i,k} \otimes K \rightharpoonup S(t^l_{k,j}) \\
&= \sum_k q^k t^l_{i,k} \otimes K \rightharpoonup {t^l_{k,j}}^* \\
&= \sum_k q^k t^l_{i,k} \otimes (K^{-1} \rightharpoonup t^l_{k,j})^* \\
&= \sum_k t^l_{i,k} \otimes {t^l_{k,j}}^* \\
&= (1 \otimes S)\Delta(t^l_{i,j})\,.
\end{aligned}$$

Then

$$\begin{aligned}
\partial_E(\omega_z(\alpha)(\beta)) &= \partial_E(\alpha_{(1)}\beta(K^{2z} \rightharpoonup S(\alpha_{(2)}))) \\
&= \partial_E(\alpha_{(1)})(K \rightharpoonup \beta)(K^{2z+1} \rightharpoonup S(\alpha_{(2)})) + (K^{-1} \rightharpoonup \alpha_{(1)})\partial_E(\beta)(K^{2z+1} \rightharpoonup S(\alpha_{(2)})) \\
&\quad + (K^{-1} \rightharpoonup \alpha_{(1)}\beta)\partial_E(K^{2z} \rightharpoonup S(\alpha_{(2)})) \\
&= \partial_E(\alpha_{(1)})(K \rightharpoonup \beta)(K^{2z+1} \rightharpoonup S(\alpha_{(2)})) + \omega_{z+1}(\alpha)(\partial_E(\beta)) \\
&\quad + (K^{-1} \rightharpoonup \alpha_{(1)}\beta)\partial_E(K^{2z} \rightharpoonup S(\alpha_{(2)}))
\end{aligned}$$

so that $\partial_E \omega_z(\alpha) - \omega_{z+1}(\alpha)\partial_E$ and $\partial_F \omega_z(\alpha) - \omega_{z+1}(\alpha)\partial_F$, similarly, are bounded on $S_+ \oplus S_-$. Furthermore,

$$\begin{aligned}
\sum_j \omega_0(t^l_{i,j})(\omega_1({t^l_{i',j}}^*)(\beta)) &= \sum_{j,k} t^l_{i,k}\omega_1({t^l_{i',j}}^*)(\beta){t^l_{j,k}}^* \\
&= \sum_{j,k,k'} q^{2j} t^l_{i,k}{t^l_{i',k'}}^* \beta t^l_{j,k'}{t^l_{j,k}}^* \\
&= \sum_{j,k,k'} q^{2k'} t^l_{i,k}{t^l_{i',k'}}^* \beta(K^{-2} \rightharpoonup t^l_{j,k'} \leftharpoonup K^2) S(t^l_{k,j}) \\
&= \sum_{j,k,k'} q^{2k'} t^l_{i,k}{t^l_{i',k'}}^* \beta(K^{-2} \rightharpoonup (t^l_{j,k'}(K^2 \rightharpoonup S(t^l_{k,j}) \leftharpoonup K^{-2})) \leftharpoonup K^2) \\
&= \sum_{j,k,k'} q^{2k'} t^l_{i,k}{t^l_{i',k'}}^* \beta(K^{-2} \rightharpoonup (t^l_{j,k'}S^{-1}(t^l_{k,j})) \leftharpoonup K^2) \\
&= \sum_k q^{2k} t^l_{i,k}{t^l_{i',k}}^* \beta(K^{-2} \rightharpoonup 1 \leftharpoonup K^2) \\
&= \sum_k q^{2k} t^l_{i,k}{t^l_{i',k}}^* \beta \\
&= \omega_{-1}(t^l_{i,i'})(1)\beta\,.
\end{aligned}$$

Let $\mu^l = \sum_{i,j} \omega_{-1/2}(t^l_{i,j})(1) \otimes \tau^l_{i,j} = \sum_{i,j,k} q^k t^l_{i,k} {t^l_{j,k}}^* \otimes \tau^l_{i,j}$. For $l = \frac{1}{2}$,

$$\mu^{\frac{1}{2}} = q^{\frac{1}{2}} T^{\frac{1}{2}}_{\frac{1}{2}} {T^{\frac{1}{2}}_{\frac{1}{2}}}^* + q^{-\frac{1}{2}} T^{\frac{1}{2}}_{-\frac{1}{2}} {T^{\frac{1}{2}}_{-\frac{1}{2}}}^* = q^{\frac{1}{2}} \begin{pmatrix} q^2 A & -B \\ -B^* & 1-A \end{pmatrix} + q^{-\frac{1}{2}} \begin{pmatrix} 1-q^2 A & B \\ B^* & A \end{pmatrix}.$$

Thus, with $P_\pm$ the projections onto the positive and negative spinors, $\mu^{1/2} = q^{1/2} P_+ + q^{-1/2} P_-$. If we regard K as an unbounded operator on $C^*(SU_q(2))$ the conformal factor is

$$\mu = W(1 \otimes K)W^*$$

where W is the multiplicative unitary of $SU_q(2)$.

We remark that μ^l is positive and $(\mu^l)^z = \sum_{i,j} \omega_{-z/2}(t^l_{i,j})(1) \otimes \tau^l_{i,j}$. Because $\mu^l \in \mathcal{O}(\mathbf{S}^2_q) \otimes M_{2l}(\mathbb{C})$, it is clear that $[D \otimes 1, \mu^l]$ is bounded. We are now in a position to see also that

$$\begin{aligned} &V^l_{\omega_1}(D \otimes 1)V^{l*}_{\omega_1} - \mu^l (D \otimes 1)\mu^{l*} \\ &\quad= \sum_{l,i,j,i',j'} \Big(\omega_1(t^l_{i,j}) D \omega_1({t^l_{i',j'}}^*) - \omega_{-1/2}(t^l_{i,j})(1) D \omega_{-1/2}(t^l_{i',j'})(1)\Big) \otimes \tau^l_{i,j}\tau^l_{j',i'} \\ &\quad= \sum_{l,i,j,i'} \Big(\omega_1(t^l_{i,j}) D \omega_1({t^l_{i',j}}^*) - \omega_{-1/2}(t^l_{i,j})(1) D \omega_{-1/2}(t^l_{i',j})(1)\Big) \otimes \tau^l_{i,i'} \\ &\quad= \sum_{l,i,j,i'} \Big(-(D\omega_0(t^l_{i,j}) - \omega_1(t^l_{i,j})D)\omega_1({t^l_{i',j}}^*) + [D, \omega_{-1/2}(t^l_{i,j})(1)]\omega_{-1/2}(t^l_{i',j})(1)\Big) \otimes \tau^l_{i,i'} \end{aligned}$$

is bounded. Finally, we obtain that $(\mathcal{O}(S^2_q), L^2(S_+ \oplus S_-), D)$ is conformally $\widehat{SU_q(2)}$-equivariant with conformal factor μ.

The locally compact quantum group $SL_q(2)$, the quantum deformation of $SL(2,\mathbb{C})$, is the Drinfeld double $SU_q(2) \bowtie \widehat{SU_q(2)}$; see e.g. [VY20, §4.4.1]. As C*-algebras,

$$C(SL_q(2)) = C(SU_q(2)) \otimes C^*(SU_q(2)).$$

The comultiplication on $C(SL_q(2))$ is

$$\Delta_{SL_q(2)} = (1 \otimes \Sigma \otimes 1)(\mathrm{id} \otimes \mathrm{ad}(W) \otimes \mathrm{id}) \circ (\Delta \otimes \hat{\Delta})$$

and the antipode is

$$S_{SL_q(2)} = \mathrm{ad}(W^*) \circ (S \otimes \hat{S}) = (S \otimes \hat{S}) \circ \mathrm{ad}(W).$$

By [BV05, Theorem 5.3] the unitary antipode is similarly

$$R_{SL_q(2)} = \mathrm{ad}(W^*) \circ (R \otimes \hat{R}) = (R \otimes \hat{R}) \circ \mathrm{ad}(W).$$

Our conventions differ from those of [NV10] in that we use right coactions rather than left ones. The translation between these is not difficult: a left coaction can be turned into a right coaction, and vice versa, by applying the unitary antipode to the C*-bialgebra leg and then flipping the legs. Taking this into account in [NV10, Proposition 3.2] the action of $SL_q(2)$ on S^2_q is given by the coaction

$$\begin{aligned} \delta_{\bowtie} &= (\Sigma \otimes 1)(1 \otimes \Sigma)(\mathrm{ad}(W^*) \otimes \mathrm{id})(R \otimes \hat{R} \otimes \mathrm{id})(1 \otimes \Sigma)(\mathrm{id} \otimes \mathrm{id} \otimes \hat{R})(\mathrm{id} \otimes \delta_{\omega_1})\Sigma(\mathrm{id} \otimes R)\delta_\Delta \\ &= (\Sigma \otimes 1)(1 \otimes \Sigma)(\mathrm{ad}(W^*) \otimes \mathrm{id})(1 \otimes \Sigma)(\Sigma \otimes 1)(1 \otimes \Sigma)(\delta_{\omega_1} \otimes \mathrm{id})\delta_\Delta \\ &= (\mathrm{id} \otimes \mathrm{ad}(W^*))(1 \otimes \Sigma)(\delta_{\omega_1} \otimes \mathrm{id})\delta_\Delta \end{aligned}$$

of $C(SL_q(2))$. Using the standard leg-numbering notation the admissible unitary is

$$V_{\bowtie} = (1 \otimes W^*)V_{\omega_1,13}(V_\Delta \otimes 1)(1 \otimes W).$$

Let $\mu_{\bowtie} = (1 \otimes W^*)\mu_{13}(1 \otimes W)$. Then

$$V_{\bowtie}(D \otimes 1)V_{\bowtie}^* - \mu_{\bowtie}(D \otimes 1)\mu_{\bowtie}^* = (1 \otimes W^*)\left(V_{\omega_1,13}(D \otimes 1 \otimes 1)V_{\omega_1,13}^* - \mu_{13}(D \otimes 1 \otimes 1)\mu_{13}^*\right)(1 \otimes W)$$

is $C(SU_q(2)) \otimes C^*(SU_q(2))$-matched because it is bounded when restricted to each of the submodules $L^2(S_+ \oplus S_-) \otimes C(SU_q(2)) \otimes M_{2l}(\mathbb{C})$. In terms of the Peter–Weyl basis,

$$\begin{aligned}\mu_{\bowtie} &= \sum_{i,j,k,l,i',j',i'',j''} q^k t^l_{i,k} {t^l_{j,k}}^* \otimes t^l_{i'',j''}{}^* t^l_{i',j'} \otimes \tau^l_{j'',i''} \tau^l_{i,j} \tau^l_{i',j'} \\ &= \sum_{i,j,k,l,m,n} q^k t^l_{i,k} {t^l_{j,k}}^* \otimes {t^l_{i,m}}^* t^l_{j,n} \otimes \tau^l_{m,n}.\end{aligned}$$

This shows that the first leg of $\mu_{\bowtie}$ is in $\mathcal{O}(\mathbf{S}^2_q)$ so that $[D \otimes 1 \otimes 1, \mu_{\bowtie}]$ is similarly $C(SL_q(2))$-matched. Regarding K as an unbounded operator on $C^*(SU_q(2))$, the conformal factor is

$$\mu_{\bowtie} = W_{23}^* W_{13}(1 \otimes 1 \otimes K)W_{13}^* W_{23}.$$

We have now demonstrated

Proposition III.3.7. *The spectral triple $(C(S^2_q), L^2(S_+ \oplus S_-), D)$ is conformally $SL_q(2)$-equivariant with conformal factor $\mu_{\bowtie}$.*

Remark III.3.8. As a consequence, applying Theorem III.3.5, the logarithmically dampened spectral triple $(C(S^2_q), L^2(S_+ \oplus S_-), L_D)$ is uniformly $SL_q(2)$-equivariant. Recalling the expressions for ∂_E and ∂_F in terms of the Peter–Weyl basis,

$$D\begin{pmatrix} t^l_{i,\frac{1}{2}} \\ t^{l'}_{i',-\frac{1}{2}} \end{pmatrix} = [l + 1/2]_q \begin{pmatrix} t^{l'}_{i',\frac{1}{2}} \\ t^l_{i,-\frac{1}{2}} \end{pmatrix}.$$

One can check that

$$\frac{[l+\frac{1}{2}]_q}{\sqrt{1+[l+\frac{1}{2}]_q^2}} \log\sqrt{1+[l+\tfrac{1}{2}]_q^2} - (l+\tfrac{1}{2})\log q^{-1}$$

converges to $\log(q^{-1} - q)$ as $l \to \infty$. Hence, up to a bounded difference, L_D is equal to $\log(q^{-1})D_1$, where D_1 is the Dirac operator on the classical 2-sphere; cf. [DDLW07].

III.4 Conformally generated cycles and twisted spectral triples

In this section, we present a new way of guaranteeing that unbounded cycles without bounded commutators in the conventional sense have well-defined bounded transforms. In particular, our approach covers all known examples of twisted spectral triples with well-defined bounded transforms. One of the features of our approach is that no ‘twist’ or automorphism of the algebra is involved, which suggests that this structure is a red herring, at least as far as KK-theory is concerned.

So far, relatively few examples of twisted spectral triples have been described in the literature. One reason for this is the difficulty in guaranteeing that the bounded transform is well-defined. The *Lipschitz regularity* condition [CM08, Definition 3.1 (3.3)], although natural in a relatively classical situation, where a pseudodifferential calculus is available, is not so satisfactory in general. Part of the motivation for developing the technical results in this Chapter was the construction of twisted spectral triples for certain badly behaved dynamical systems, for which Lipschitz regularity becomes intractable.

The framework of conformally generated cycles is applicable to all examples of twisted spectral triples with topological content in the literature, as far as we are aware. Among those examples to which it can be applied are

- Conformal perturbations of spectral triples (or Kasparov modules) of the $D \rightsquigarrow kDk$ type [CM08, §2.2];
- Crossed products by groups of conformal diffeomorphisms [CM08, §2.3] [Mos10, §3.1] (and, more generally, the dual Green–Julg map of conformally equivariant unbounded Kasparov modules);
- Cuntz–Krieger algebras, as in [Haw13, Chapter 6];
- Unbounded modular cycles, in the sense of [Kaa21, Definition 3.1]; and
- Pseudodifferential calculus on the Podleś sphere and other examples with *diagonalisable twist*, as treated in [MY19].

The multiplicative perturbation theory developed in §III.1.3 was partly inspired by [MY19]. In principle, the techniques here could be used to build pseudodifferential calculi, mimicking the approach in [MY19]. Examples of twisted spectral triples to which our methods do not apply are

- The quantum statistical mechanics constructions of [GMT14] which are not Lipschitz regular and, indeed, whose bounded transform is manifestly not a Fredholm module;
- The Lorentzian geometry constructions of [DFLM18], whose twist is an involution and not relevant to the topology; and
- Examples without (locally) compact resolvent, such as those in [KS12] and [IM16].

To formulate a framework sufficient to describe the examples, we will again use the notions of matched operators and compactly supported states from Appendices A.1.2 and A.1.3. Recall from Proposition A.1.22 the $*$-algebra of matched operators $\mathrm{Mtc}^*(F,C)$ on the module F with respect to the algebra C.

Definition III.4.1. A *conformally generated A-B-cycle* $(A, E_B, D; C, \mu)$ is an A-B-correspondence E, a regular operator D on E, a C*-algebra C, and a pair $\mu = (\mu_L, \mu_R)$ of (even) C-matched operators on $E \otimes C$, whose inverses are also C-matched, such that

1. D is self-adjoint;
2. $(1 + D^2)^{-1}a$ is compact for all $a \in A$; and
3. With $\mathscr{L}$ the set of $a \in \mathrm{Mtc}^*(E \otimes C, C)$ such that

$$[D \otimes 1, a] \quad [\mu_L(D \otimes 1)\mu_L^*, a] \quad [D \otimes 1, \mu_L^* a] \quad [D \otimes 1, \mu_L^{-1} a] \quad [D \otimes 1, a\mu_L] \quad [D \otimes 1, a\mu_L^{-1*}]$$

are C-matched, with $\mathscr{R}$ the set of $a \in \mathrm{Mtc}^*(E \otimes C, C)$ such that

$$[D \otimes 1, a] \quad [\mu_R(D \otimes 1)\mu_R^*, a] \quad [D \otimes 1, \mu_R^* a] \quad [D \otimes 1, \mu_R^{-1} a] \quad [D \otimes 1, a\mu_R] \quad [D \otimes 1, a\mu_R^{-1*}]$$

are C-matched, and with

$$\mathscr{T} = \left\{ a \in \mathrm{Mtc}^*(E \otimes C, C) \,\middle|\, \mu_L(D \otimes 1)\mu_L^* a - a\mu_R(D \otimes 1)\mu_R^* \in \mathrm{Mtc}^*(E \otimes C, C) \right\},$$

the algebra A is contained in $C^*((1 \otimes \psi)(\mathscr{L}\mathscr{T}\mathscr{R}) \mid \psi \in \mathcal{S}_c(C))$, where $\mathcal{S}_c(C)$ are the compactly supported states on C.

If E is a $\mathbb{Z}/2\mathbb{Z}$-graded A-B-correspondence (that is, with A acting by even operators), we require that D be an odd operator and that μ_L and μ_R be even and call $(A, E_B, D; C, \mu)$ an *even* conformally generated cycle. If E is ungraded, $(A, E_B, D; C, \mu)$ is *odd*.

Remarks III.4.2.

1. The spaces $\mathscr{L}$ and $\mathscr{R}$ are $*$-algebras. The space $\mathscr{T}$ is a ternary ring of C-matched operators. We have $\mathscr{L}\mathscr{T} \subseteq \mathscr{T}$ and $\mathscr{T}\mathscr{R} \subseteq \mathscr{T}$, and $\mathscr{L}\mathscr{T}\mathscr{R}$ is also a ternary ring of C-matched operators.

2. Proposition A.1.30 shows that the application of a compactly supported state on C to a C-matched operator is well-defined. By Proposition A.1.31, $\mathcal{S}_c(C)$ in condition 3. of Definition III.4.1 could be replaced with $\mathcal{S}(C)$, the set of all states on C, at least to those elements of $\mathscr{L}\mathscr{T}\mathscr{R}$ which are adjointable.
3. Any unbounded Kasparov module (A, E_B, D) can be regarded as a conformally generated cycle $(A, E_B, D; \mathbb{C}, (1, 1))$.

One should think of conformally generated cycles as having a dynamical quality, in addition to a strictly geometrical one, with the C*-algebra C as a 'dynamical direction'. In examples, the elements of $\mathscr{T}$ correspond to endomorphisms with bounded 'twisted' commutators with D, as we will see in Theorem III.4.5. Elements of $\mathscr{L}, \mathscr{R}$ encode the regularity of the 'conformal factors' μ_L, μ_R.

Definition III.4.1 could be extended to higher order cycles but, in the interests of readability, we do not pursue this here.

Remark III.4.3. Using Proposition A.1.24, we may specialise Definition III.4.1 to the case when $C = C_0(X)$ for a locally compact Hausdorff space X. Consider a conformally generated A-B-cycle $(A, E_B, D; C_0(X), \mu)$. We may interpret $\mu = (\mu_L, \mu_R)$ as a pair of $*$-strongly continuous families $(\mu_{L,x})_{x\in X}$ and $(\mu_{R,x})_{x\in X}$ of (even) invertible adjointable operators over X. Condition 3. of Definition III.4.1 becomes:

3'. With $\mathscr{L}$ the set of $*$-strongly continuous maps $a : X \to \mathrm{End}^*(E)$ such that the maps

$$x \mapsto [D, a_x] \qquad x \mapsto [\mu_{L,x} D \mu^*_{L,x}, a_x]$$
$$x \mapsto [D, \mu^*_{L,x} a_x] \qquad x \mapsto [D, \mu^{-1}_{L,x} a_x] \qquad x \mapsto [D, a_x \mu_{L,x}] \qquad x \mapsto [D, a_x \mu^{-1*}_{L,x}]$$

are $*$-strongly continuous to $\mathrm{End}^*(E)$, with $\mathscr{R}$ the set of $*$-strongly continuous maps $a : X \to \mathrm{End}^*(E)$ such that the maps

$$x \mapsto [D, a_x] \qquad x \mapsto [\mu_{R,x} D \mu^*_{R,x}, a_x]$$
$$x \mapsto [D, \mu^*_{R,x} a_x] \qquad x \mapsto [D, \mu^{-1}_{R,x} a_x] \qquad x \mapsto [D, a_x \mu_{R,x}] \qquad x \mapsto [D, a_x \mu^{-1*}_{R,x}]$$

are $*$-strongly continuous to $\mathrm{End}^*(E)$, and with

$$\mathscr{T} = \{a \in C(X, \mathrm{End}^*(E)_{*-s}) \,|\, x \mapsto \mu_{L,x} D \mu^*_{L,x} a_x - a_x \mu_{R,x} D \mu^*_{R,x} \in C(X, \mathrm{End}^*(E)_{*-s})\},$$

the algebra A is contained in $C^*((1 \otimes m)(\mathscr{L}\mathscr{T}\mathscr{R}) \,|\, m \in \mathcal{M}_c(X))$, where $\mathcal{M}_c(X)$ is the set of compactly supported Radon measures on X.

An important special case is when X is a discrete set (and, in particular, when X is a point). In this case, Condition 3. of Definition III.4.1 becomes:

3". With $\mathscr{L}_x$ the set of $a \in \mathrm{End}^*(E)$ such that

$$[D, a] \qquad [\mu_{L,x} D \mu^*_{L,x}, a] \qquad [D, \mu^*_{L,x} a] \qquad [D, \mu^{-1}_{L,x} a] \qquad [D, a\mu_{L,x}] \qquad [D, a\mu^{-1*}_{L,x}]$$

are adjointable, with $\mathscr{R}_x$ the set of $a \in \mathrm{End}^*(E)$ such that

$$[D, a] \qquad [\mu_{R,x} D \mu^*_{R,x}, a] \qquad [D, \mu^*_{R,x} a] \qquad [D, \mu^{-1}_{R,x} a] \qquad [D, a\mu_{R,x}] \qquad [D, a\mu^{-1*}_{R,x}]$$

are adjointable, and with

$$\mathscr{T}_x = \{a \in \mathrm{End}^*(E) \,|\, \mu_{L,x} D \mu^*_{L,x} a - a \mu_{R,x} D \mu^*_{R,x} \in \mathrm{End}^*(E)\},$$

the algebra A is contained in the C*-algebra $C^*(\mathscr{L}_x \mathscr{T}_x \mathscr{R}_x \,|\, x \in X)$.

Theorem III.4.4. *Let $(A, E_B, D; C, \mu)$ be a conformally generated A-B-cycle. Then (A, E_B, F_D) is a bounded Kasparov module of the same parity.*

Proof. The main point to check is that $[F_D, a]$ is compact for all $a \in A$. Let c be a positive element of the Pedersen ideal K_C, so that, by Proposition A.1.20, the restriction of μ to the $B \otimes \overline{\mathrm{span}}(CcC)$-module $E \otimes \overline{\mathrm{span}}(CcC)$ is bounded. From now on, we work on the module $E \otimes \overline{\mathrm{span}}(CcC)$. Let $l_1, l_2 \in \mathscr{L}$ and $r_1, r_2, r_3 \in \mathscr{R}$. Omitting instances of $\otimes 1$ for simplicity, Theorem III.1.33 shows that

$$(F_{\mu_L D\mu_L^*} - F_D) l_1 l_2 \langle \mu_L D \mu_L^* \rangle^\beta \qquad (F_{\mu_R D\mu_R^*} - F_D) r_1 r_2 r_3 \langle D \rangle^\beta$$

are bounded for $\beta < 1$. With $l = l_1 l_2$ and $r = r_1 r_2 r_3$,

$$(F_D l - l F_{\mu_L D\mu_L^*}) \langle \mu_L D \mu_L^* \rangle^\beta \qquad (F_{\mu_R D\mu_R^*} r - r F_D) \langle D \rangle^\beta$$

are hence bounded. Let $t \in \mathscr{T}$. By Proposition I.1.1,

$$(F_{\mu_L D\mu_L^*} t - t F_{\mu_R D\mu_R^*}) \langle \mu_R D \mu_R^* \rangle^\beta$$

is bounded and we have

$$[F_D, ltr] = (F_D l - l F_{\mu_L D\mu_L^*}) tr + l(F_{\mu_L D\mu_L^*} t - t F_{\mu_R D\mu_R^*}) r + lt(F_{\mu_R D\mu_R^*} r - r F_D).$$

We see that $[F_D \otimes 1, ltr]\langle D \rangle^\beta \otimes 1$ is bounded on the module $E \otimes \overline{\mathrm{span}}(CcC)$. This is the case for every positive $c \in K_C$ so, by Proposition A.1.20, $[F_D \otimes 1, ltr]\langle D \rangle^\beta \otimes 1$ is a C-matched operator on $E \otimes C$. Let ψ be a compactly supported state on C. By Proposition A.1.30, we may apply $1 \otimes \psi$ to $[F_D \otimes 1, ltr]\langle D \rangle^\beta \otimes 1$ to obtain the bounded operator

$$(1 \otimes \psi)([F_D \otimes 1, ltr]\langle D \rangle^\beta \otimes 1) = [F_D, (1 \otimes \psi)(ltr)]\langle D \rangle^\beta.$$

For $a \in A$ the operator

$$[F_D, 1 \otimes \psi(ltr)]a = [F_D, (1 \otimes \psi)(ltr)]\langle D \rangle^\beta \langle D \rangle^{-\beta} a$$

is compact. Using the Leibniz rule and taking norm limits, $[F_D, b]$ is compact for all $b \in C^*((1 \otimes \psi)(\mathscr{L}\mathscr{T}\mathscr{R}) \mid \psi \in \mathcal{S}_c(C))$, which includes A. □

We now consider conformal perturbations of unbounded Kasparov modules, which include the conformal perturbations of noncommutative tori [CM08, §2.2].

Theorem III.4.5. *Let (A, E_B, D) be an unbounded Kasparov module. Let k be an invertible normal element of $\mathrm{End}^*(E)$. Suppose that $\overline{\mathrm{span}}(\mathscr{M} A \mathscr{M}) \supseteq A$ where $\mathscr{M}$ is the set of $a \in \mathrm{End}^*(E)$ such that*

$$[kDk^*, a] \qquad [D, a] \qquad [D, k^*]a \qquad [D, k^*k]a \qquad a[D, k] \qquad a[D, k^*k]$$

are bounded. Then $(A, E_B, kDk^*; \mathbb{C}, (k^{-1}, k^{-1}))$ is a conformally generated cycle. In particular, if k is normal and invertible and (A, E_B, D) is an unbounded Kasparov module with $[D, k]$ bounded then the data $(A, E_B, kDk^*; \mathbb{C}, (k^{-1}, k^{-1}))$ define a conformally generated cycle. Hence (A, E_B, F_{kDk^*}) is a Kasparov module and $[(A, E_B, F_{kDk^*})] = [(A, E_B, F_D)] \in KK(A, B)$.

Proof. It is straightforward to check that, for all $a \in \mathscr{M}$,

$$[kDk^*, a] \qquad [D, a] \qquad [kDk^*, k^{-1*}a] \qquad [kDk^*, ka] \qquad [kDk^*, ak^{-1}]$$

are bounded so that $\mathscr{M} \subseteq \mathscr{L} \cap \mathscr{R}$ where $\mathscr{L}, \mathscr{R}$ are as in Definition III.4.1. As $A \subseteq \overline{\mathscr{T}} = \overline{\mathrm{Lip}_0^*(D)}$, we are done. For the final statements, if $[D, k]$ is bounded then $\mathscr{M}$ contains scalar multiples of the identity and so $\overline{\mathrm{span}}(\mathscr{M} A \mathscr{M}) \supseteq A$. An application of Theorem III.1.33 gives the equality of the Kasparov classes. □

Example III.4.6. We recall the noncommutative torus $C(\mathbb{T}^2_\alpha)$ from Example III.1.13 and the spectral triple

$$\left(C(\mathbb{T}^2_\alpha), L^2(\mathbb{T}^2_\alpha) \otimes \mathbb{C}^2, D := \begin{pmatrix} & \delta_1+\tau\delta_2 \\ \delta_1+\overline{\tau}\delta_2 & \end{pmatrix}\right).$$

As in Example III.1.13, choose a positive invertible element $k \in C(\mathbb{T}^2_\alpha)$ in the domains of δ_1 and δ_2. Using left multiplication by k yields a conformally generated cycle

$$\left(C(\mathbb{T}^2_\alpha), L^2(\mathbb{T}^2_\alpha) \otimes \mathbb{C}^2, kDk; \mathbb{C}, (k^{-1}, k^{-1})\right).$$

Thus the classes defined by F_D and F_{kDk} in $KK(C(\mathbb{T}^2_\alpha), \mathbb{C})$ coincide.

The unbounded Kasparov module $(C(\mathbb{T}^2_\alpha), L^2(C(\mathbb{T}^2_\alpha), \Phi)_{C(\mathbb{T})}, \delta_2)$ also gives rise to a conformally generated cycle

$$(C(\mathbb{T}^2_\alpha), L^2(C(\mathbb{T}^2_\alpha), \Phi)_{C(\mathbb{T})}, k\delta_2 k; \mathbb{C}, (k^{-1}, k^{-1}))$$

where $k \in C(\mathbb{T}^2_\alpha)$ is now a positive invertible element in the domain of δ_2. Thus the classes defined by F_{δ_2} and $F_{k\delta_2 k}$ in $KK(C(\mathbb{T}^2_\alpha), C(\mathbb{T}))$ coincide.

Next we consider unbounded modular cycles in the sense of [Kaa21, Definition 3.1] [Kaa24, Definition 8.1]. Using our methods the bounded transform can be achieved in greater generality. Compare Proposition III.1.42.

Proposition III.4.7. *Let E be an A-B correspondence. Let D be a self-adjoint regular operator and Δ_+ and Δ_- a pair of commuting positive adjointable operators on E such that*

- *For all $a \in A$, $(1 + D^2)^{-1}a$ is compact and the sequence $(a(\Delta_+ + \Delta_-)(\Delta_+ + \Delta_- + \frac{1}{n})^{-1})_{n=1}^\infty$ converges in norm to (the representation of) a;*
- *$\{\Delta_+, \Delta_-\} \operatorname{dom} D \subseteq \operatorname{dom} D$ and $[D, \Delta_+]$, $[D, \Delta_-]$ are bounded; and*
- *$A \subseteq \overline{\mathscr{N}}$, where $\mathscr{N}$ is the set of $a \in \operatorname{End}^*(E)$ such that $a \operatorname{dom} D \subseteq \operatorname{dom} D$ and $\Delta_- D a \Delta_+ - \Delta_+ a D \Delta_-$ is bounded.*

Let $(h_n)_{n\in\mathbb{N}_{\geq 1}} \subseteq C_b^\infty(\mathbb{R}_+^\times)$ be any sequence of positive functions with bounded reciprocals which agree with the function $x \mapsto x^{-1/2}$ on the interval $[\frac{1}{n}, n]$. Then, with $\mu_{L,n} = \mu_{R,n} = h_n(\Delta_+)h_n(\Delta_-)^{-1}$, the data $(A, E_B, D; C_0(\mathbb{N}_{\geq 1}), \mu)$ define a conformally generated cycle.

Proof. First, by the smooth functional calculus of Theorem A.4.18, $[D, h_n(\Delta_+)h_n(\Delta_-)^{-1}]$ is bounded so $1 \in \mathscr{L}_n, \mathscr{R}_n$ for every $n \in \mathbb{N}_{\geq 1}$. Second, $\mathscr{T}_n$ consists of those $b \in \operatorname{End}^*(E)$ such that

$$[h_n(\Delta_+)h_n(\Delta_-)^{-1} D h_n(\Delta_+)h_n(\Delta_-)^{-1}, b]$$

extends to an adjointable operator. Let $f_1, f_2, f_3, f_4 \in C_c^\infty((\frac{1}{n}, n))$ and $a \in \mathscr{N}$ and define $b \in \operatorname{End}^*(E)$ to be the product

$$f_1(\Delta_+)f_2(\Delta_-)af_3(\Delta_+)f_4(\Delta_-) \in C_0((\tfrac{1}{n}, n))(\Delta_+)C_0((\tfrac{1}{n}, n))(\Delta_-)\mathscr{N}C_0((\tfrac{1}{n}, n))(\Delta_+)C_0((\tfrac{1}{n}, n))(\Delta_-).$$

Then $bh_n(\Delta_+)h_n(\Delta_-)^{-1} = b\Delta_+^{-1/2}\Delta_-^{1/2}$ and, again using the smooth functional calculus,

$$\begin{aligned}
&[h_n(\Delta_+)h_n(\Delta_-)^{-1} D h_n(\Delta_+)h_n(\Delta_-)^{-1}, b] \\
&\quad = f_1(\Delta_+)f_2(\Delta_-)\Delta_+^{-1}(\Delta_- D a \Delta_+ - \Delta_+ a D \Delta_-)\Delta_+^{-1} f_3(\Delta_+)f_4(\Delta_-) \\
&\qquad + h_n(\Delta_+)h_n(\Delta_-)^{-1}\left[D, \Delta_+^{-1/2}\Delta_-^{1/2} f_1(\Delta_+)f_2(\Delta_-)\right] a f_3(\Delta_+)f_4(\Delta_-) \\
&\qquad + f_1(\Delta_+)f_2(\Delta_-)a\left[D, \Delta_-^{1/2}\Delta_+^{-1/2} f_3(\Delta_+)f_4(\Delta_-)\right] h_n(\Delta_+)h_n(\Delta_-)^{-1}
\end{aligned}$$

is bounded. The closure of $C_0((\frac{1}{n}, n))(\Delta_+)C_0((\frac{1}{n}, n))(\Delta_-)$ is $C^*(\Delta_+, \Delta_-)$. By Lemma III.1.43, $A \subseteq \overline{AC^*(\Delta_+, \Delta_-)}$ so that

$$\overline{\mathscr{L}\mathscr{T}\mathscr{R}} \supseteq \overline{C^*(\Delta_+, \Delta_-)\mathscr{N}C^*(\Delta_+, \Delta_-)} \supseteq \overline{C^*(\Delta_+, \Delta_-)AC^*(\Delta_+, \Delta_-)} \subseteq A$$

and we are done. □

As a last application we consider again the relation to the logarithmic transform.

Proposition III.4.8. *cf. [GMR19, Corollary 1.20] Let (A, E_B, D) consist of a C^*-algebra A represented on a Hilbert B-module E and a regular operator D on E, such that*

- *D is self-adjoint;*
- *$(1 + D^2)^{-1/2}a$ is compact for all $a \in A$; and*
- *There is a dense subset of $a \in A$ such that $a \operatorname{dom} D \subseteq \operatorname{dom} D$ and $[F_D, a]\log\langle D\rangle$ is bounded.*

Then, with $L_D = F_D \log\langle D\rangle$, the triple (A, E_B, L_D) is an unbounded Kasparov module whose bounded transform is equal to (A, E_B, F_D) up to a locally compact difference.

Theorem III.4.9. *Let $(A, E_B, D; C, (\mu_L, \mu_R))$ be a conformally generated cycle. Then (A, E_B, L_D) is an unbounded Kasparov module of the same parity.*

Proof. By the Proof of Theorem III.4.4, $[F_D, (1 \otimes \psi)(ltr)]\langle D\rangle^\beta$ is bounded for $\psi \in \mathscr{S}_c(C)$, $l \in \mathscr{L}^2$, $t \in \mathscr{T}$, $r \in \mathscr{R}^3$, and $\beta < 1$. We have

$$\begin{aligned} ltr \operatorname{dom}(D \otimes 1)(1 \otimes K_C) &\subseteq lt\mu_R^{-1*} \operatorname{dom}(D \otimes 1)(1 \otimes K_C) \subseteq l\mu_L^{-1*} \operatorname{dom}(D \otimes 1)(1 \otimes K_C) \\ &\subseteq \operatorname{dom}(D \otimes 1)(1 \otimes K_C). \end{aligned}$$

Hence $(\langle D\rangle \otimes 1)ltr(\langle D\rangle^{-1} \otimes 1)$ is C-matched. Applying Proposition A.1.30,

$$(1 \otimes \psi)((\langle D\rangle \otimes 1)ltr(\langle D\rangle^{-1} \otimes 1) = \langle D\rangle(1 \otimes \psi)(ltr)\langle D\rangle^{-1}$$

is an adjointable operator on E, and so $(1 \otimes \psi)(ltr) \operatorname{dom} D \subseteq \operatorname{dom} D$. By Proposition III.1.35, the commutator $[L_D, (1 \otimes \psi)(ltr)]$ is bounded. By the Leibniz rule, $[L_D, b]$ is bounded for all b in the $*$-algebra generated by $\{(1 \otimes \psi)(\mathscr{L}\mathscr{T}\mathscr{R}) \mid \psi \in \mathscr{S}_c(C)\}$. This is dense in $C^*((1 \otimes \psi)(\mathscr{L}\mathscr{T}\mathscr{R}) \mid \psi \in \mathscr{S}_c(C))$, which includes A. □

In principle, the logarithmic transform, if carried out piece-by-piece, could be used to produce KK-classes from ‘multi-twisted’ spectral triples which have appeared in the literature, such as for quantum groups [KK20] and dynamical systems [KK25]. (See also [DS22], where an approach similar to that of [Sit15] is used to obtain ordinary spectral triples from partial conformal rescalings.) The construction of §II.4.1 is really an example of this. In the next Chapter, we shall develop a framework accounting for a different kind of ‘multidirectional’ behaviour, which we call *tangled cycles*. However, it is not clear whether conformally generated cycles and tangled cycles can be reconciled; we leave this for the future.

III.4.1 Descent and the dual Green–Julg map for conformal equivariance

In the conformally equivariant setting, the descent map and the dual Green–Julg map produce conformally generated cycles.

Proposition III.4.10. *Let G be a locally compact group and let (A, E_B, D) be a $(\mu_g)_{g\in G}$-conformally G-equivariant unbounded Kasparov module. Then, for $t \in \{u, r\}$,*

$$(A \rtimes_t G, (E \rtimes_t G)_{B\rtimes_t G}, \tilde{D}; C_0(G), (1, \tilde{\mu}_g)_{g\in G})$$

is a conformally generated cycle, where $\tilde{D}$ is the regular operator given on $\xi \in C_c(G, E) \subseteq E \rtimes_t G$ by $(\tilde{D}\xi)(h) = D(\xi(h))$ and $(\tilde{\mu}_g)_{g\in G}$ are given by $(\tilde{\mu}_g\xi)(h) = \mu_g(\xi(h))$.

Proof. The local compactness of the resolvent is the same as in the uniform case, Proposition I.2.16. Recall the spaces $\mathscr{L}$, $\mathscr{T}$, and $\mathscr{R}$ of Remark III.4.3. It is straightforward to verify that the constant families $(\tilde{d})_{g\in G} \in \mathscr{L}$ and $(\tilde{b}^*\tilde{c})_{g\in G} \in \mathscr{R}$ for all $d \in \mathrm{Lip}^*_0(D)$ and $b, c \in \mathcal{Q}$. Let $(u_g)_{g\in G} \subseteq \mathrm{End}^*_{B\rtimes_t G}(E\rtimes_t G)$ be the canonical unitaries implementing the group action, given by

$$(u_h\xi)(g) = U_h\xi(h^{-1}g)$$

on $\xi \in C_c(G, E)$ (where we recall the notation of Definition I.2.11). A family of operators t is in $\mathscr{T}$ if $g \mapsto \tilde{D}t_g - t_g\tilde{\mu}_g\tilde{D}\tilde{\mu}^*_g$ is $*$-strongly continuous into bounded operators. Using the condition for conformal equivariance that for $a \in \mathcal{Q}$ the map

$$g \mapsto U_g D U^*_g a - a\mu_g D\mu^*_g$$

is $*$-strongly continuous into bounded operators, we see that $g \mapsto u^*_g\tilde{a}$ is in $\mathscr{T}$. So, $g \mapsto \tilde{d}u^*_g\tilde{a}\tilde{b}^*\tilde{c}$ is in $\mathscr{L}\mathscr{T}\mathscr{R}$.

We now evaluate $\mathscr{L}\mathscr{T}\mathscr{R}$ on compactly supported Radon measures on G and ask if this generates $A \rtimes_t G$. It will suffice to integrate the paths $g \mapsto \tilde{d}u^*_g\tilde{a}\tilde{b}^*\tilde{c}$, which are constant apart from u^*_g, against compactly supported continuous functions on G. Proceeding step-by-step,

$$\begin{aligned}
\overline{\mathrm{span}}(\mathrm{Lip}^*_0(D)C^*_c(G)\mathcal{Q}\mathcal{Q}^*\mathcal{Q}) &\supseteq \overline{\mathrm{span}}(AC^*_t(G)\mathcal{Q}\mathcal{Q}^*\mathcal{Q}) \\
&= \overline{\mathrm{span}}((A \rtimes_t G)\mathcal{Q}\mathcal{Q}^*\mathcal{Q}) \\
&= \overline{\mathrm{span}}(C^*_t(G)A\mathcal{Q}\mathcal{Q}^*\mathcal{Q}) \\
&\supseteq \overline{\mathrm{span}}(C^*_t(G)A) \\
&= A \rtimes_t G
\end{aligned}$$

as required. □

Proposition III.4.11. *Let (A, E_B, D) be a $(\mu_g)_{g\in G}$-conformally G-equivariant unbounded Kasparov module, with G acting trivially on B. Then*

$$(A \rtimes_u G, E_B, D; C_0(G), (1, \mu_g)_{g\in G})$$

is a conformally generated cycle, with the integrated representation of $A \rtimes_u G$.

Proof. The local compactness of the resolvent is the same as in the uniform case, Proposition I.2.17. Recall the spaces $\mathscr{L}$, $\mathscr{T}$, and $\mathscr{R}$ of Remark III.4.3. It is straightforward to verify that the constant families $(d)_{g\in G} \in \mathscr{L}$ and $(b^*c)_{g\in G} \in \mathscr{R}$ for all $d \in \mathrm{Lip}^*_0(D)$ and $b, c \in \mathcal{Q}$. A path of operators t is in $\mathscr{T}$ if

$$g \mapsto Dt_g - t_g\mu_g D\mu^*_g$$

is $*$-strongly continuous into bounded operators. Using the condition for conformal equivariance that $g \mapsto U_g D U^*_g a - a\mu_g D\mu^*_g$ is $*$-strongly continuous into bounded operators for $a \in \mathcal{Q}$, we see that $g \mapsto U^*_g a$ is in $\mathscr{T}$. So, $g \mapsto dU^*_g ab^*c$ is in $\mathscr{L}\mathscr{T}\mathscr{R}$. As in the Proof of Proposition III.4.10, the closed span of $\mathrm{Lip}^*_0(D)C^*_c(G)\mathcal{Q}\mathcal{Q}^*\mathcal{Q}$ includes $A \rtimes_u G$. □

Remark III.4.12. It is clear that the bounded transform $(A \rtimes_t G, (E \rtimes_t G)_{B \rtimes_t G}, F_{\tilde{D}} = \tilde{F_D})$ of the descent

$$(A \rtimes_t G, (E \rtimes_t G)_{B \rtimes_t G}, \tilde{D}; C_0(G), (1, \tilde{\mu}_g)_{g \in G})$$

of a conformally G-equivariant cycle (A, E_B, D) is exactly the descent of the bounded transform (A, E_B, F_D). The same is true for the dual Green–Julg map.

We recall the identity

$$2A^*CB = (A+B)^*C(A+B) - i(A+iB)^*C(A+iB) + (-1+i)(B^*CB + A^*CA)$$

for elements A, B, and C of a $*$-algebra, which implies that

$$\operatorname{span}\{x^*Cx \mid x \in \operatorname{span}\{A,B\}\} = \operatorname{span}\{x^*Cy \mid x, y \in \operatorname{span}\{A,B\}\}.$$

Proposition III.4.13. *Let $\mathbb{G}$ be a locally compact quantum group and let (A, E_B, D) be a μ-conformally $\mathbb{G}$-equivariant unbounded Kasparov module. For $t \in \{u, r\}$, let ι be the inclusion* $\operatorname{End}^0(E) \to M(\operatorname{End}^0(E) \rtimes_t \mathbb{G}) \cong \operatorname{End}^*_{B \rtimes_t \mathbb{G}}(E \rtimes_t \mathbb{G})$*. Then*

$$(A \rtimes_t \mathbb{G}, (E \rtimes_t \mathbb{G})_{B \rtimes_t \mathbb{G}}, \iota(D); C_0^r(\mathbb{G}), (1, (\iota \otimes \mathrm{id})(\mu)))$$

is a conformally generated cycle.

Proof. The compactness of the resolvent is as in the Proof of Proposition I.3.18. Recall the spaces $\mathscr{L}$, $\mathscr{T}$, and $\mathscr{R}$ of Definition III.4.1. It is straightforward to verify that $\iota(d) \otimes 1 \in \mathscr{L}$ and $\iota(b^*c) \otimes s_2^* s_3 \in \mathscr{R}$ for all $d \in \operatorname{Lip}_0^*(D)$, $b, c \in \mathscr{Q}$, and $s_2, s_3 \in \mathscr{S}_b, \mathscr{S}_c$.

By the universality of the crossed product, see [Ver02, §4.1] [Vae05, §2.3], the morphism

$$\operatorname{End}^0(E) \rtimes_u \mathbb{G} \to \operatorname{End}^0(E) \rtimes_t \mathbb{G}$$

gives rise both to the morphism

$$\iota : \operatorname{End}^0(E) \to M(\operatorname{End}^0(E) \rtimes_t \mathbb{G}) \cong \operatorname{End}^*(E \rtimes_t \mathbb{G})$$

and a unitary $X \in M((\operatorname{End}^0(E) \rtimes_t \mathbb{G}) \otimes C_0^r(\mathbb{G})) \cong \operatorname{End}^*((\operatorname{End}^0(E) \rtimes_t \mathbb{G}) \otimes C_0^r(\mathbb{G}))$ such that

$$X(\iota(T) \otimes 1)X^* = (\iota \otimes \mathrm{id})(V_E(T \otimes_{\delta_B} 1)V_E^*)$$

for $T \in \operatorname{End}^0(E)$. Let $a \in \mathscr{Q}$ and $s_1 \in \mathscr{S}_a$; then $X^*(\iota(a) \otimes s_1) \in \mathscr{T}$ because

$$\begin{aligned}
&(\iota(D) \otimes 1)X^*(\iota(a) \otimes s_1) - X^*(\iota(a) \otimes s_1)(\iota \otimes \mathrm{id})(\mu)(\iota(D) \otimes 1)(\iota \otimes \mathrm{id})(\mu)^* \\
&\quad = X^*\left(X(\iota(D) \otimes 1)X^*(\iota(a) \otimes s_1) - (\iota \otimes \mathrm{id})\left((a \otimes s_1)\mu(D \otimes 1)\mu^*\right)\right) \\
&\quad = X^*(\iota \otimes \mathrm{id})\left(V_E(D \otimes_{\delta_B} 1)V_E^*(a \otimes s_1) - (a \otimes s_1)\mu(D \otimes 1)\mu^*\right)
\end{aligned}$$

is $C_0^r(\mathbb{G})$-matched. So, $(d \otimes 1)X^*(\iota(ab^*c) \otimes s_1 s_2^* s_3)$ is in $\mathscr{L}\mathscr{T}\mathscr{R}$.

We need to show that $A \rtimes_t \mathbb{G}$ is contained in $C^*((1\otimes\omega)(\mathscr{L}\mathscr{T}\mathscr{R}) \mid \omega \in \mathcal{S}_c(C))$. Proceeding step-by-step,

$$\begin{aligned}
&C^*((1\otimes\omega)(\mathscr{L}\mathscr{T}\mathscr{R}) \mid \omega \in \mathcal{S}_c(C))\\
&\quad\supseteq \overline{\operatorname{span}}\Big\{(1\otimes\omega)\big((\iota(d)\otimes 1)X^*(\iota(ab^*c)\otimes s_1s_2^*s_3)\big) = \iota(d)(1\otimes\omega)\big((1\otimes s_3^*s_2s_1^*)X\big)^*\iota(ab^*c)\\
&\qquad\qquad \Big| \, a,b,c\in\mathcal{Q}; d\in \operatorname{Lip}_0^*(D); s_1\in\mathcal{S}_a; s_2\in\mathcal{S}_b; s_3\in\mathcal{S}_c; \omega\in\mathcal{S}_c(C_0^r(\mathbb{G}))\Big\}\\
&\quad= \overline{\operatorname{span}}\Big(\iota(\operatorname{Lip}_0^*(D))\Big\{(1\otimes\omega)\big((1\otimes s_3^*s_2s_1^*)X\big) \mid s_1\in\mathcal{S}_a; s_2\in\mathcal{S}_b; s_3\in\mathcal{S}_c; \omega\in\mathcal{S}_c(C_0^r(\mathbb{G}))\Big\}^*\iota(\mathcal{Q})\Big)\\
&\quad\supseteq \overline{\operatorname{span}}\Big(\iota(\operatorname{Lip}_0^*(D))\Big\{(1\otimes\eta^*s_4^*s_3^*s_2s_1^*)X(1\otimes s_4\eta)\\
&\qquad\qquad \Big| \, s_1\in\mathcal{S}_a; s_2\in\mathcal{S}_b; s_3\in\mathcal{S}_c; s_4\in K_{C_0^r(\mathbb{G})}; \eta\in L^2(C_0^r(\mathbb{G}))\Big\}^*\iota(\mathcal{Q})\Big)\\
&\quad= \overline{\operatorname{span}}\Big(\iota(\operatorname{Lip}_0^*(D))\Big\{(1\otimes\eta_1^*s_4^*s_3^*s_2s_1^*)X(1\otimes s_5\eta_2)\\
&\qquad\qquad \Big| \, s_1\in\mathcal{S}_a; s_2\in\mathcal{S}_b; s_3\in\mathcal{S}_c; s_4,s_5\in K_{C_0^r(\mathbb{G})}; \eta_1,\eta_2\in L^2(C_0^r(\mathbb{G}))\Big\}^*\iota(\mathcal{Q})\Big)\\
&\quad= \overline{\operatorname{span}}\big(\iota(\operatorname{Lip}_0^*(D))\{(1\otimes\eta_1^*)X(1\otimes\eta_2) \mid \eta_1,\eta_2\in L^2(C_0^r(\mathbb{G}))\}^*\iota(\mathcal{Q})\big)\\
&\quad= \overline{\operatorname{span}}\big(\iota(\operatorname{Lip}_0^*(D))\{(1\otimes\omega)(X) \mid \omega\in L^1(\mathbb{G})\}^*\iota(\mathcal{Q})\big)\\
&\quad= \overline{\operatorname{span}}(\iota(\operatorname{Lip}_0^*(D))C_u^*(\mathbb{G})\iota(\mathcal{Q}))\\
&\quad\supseteq \overline{\operatorname{span}}(\iota(A)C_u^*(\mathbb{G})\iota(\mathcal{Q})) = \overline{\operatorname{span}}((A\rtimes_u\mathbb{G})\iota(\mathcal{Q})) = \overline{\operatorname{span}}(C_u^*(\mathbb{G})\iota(A\mathcal{Q}))\\
&\quad\supseteq \overline{\operatorname{span}}(C_u^*(\mathbb{G})\iota(A)) = A\rtimes_u\mathbb{G}
\end{aligned}$$

by the density of $L^2(\mathbb{G})K_{C_0^r(\mathbb{G})}\mathcal{S}_c^*\mathcal{S}_b\mathcal{S}_a^* \subseteq L^2(\mathbb{G})$ and the inclusion $A\subseteq\overline{\operatorname{span}}(A\mathcal{Q})$. □

Proposition III.4.14. *Let $\mathbb{G}$ be a locally compact quantum group and let (A, E_B, D) be a conformally $\mathbb{G}$-equivariant unbounded Kasparov module, with $\mathbb{G}$ acting trivially on B. Then*

$$(A\rtimes_u\mathbb{G}, E_B, D; C_0^r(\mathbb{G}), (1,\mu))$$

is a conformally generated cycle, with the integrated representation of $A\rtimes_u\mathbb{G}$.

Proof. Recall the spaces $\mathscr{L}$, $\mathscr{T}$, and $\mathscr{R}$ of Definition III.4.1. It is straightforward to verify that $d\otimes 1\in\mathscr{L}$ and $b^*c\otimes s_2^*s_3\in\mathscr{R}$ for all $d\in\operatorname{Lip}_0^*(D)$, $b,c\in\mathcal{Q}$, and $s_2,s_3\in\mathcal{S}_b,\mathcal{S}_c$. Let $a\in\mathcal{Q}$ and $s_1\in\mathcal{S}_a$; then, by Definition III.3.1,

$$(D\otimes 1)V_E^*(a\otimes s) - V_E^*(a\otimes s)\mu(D\otimes 1)\mu^*$$

is $C_0^r(\mathbb{G})$-matched and $V_E^*(a\otimes s_1)\in\mathscr{T}$. So $(d\otimes 1)V_E^*(ab^*c\otimes s_1s_2^*s_3)$ is in $\mathscr{L}\mathscr{T}\mathscr{R}$.

We need to show that $A\rtimes_u\mathbb{G}$ is contained in $C^*((1\otimes\omega)(\mathscr{L}\mathscr{T}\mathscr{R}) \mid \omega\in\mathcal{S}_c(C))$. The same manipulations as in the proof of Proposition III.4.13, with V_E in place of X, show that

$$C^*((1\otimes\omega)(\mathscr{L}\mathscr{T}\mathscr{R}) \mid \omega\in\mathcal{S}_c(C)) \supseteq \overline{\operatorname{span}}(\operatorname{Lip}_0^*(D)C_u^*(\mathbb{G})\mathcal{Q}) \supseteq A\rtimes_u\mathbb{G},$$

as required. □

Remark III.4.15. It is again clear that the bounded transform $(A\rtimes_t\mathbb{G}, (E\rtimes_t\mathbb{G})_{B\rtimes_t\mathbb{G}}, F_{\iota(D)} = \iota(F_D))$ of the descent

$$(A\rtimes_t\mathbb{G}, (E\rtimes_t\mathbb{G})_{B\rtimes_t\mathbb{G}}, \iota(D); C_0^r(\mathbb{G}), (1, (\iota\otimes\mathrm{id})(\mu)))$$

of a conformally $\mathbb{G}$-equivariant cycle (A, E_B, D) is exactly the descent of the bounded transform (A, E_B, F_D). The same is true for the dual Green–Julg map.

III.4.2 An equivalence relation on conformally generated cycles

In this section, we consider an equivalence relation on conformally generated cycles making the equivalence classes an abelian group, following §I.1.

Remark III.4.16. The direct sum of two conformally generated cycles $(A, E_{1,B}, D_1; C_1, \mu_1)$ and $(A, E_{2,B}, D_2; C_2, \mu_2)$ is

$$(A, E_{1,B} \oplus E_{2,B}, D_1 \oplus D_2; C_1 \oplus C_2, \mu_1 \oplus 1 \oplus 1 \oplus \mu_2)$$

where $\mu_1 \oplus 1 \oplus 1 \oplus \mu_2 \in (E_1 \otimes C_1 \oplus E_2 \otimes C_1 \oplus E_1 \otimes C_2 \oplus E_2 \oplus C_2)^2$. If $C_1 = C_2$ or, more generally, if C_1 and C_2 have a common ideal J, one could write the direct sum in a smaller way. In practice, also, it is often possible to change C and μ without affecting the validity of a cycle $(A, E_B, D; C, \mu)$. One should therefore think of conformally generated cycles $(A, E_B, D; C_1, \mu_1)$ and $(A, E_B, D; C_2, \mu_2)$ as equivalent.

Note that the external product of conformally generated cycles is not constructive.

Definition III.4.17. Two conformally generated cycles $(A, E_{1,B}, D_1; C_1, \mu_1)$ and $(A, E_{2,B}, D_2; C_2, \mu_2)$ are *cobordant* if there exists a conformally generated cycle $(A, E_B, D; C, \mu)$ and an even partial isometry $v \in \mathrm{End}^*(E)$ such that

1. v commutes with (the representation of) A, and vv^* and v^*v commute with D;
2. $vA \subseteq C^*((1 \otimes \psi)(\mathscr{L}\mathscr{T}\mathscr{R}) \mid \psi \in \mathscr{S}_c(C))$;
3. $(A, (1 - vv^*)E_B, (1 - vv^*)D(1 - vv^*))$ is unitarily equivalent to $(A, E_{1,B}, D_1)$; and
4. $(A, (1 - v^*v)E_B, (1 - v^*v)D(1 - v^*v))$ is unitarily equivalent to $(A, E_{2,B}, D_2)$.

Example III.4.18. Let $(A, E_B, D; v)$ be a cobordism between unbounded Kasparov modules (A, E'_B, D_1) and (A, E''_B, D_2). Then

$$(A, E_B, D; \mathbb{C}, (1, 1); v)$$

is a cobordism between $(A, E'_B, D_1; \mathbb{C}, (1, 1))$ and $(A, E''_B, D_2; \mathbb{C}, (1, 1))$.

When applied to ordinary Kasparov modules, Definition III.4.17 also encompasses conformal transformations and singular conformal transformations.

Example III.4.19. Suppose that (U, μ) is a conformal transformation from the unbounded Kasparov module (A, E_B, D_1) to (A, E'_B, D_2). Then

$$\left(A, (E \oplus E')_B, \begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix}; \mathbb{C}^2, \left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \oplus \begin{pmatrix} \mu & \\ & 1 \end{pmatrix}\right); \begin{pmatrix} & 0 \\ U & \end{pmatrix}\right)$$

is a cobordism between $(A, E_B, D_1; \mathbb{C}, (1, 1))$ and $(A, E'_B, D_2; \mathbb{C}, (1, 1))$. We leave the demonstration of this as a special case of Example III.4.20.

More generally,

Example III.4.20. Let $(U, (\mu_i)_{i \in I})$ be a singular conformal transformation from one unbounded Kasparov module, (A, E_B, D_1), to another, (A, E'_B, D_2), as in Definition III.1.38. We will show that

$$\left(A, (E \oplus E')_B, \begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix}; C_0(\{\mathrm{pt}\} \sqcup I), \left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}_{i \in I}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \oplus \begin{pmatrix} \mu_i & \\ & 1 \end{pmatrix}_{i \in I}\right); \begin{pmatrix} & 0 \\ U & \end{pmatrix}\right)$$

is a cobordism between $(A, E_B, D_1; \mathbb{C}, (1, 1))$ and $(A, E'_B, D_2; \mathbb{C}, (1, 1))$. Here, I is treated as a discrete set. For $a \in \mathscr{M}_i$, we can check that

$$\begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix}\begin{pmatrix} & 0 \\ Ua & \end{pmatrix} - \begin{pmatrix} & 0 \\ Ua & \end{pmatrix}\begin{pmatrix} \mu_i & \\ & 1 \end{pmatrix}\begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix}\begin{pmatrix} \mu_i & \\ & 1 \end{pmatrix}^* = \begin{pmatrix} & 0 \\ U(U^* D_2 U a - a\mu_i D_1 \mu_i^*) & \end{pmatrix}$$

is bounded, so that $\begin{pmatrix} & 0 \\ U & \end{pmatrix}\begin{pmatrix} \mathscr{M} & \\ & 0 \end{pmatrix} \in \mathscr{T}_i$. One can check that $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \in \mathscr{L}_i$ and that $\mathscr{R}_i$ contains $\begin{pmatrix} \mathscr{M}_i^*\mathscr{M}_i & \\ & 0 \end{pmatrix}$. Furthermore, $\mathscr{L}_{\mathrm{pt}}$, $\mathscr{T}_{\mathrm{pt}}$, and $\mathscr{R}_{\mathrm{pt}}$ all contain $\begin{pmatrix} \mathrm{Lip}_0^*(D_1) & \\ & \mathrm{Lip}_0^*(D_2) \end{pmatrix}$. Hence

$$\begin{aligned}\begin{pmatrix} & 0 \\ U & \end{pmatrix} A \subseteq \overline{\operatorname{span}}_{i\in I}\left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\begin{pmatrix} & 0 \\ U & \end{pmatrix}\begin{pmatrix} \mathscr{M} & \\ & 0 \end{pmatrix}\begin{pmatrix} \mathscr{M}_i^*\mathscr{M}_i & \\ & 0 \end{pmatrix}\begin{pmatrix} \mathrm{Lip}_0^*(D_1) & \\ & 0 \end{pmatrix}\right) \\ \subseteq \overline{\operatorname{span}}_{i\in I}\left(\mathscr{L}_i\mathscr{T}_i\mathscr{R}_i\mathscr{L}_{\mathrm{pt}}\mathscr{T}_{\mathrm{pt}}\mathscr{R}_{\mathrm{pt}}\right) \subseteq C^*(\mathscr{L}_x\mathscr{T}_x\mathscr{R}_x \mid x \in \{\mathrm{pt}\} \sqcup I)\end{aligned}$$

and we are done.

We are led to the following definition.

Definition III.4.21. Two unbounded Kasparov modules (A, E_B, D_1) and (A, E'_B, D_2) are *conformant* if there exists if there exists a conformally generated cycle $(A, E_B, D; C, \mu)$ and an even partial isometry $v \in \mathrm{End}^*(E)$ making the conformally generated cycles $(A, E_B, D_1; \mathbb{C}, (1, 1))$ and $(A, E'_B, D_2; \mathbb{C}, (1, 1))$ cobordant. We call the data $(A, E_B, D; C, \mu; v)$ a *conformism* between (A, E_B, D_1) and (A, E'_B, D_2).

Example III.4.22. We pick up from the setting of Theorem III.4.5, adopting the notation there. We will show that the conformally generated cycles

$$(A, E_B, D; \mathbb{C}, (1, 1)) \qquad (A, E_B, kDk^*; \mathbb{C}, (k^{-1}, k^{-1}))$$

are cobordant. A suitable cobordism is

$$\left(A, (E \oplus E)_B, \begin{pmatrix} D & \\ & kDk^* \end{pmatrix}; \mathbb{C}, \left(\begin{pmatrix} 1 & \\ & k^{-1} \end{pmatrix}, \begin{pmatrix} 1 & \\ & k^{-1} \end{pmatrix}\right); \begin{pmatrix} & 0 \\ 1 & \end{pmatrix}\right).$$

We check that

$$\begin{pmatrix} 1 & \\ & k^{-1} \end{pmatrix}\begin{pmatrix} D & \\ & kDk^* \end{pmatrix}\begin{pmatrix} 1 & \\ & k^{-1} \end{pmatrix}\begin{pmatrix} & 0 \\ 1 & \end{pmatrix} - \begin{pmatrix} & 0 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & \\ & k^{-1} \end{pmatrix}\begin{pmatrix} D & \\ & kDk^* \end{pmatrix}\begin{pmatrix} 1 & \\ & k^{-1} \end{pmatrix}^* = 0$$

so that $\begin{pmatrix} & 0 \\ 1 & \end{pmatrix} \in \mathscr{T}$. Both $\mathscr{L}$ and $\mathscr{R}$ contain $\mathbb{C}1 \oplus \mathscr{M}$. We remark that $\mathscr{M}$ is a $*$-algebra of operators, so $\overline{\operatorname{span}}(\mathscr{M}^2) = \overline{\mathscr{M}}$. We have

$$\begin{pmatrix} & 0 \\ 1 & \end{pmatrix} A \subseteq \overline{\operatorname{span}}\left(\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\begin{pmatrix} & 0 \\ 1 & \end{pmatrix}\begin{pmatrix} 0 & \\ & \mathscr{M}^2 \end{pmatrix} A \begin{pmatrix} 0 & \\ & \mathscr{M} \end{pmatrix}\right) \subseteq \overline{\operatorname{span}}(\mathscr{L}\mathscr{T}\mathscr{R}\mathscr{L}\mathscr{T}\mathscr{R})$$

and we are done.

Proposition III.4.23. *Cobordism of conformally generated cycles is an equivalence relation and is compatible with direct sums.*

Proof. For reflexivity, we take $v = 0 \in \mathrm{End}^*(E)$ to see that $(A, E_B, D; C, \mu)$ is cobordant to itself.

For symmetry, note that $v^*A = (vA)^* \subseteq C^*((1 \otimes \psi)(\mathscr{L}\mathscr{T}\mathscr{R}) \mid \psi \in \mathscr{S}_c(C))$ so that making the substitution of v^* for v reverses the roles of $(A, E_{1,B}, D_1; C_1, \mu_1)$ and $(A, E_{2,B}, D_2; C_2, \mu_2)$.

For transitivity, suppose that $(A, E_B, D; C, \mu; v)$ is a cobordism between $(A, E_{1,B}, D_1; C_1, \mu_1)$ and $(A, E_{2,B}, D_2; C_2, \mu_2)$, and $(A, E'_B, D'; C', \mu'; v')$ is a cobordism between $(A, E_{2,B}, D_2; C_2, \mu_2)$ and $(A, E_{3,B}, D_3; C_3, \mu_3)$. Let $U : (1 - v^*v)E \to E_2$ and $U' : (1 - v'v'^*)E \to E_2$ be the unitary equivalences between the cycles

$$(A, (1 - v^*v)E_B, (1 - v^*v)D(1 - v^*v)) \quad \text{and} \quad (A, (1 - v'v'^*)E'_B, (1 - v'v'^*)D'(1 - v'v'^*))$$

and the cycle $(A, E_{2,B}, D_2)$. Then

$$(A, (E \oplus E')_B, D \oplus D'; C \oplus C_2 \oplus C', \mu \oplus 1 \oplus v^*v + U^*\mu_2 U \oplus v'v'^* + U'^*\mu_2 U' \oplus 1 \oplus \mu'; v + U'^*U + v'),$$

is a cobordism between $(A, E_{1,B}, D_1; C_1, \mu_1)$ and $(A, E_{3,B}, D_3; C_3, \mu_3)$, where

$$\mu \oplus 1 \oplus v^*v + U^*\mu_2 U \oplus v'v'^* + U'^*\mu_2 U' \oplus 1 \oplus \mu' \\ \in (E \otimes C) \oplus (E' \otimes C) \oplus (E \otimes C_2) \oplus (E' \otimes C_2) \oplus (E \otimes C') \oplus (E' \otimes C').$$

We have

$$(v + U'^*U + v')(v + U'^*U + v')^* = vv^* \oplus 1 \qquad (v + U'^*U + v')^*(v + U'^*U + v') = 1 \oplus v'^*v'.$$

Let $\mathscr{L}''$, $\mathscr{T}''$, and $\mathscr{R}''$ be the spaces of Definition III.4.1, corresponding to this cycle. We have

$$\mathscr{L} \oplus \mathscr{L}' \subseteq \mathscr{L}'' \qquad \mathscr{T} \oplus \mathscr{T}' \subseteq \mathscr{T}'' \qquad \mathscr{R} \oplus \mathscr{R}' \subseteq \mathscr{R}'',$$

so that $(v+v')A \subseteq C^*((1 \otimes \psi)(\mathscr{L}''\mathscr{T}''\mathscr{R}'') \mid \psi \in \mathcal{S}_c(C \oplus C_2 \oplus C'))$. Because D commutes with $(1-v^*v)$ and D' commutes with $(1 - v'v'^*)$, $D'U'^*U = U'^*D_2U = U'^*UD$ on $E \oplus E'$. Hence

$$U'^*\mathscr{L}_2\mathscr{T}_2\mathscr{R}_2 U \subseteq \mathscr{L}''\mathscr{T}''\mathscr{R}''$$

and

$$U'^*UA = U'^*AU \subseteq U'^*C^*((1 \otimes \psi)(\mathscr{L}_2\mathscr{T}_2\mathscr{R}_2) \mid \psi \in \mathcal{S}_c(C_2))U \\ \subseteq C^*((1 \otimes \psi)(\mathscr{L}''\mathscr{T}''\mathscr{R}'') \mid \psi \in \mathcal{S}_c(C \oplus C_2 \oplus C'))$$

as required. □

Unlike additive perturbations of unbounded Kasparov modules, conformal transformations are not necessarily reversible nor composable. The extra room in the definition of conformism circumvents this issue. As a special case of Proposition III.4.23, we have

Corollary III.4.24. *Conformism of unbounded Kasparov modules is an equivalence relation and is compatible with direct sums.*

Proposition III.4.25. *Given two cobordant conformally generated cycles $(A, E_{1,B}, D_1; C_1, \mu_1)$ and $(A, E_{2,B}, D_2; C_2, \mu_2)$, their bounded transforms $(A, E_{1,B}, F_{D_1})$ and $(A, E_{2,B}, F_{D_2})$ are cobordant and so define the same element in $KK(A, B)$.*

Proof. Let $(A, E_B, D; C, \mu; v)$ be a cobordism between $(A, E_{1,B}, D_1; C_1, \mu_1)$ and $(A, E_{2,B}, D_2; C_2, \mu_2)$. By Theorem III.4.4, (A, E_B, F_D) is a bounded Kasparov module and $[F_D, vA] \subseteq \mathrm{End}^0(E)$. By Lemma I.1.7, $F_{(1-vv^*)D(1-vv^*)} = (1 - vv^*)F_D(1 - vv^*)$ on the module $(1 - vv^*)E$ and $F_{(1-v^*v)D(1-v^*v)} = (1 - v^*v)F_D(1 - v^*v)$ on the module $(1 - v^*v)E$. Hence $(A, E_B, F_D; v)$ is a bounded cobordism between (A, E'_B, F_{D_1}) and (A, E''_B, F_{D_2}). □

In the following, we use the notation $Z_{\mathscr{X}}(T) = \{x \in \mathscr{X} \mid [T, x] = 0\}$ for the centraliser in a subspace $\mathscr{X} \subseteq \mathrm{Mtc}^*(E \otimes C, C)$ of an adjointable operator T on $E \otimes C$.

Definition III.4.26. A conformally generated cycle $(A, E_B, D; C, \mu)$ is *positively degenerate* if there exists a self-adjoint unitary $s \in \mathrm{End}^*(E)$ (odd if the cycle is of even parity), preserving the domain of D, such that

- The anticommutator $Ds + sD$ is semibounded below, i.e. $Ds + sD \geq -c$ for some $c > 0$;
- $[\mu, s \otimes 1] = 0$; and
- $A \subseteq C^*((1 \otimes \psi)(Z_{\mathscr{L}}(s \otimes 1)Z_{\mathscr{T}}(s \otimes 1)Z_{\mathscr{R}}(s \otimes 1)) \mid \psi \in \mathcal{S}_c(C))$.

Proposition III.4.27. *A positively degenerate conformally generated cycle $(A, E_B, D; C, \mu)$ is cobordant to the zero cycle $(A, 0_B, 0; 0, 0)$.*

Proof. Let $s \in \mathrm{End}^*(E)$ be a symmetry implementing the degeneracy. Let N be the number operator and S the unilateral shift on $\ell^2(\mathbb{N}_{\geq 0})$. Then

$$(A, E_B \otimes \ell^2(\mathbb{N}_{\geq 0}), D \otimes 1 + s \otimes N; C \oplus \mathbb{C}, (\mu_L \otimes 1 \oplus 1 \otimes 1, \mu_R \otimes 1 \oplus 1 \otimes 1); 1 \otimes S) \tag{III.4.28}$$

is a cobordism from $(A, E_B, D; C, \mu)$ to $(A, 0_B, 0; 0, 0)$. The compactness of the resolvent is as in Proposition I.1.13.

Let $\mathscr{L}'$, $\mathscr{T}'$, and $\mathscr{R}'$ be the spaces of Definition III.4.1, corresponding to the cycle (III.4.28). Using the relation $NS = S(N+1)$, we check that

$$(D \otimes 1 + s \otimes N)(1 \otimes S) - (1 \otimes S)(D \otimes 1 + s \otimes N) = s \otimes [N, S] = s \otimes S$$

is bounded. Hence, noting that $[\mu, s \otimes 1] = 0$,

$$\begin{aligned} \mathscr{L}' &\supseteq Z_{\mathscr{L}}(s \otimes 1) \oplus \mathbb{C}1 \otimes \mathrm{span}\{1, S\} \\ \mathscr{R}' &\supseteq Z_{\mathscr{R}}(s \otimes 1) \oplus \mathbb{C}1 \otimes \mathrm{span}\{1, S\} \\ \mathscr{T}' &\supseteq Z_{\mathscr{T}}(s \otimes 1) \oplus \mathbb{C}1 \otimes \mathrm{span}\{1, S\} \end{aligned}$$

and $(1 \otimes S)A \subseteq C^*((1 \otimes \psi)(\mathscr{L}'\mathscr{T}'\mathscr{R}') \mid \psi \in \mathcal{S}_c(C \oplus \mathbb{C}))$, as required. □

Corollary III.4.29. *Given a conformally generated cycle* $(A, E_B, D; C, \mu)$,

$$(A, E_B, D; C, \mu) \oplus (A, E_B^{(\mathrm{op})}, -D; C, \mu) = \left(A, (E \oplus E^{(\mathrm{op})})_B, \begin{pmatrix} D & \\ & -D \end{pmatrix}; C \oplus C, \mu \oplus 1 \oplus 1 \oplus \mu\right),$$

where $E^{(\mathrm{op})}$ *is* E *with the opposite grading if* E *is graded, is cobordant to* $(A, 0_B, 0; 0, 0)$.

Proof. Using the observations of Remark III.4.16, we may replace the direct sum cycle with

$$\left(A, (E \oplus E^{(\mathrm{op})})_B, \begin{pmatrix} D & \\ & -D \end{pmatrix}; C, \mu\right)$$

and the symmetry $s = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ makes this positively degenerate. □

We thus obtain

Theorem III.4.30. *Cobordism classes of conformally generated A-B-cycles form a* $\mathbb{Z}/2\mathbb{Z}$*-graded abelian group which surjects onto* $KK(A, B)$*. Similarly, conformism classes of unbounded Kasparov A-B-modules form a* $\mathbb{Z}/2\mathbb{Z}$*-graded abelian group which surjects onto* $KK(A, B)$.

Chapter IV

Parabolic noncommutative geometry

In this Chapter, we introduce the notion of tangled cycle, which encompasses the anisotropies arising in parabolic geometry as well as the parabolic commutator bounds arising in so-called 'bad Kasparov products'. Tangled cycles incorporate anisotropy by replacing the unbounded operator in a higher order cycle that mimics a Dirac operator with several unbounded operators mimicking directional Dirac operators. We allow for varying and dependent orders in different directions, controlled by a weighted graph. We study the conformal equivariance of tangled cycles as well as how they fit into KK-theory by means of producing higher order cycles. Our main examples fit into two classes: hypoelliptic spectral triples constructed from Rockland complexes on parabolic geometries and Kasparov product spectral triples for nilpotent group C*-algebras and crossed product C*-algebras of parabolic dynamical systems.

IV.1 Strictly tangled cycles

The operator D in an unbounded cycle will in a strictly tangled cycle be replaced by a collection of operators $\boldsymbol{D}$. We first introduce some terminology, make some preliminary observations and provide some examples that motivate our definition of strictly tangled cycles.

Definition IV.1.1. A collection of self-adjoint regular operators $\boldsymbol{D} = (D_j)_{j\in I}$ on a Hilbert B-module E is said to be strictly anticommuting if, for all $j \neq k$,

$$D_j D_k + D_k D_j = 0$$

on a common core for $(D_j)_{j\in I}$.

Examples IV.1.2.

1. Assume that M_1 and M_2 are two oriented, compact, Riemannian manifolds, with Clifford bundles $E_1 \to M_1$ and $E_2 \to M_2$ with Dirac operators $\not{D}_1$ and $\not{D}_2$ thereon. For simplicity, we assume that the manifolds are even dimensional so all Clifford bundles and Dirac operators are graded. We write $E_1 \hat{\boxtimes} E_2 \to M_1 \times M_2$ for their graded exterior tensor product. By construction, the pair of operators
$$D_1 := \not{D}_1 \hat{\otimes} 1_{E_2} \qquad D_2 := 1_{E_1} \hat{\otimes} \not{D}_2$$
form a strictly anticommuting collection on the Hilbert space $L^2(M_1 \times M_2, E_1 \tilde{\boxtimes} E_2)$. Here the domain of D_1 is the (graded) Hilbert space tensor product $H^1(M_1, E_1) \hat{\otimes} L^2(M_2, E_2)$ and the domain of D_2 is the (graded) Hilbert space tensor product $L^2(M_1, E_1) \hat{\otimes} H^1(M_2, E_2)$. Here $D := D_1 + D_2$ is a Dirac operator on the Clifford bundle $E_1 \tilde{\boxtimes} E_2 \to M_1 \times M_2$. A similar construction can also be made for a foliated manifold [CS84, Kor08], with D_1 being a tangential Dirac operator and D_2 a transversal Dirac operator but in this case $D_1 D_2 + D_2 D_1$ is generally not zero, and only lower order if the foliation is Riemannian.
2. We can more generally consider the (constructive) external Kasparov product. If $(\mathscr{A}_1, H_1, D_1)$ and $(\mathscr{A}_2, H_2, D_2)$ are two higher order spectral triples, their external Kasparov product is constructed as $(\mathscr{A}_1 \otimes \mathscr{A}_2, H_1 \tilde{\otimes} H_2, D_1 \tilde{\otimes} 1 + 1 \tilde{\otimes} D_2)$. Here $(D_1 \tilde{\otimes} 1, 1 \tilde{\otimes} D_2)$ form a strictly anticommuting collection on the Hilbert space $H_1 \tilde{\otimes} H_2$ and their sum is the operator in the external Kasparov product. This example goes back to Baaj–Julg's seminal paper [BJ83] where the unbounded picture was first introduced. The two pairs of strictly anticommuting operators discussed in this example will fit into the framework of ST^{2}s discussed in the next section (see Definition IV.1.7).
3. A more simple-minded example is the direct sum of two higher order spectral triples. If $(\mathscr{A}_1, H_1, D_1)$ and $(\mathscr{A}_2, H_2, D_2)$ are two higher order spectral triples, their direct sum is $(\mathscr{A}_1 \oplus \mathscr{A}_2, H_1 \oplus H_2, D_1 \oplus D_2)$. Albeit in a somewhat trivial way, $(D_1 \oplus 0, 0 \oplus D_2)$ form a strictly anticommuting collection on the Hilbert space $H_1 \oplus H_2$.
4. Let M be a compact Kähler manifold. Write d for the complex dimension of M. We can consider the Dolbeault complex
$$0 \to C^\infty(M) \xrightarrow{\overline{\partial}_1} \Gamma^\infty(\Lambda^1 T^{0,1}M) \xrightarrow{\overline{\partial}_2} \Gamma^\infty(\Lambda^2 T^{0,1}M) \xrightarrow{\overline{\partial}_3} \cdots$$
$$\cdots \xrightarrow{\overline{\partial}_{d-1}} \Gamma^\infty(\Lambda^{d-1} T^{0,1}M) \xrightarrow{\overline{\partial}_d} \Gamma^\infty(\Lambda^{d} T^{0,1}M) \to 0.$$
Here $\Lambda^* T^{0,1}M$ denotes the (complex) exterior algebra of the $(0,1)$-forms. The operators $\not{\partial}_j$ obtained as the closure of $\overline{\partial}_j + \overline{\partial}_j^*$ on $L^2(\Lambda^* T^{0,1}M)$ satisfy for $j \neq k$
$$\not{\partial}_j \not{\partial}_k = 0 = -\not{\partial}_k \not{\partial}_j.$$
In particular, the collection $(\not{\partial}_j)_{j=1}^d$ is a strictly anticommuting collection of operators on $L^2(\Lambda^* T^{0,1}M)$. The collection of strictly anticommuting operators discussed in this example will fit into the framework of ST^{2}s discussed in the next section (see Definition IV.1.7).

We can in fact for any partition $\{1, 2, \dots, d-1, d\} = S_1 \sqcup \cdots \sqcup S_n$ form $D_l := \sum_{j \in S_l} \not{\partial}_j$ and the collection $\boldsymbol{D} = (D_l)_{l=1}^n$ also forms a strictly anticommuting collection. In both of these constructions

$$\Delta^{\boldsymbol{D}}_{(1,1,\dots,1)} = \sum_{j=1}^d \not{\partial}_j^2 = \sum_{j=1}^d \overline{\partial}_j^* \overline{\partial}_j + \overline{\partial}_j \overline{\partial}_j^* = \sum_{l=1}^n D_l^2$$

is the Kodaira Laplacian. In later examples arising from complexes, we see that the orders of the differentials affect which partitions we can choose when building an ST^2; see in particular Remark IV.2.18.

As a simple consequence of the functional calculus, we have

Lemma IV.1.3. *If $\boldsymbol{D} = (D_j)_{j \in I}$ is a strictly anticommuting collection of self-adjoint regular operators on E_B and $\boldsymbol{t} \in (0, \infty)^I$, $(\mathrm{sgn}(D_j)|D_j|^{t_j})_{j \in I}$ is also a strictly anticommuting collection of self-adjoint regular operators.*

We have the following consequence of [LM19, Theorems 2.6, 5.1, 5.4].

Lemma IV.1.4. *If $\boldsymbol{D} = (D_j)_{j \in I}$ is a strictly anticommuting collection of self-adjoint regular operators on E_B and $\boldsymbol{t} \in (0, \infty)^I$, then the operator*

$$\overline{D}_{\boldsymbol{t}} := \sum_{j \in I} \mathrm{sgn}(D_j)|D_j|^{t_j}$$

is self-adjoint and regular with $\mathrm{dom}(\overline{D}_{\boldsymbol{t}}) = \bigcap_{j \in I} \mathrm{dom}\, |D_j|^{t_j}$. Further,

$$\overline{D}_{\boldsymbol{t}}^2 = \sum_{j \in I} |D_j|^{2t_j},$$

with domain $\bigcap_{j \in I} \mathrm{dom}\, |D_j|^{2t_j}$. We also use the notation $\Delta^{\boldsymbol{D}}_{\boldsymbol{t}} = \overline{D}_{\boldsymbol{t}}^2$.

Lemma IV.1.4 certainly does not use the full power of [LM19]. We leave to the future the problem of generalising Definition IV.1.7 to weakly anticommuting collections of operators.

We record the notation $\Delta^{\boldsymbol{D}}_{\boldsymbol{t}} := \overline{D}_{\boldsymbol{t}}^2$.

Lemma IV.1.5. *Let $\boldsymbol{D} = (D_j)_{j \in I}$ be a strictly anticommuting collection of self-adjoint regular operators on E_B. For $\boldsymbol{s}, \boldsymbol{t} \in (0, \infty)^I$ and $\sigma, \tau \in (0, \infty)$ such that $\sigma s_j \geq \tau t_j$ for all $j \in I$, there exists a constant $C > 0$ for which*

$$(1 + \Delta^{\boldsymbol{D}}_{\boldsymbol{s}})^{-\sigma} \leq C(1 + \Delta^{\boldsymbol{D}}_{\boldsymbol{t}})^{-\tau}$$

as positive elements of $\mathrm{End}^*_B(E)$. As a consequence, the following are equivalent for $a \in \mathrm{End}^*_B(E)$:

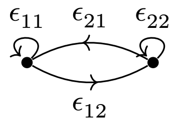
- *$a(1 + \Delta^{\boldsymbol{D}}_{\boldsymbol{t}})^{-1} \in \mathrm{End}^0_B(E)$ for every $\boldsymbol{t} \in (0, \infty)^I$; and*
- *$a(1 + \Delta^{\boldsymbol{D}}_{\boldsymbol{t}})^{-1} \in \mathrm{End}^0_B(E)$ for some $\boldsymbol{t} \in (0, \infty)^I$.*

Proof. First, for $(x_j)_{j \in I} \in [0, \infty)^I$, one can check that

$$\Big(1 + \sum_{j \in I} x_j^{2s_j}\Big)^{-\sigma} \leq C\Big(1 + \sum_{j \in I} x_j^{2t_j}\Big)^{-\tau}$$

for some constant $C > 0$ depending on $\boldsymbol{s}$, $\boldsymbol{t}$, σ, and τ. Noting that $(D_j^2)_{j \in I}$ is a strictly commuting collection of self-adjoint regular operators, the Lemma follows from functional calculus of several commuting operators. $\square$

We will encode the orders (relating to commutator properties as in a higher order spectral triple) of the components in a finite collection $\boldsymbol{D} = (D_j)_{j\in I}$ of self-adjoint operators in a matrix $\boldsymbol{\epsilon} = (\epsilon_{ij})_{i,j\in I} \in M_I([0,\infty))$. Here we write M_I for the matrices indexed by a finite set I. We think of $\boldsymbol{\epsilon}$ pictorially as a weighted directed graph. The weighted directed graph has vertices labelled by I and there is an edge from i to j labelled by ϵ_{ij} whenever $\epsilon_{ij} > 0$. For instance, the diagram



pictorially describes a 2×2 matrix $\boldsymbol{\epsilon} = (\epsilon_{ij})_{i,j=1}^2 \in M_2([0,\infty))$, and, if the reader imagines a larger collection, they may begin to understand our use of the word ‘tangled’ for the main concept of this Chapter.

Definition IV.1.6. We say that a matrix $\boldsymbol{\epsilon} = (\epsilon_{ij})_{i,j\in I} \in M_I([0,\infty))$ satisfies the *decreasing cycle condition* if for any k and $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_k) \in I^k$ with $\gamma_1 = \gamma_k$ we have that

$$\prod_{j=1}^{k} \epsilon_{\gamma_j\gamma_{j+1}} < 1.$$

The decreasing cycle condition means that the total weight along any cycle in the weighted directed graph should be < 1. The condition that $\prod_{j=1}^{k} \epsilon_{\gamma_j\gamma_{j+1}} < 1$ is indeed only a condition appearing along the cycles in the weighted digraph associated with ϵ since $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_k) \in I^k$ represents a cycle if and only if $\prod_{j=1}^{k} \epsilon_{\gamma_j\gamma_{j+1}} > 0$. In particular, if the weighted digraph associated with ϵ has no cycles then ϵ automatically satisfies the decreasing cycle condition. It follows from [Jos21, Lemma 3.23] that $\boldsymbol{\epsilon} \in M_I([0,\infty))$ satisfies the decreasing cycle condition if and only if the convex cone

$$\Omega(\boldsymbol{\epsilon}) := \{\boldsymbol{t} = (t_j) \in (0,\infty)^I : \epsilon_{ij} t_i < t_j \ \forall i,j\}$$

is nonempty.

We can interpret $\boldsymbol{\epsilon}$ as a matrix valued in the *tropical semiring*, in which context $\Omega(\boldsymbol{\epsilon})$ is a well-studied object. The tropical semiring, in the multiplicative convention, is $[0,\infty)$ with addition $\oplus$ given by $x \oplus y = \max\{x,y\}$ and multiplication $\times$ defined just as usual. Remark that 0 is the additive identity, 1 is the multiplicative identity, and multiplication distributes over addition. The reader can find more details on matrices in the tropical semiring and their relationship to weighted directed graphs in [Jos21] (where an additive convention is used for the tropical semiring, related to our multiplicative convention by the logarithm). It seems likely that there is more to be gleaned from interpreting $\boldsymbol{\epsilon}$ as a matrix over the tropical semiring but, for the purposes of this Chapter, it suffices to remember the nonemptiness of the cone $\Omega(\boldsymbol{\epsilon})$ as the antecedent of the decreasing cycle condition.

We now come to the main definition of this Chapter.

Definition IV.1.7. A strictly tangled A-B-cycle consists of an A-B-correspondence E and a finite collection $\boldsymbol{D} = (D_j)_{j\in I}$ of regular operators on E such that for a matrix $\boldsymbol{\epsilon} \in M_I([0,\infty))$ satisfying the decreasing cycle condition (see Definition IV.1.6) we have that

1. every D_j is self-adjoint and $\boldsymbol{D} = (D_j)_{j\in I}$ is strictly-anticommuting
2. for every $a \in A$, $(1 + \Delta_{\boldsymbol{t}}^{\boldsymbol{D}})^{-1} a$ is compact for some $\boldsymbol{t} \in (0,\infty)^I$ (and so for all $\boldsymbol{t} \in (0,\infty)^I$ by Lemma IV.1.5); and
3. A is contained in the closure of $\mathscr{Q}$, the set of $a \in \mathrm{End}^*(E)$ such that, for all $i \in I$, $\{a, a^*\} \operatorname{dom} D_i \subseteq \operatorname{dom} D_i$ and

$$\Big(1 + \sum_{j\in I} |D_j|^{\epsilon_{ij}}\Big)^{-1} [D_i, a] \qquad [D_i, a] \Big(1 + \sum_{j\in I} |D_j|^{\epsilon_{ij}}\Big)^{-1}$$

extend to adjointable operators on $\mathrm{End}^*(E)$.

We refer to $\boldsymbol{\epsilon}$ as a bounding matrix or graph. If E is graded, we require all operators in $\boldsymbol{D}$ to be odd and call $(A, E_B, \boldsymbol{D})$ an *even* strictly tangled cycle. If E is ungraded, $(A, E_B, \boldsymbol{D})$ is *odd.*

If, for a dense $*$-subalgebra $\mathscr{A} \subseteq A$, $\mathscr{A} \subseteq \mathscr{Q}$, we will say that $(\mathscr{A}, E_B, \boldsymbol{D})$ is a strictly tangled $\mathscr{A}$-B-cycle.

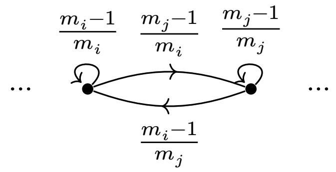
When $B = \mathbb{C}$, we will use the term *strictly tangled spectral triple* or ST^2.

In §IV.1.2, we will show that, from any strictly tangled cycle $(A, E_B, \boldsymbol{D})$, one can build a higher order cycle $(A, E_B, \overline{D}_{\boldsymbol{t}} := \sum_{j\in I} \operatorname{sgn}(D_j)|D_j|^{t_j})$, for suitable $\boldsymbol{t} \in \Omega(\boldsymbol{\epsilon})$. Before this, we make a few observations and give some motivating examples.

Remark IV.1.8. A strictly tangled cycle $(A, E_B, \boldsymbol{D})$ with $n = 1$ is the same as a higher order cycle. Indeed, $\boldsymbol{\epsilon} \in [0, \infty)$ satisfies the decreasing cycle condition if and only if $\boldsymbol{\epsilon} < 1$. In this case, if $\boldsymbol{\epsilon} \in [0, 1)$ is the bounding matrix then $(A, E_B, \boldsymbol{D})$ is an unbounded cycle of order $m = (1 - \boldsymbol{\epsilon})^{-1}$. Furthermore, we point out that there is an implicit lower bound $m \geq 1$ on the order of our higher order spectral triples originating in the requirement on $\boldsymbol{\epsilon}$ to have coefficients in $[0, \infty)$.

Remark IV.1.9. We note that our definition of a strictly tangled spectral triple is somewhat restrictive in requiring the elements of the collection $\boldsymbol{D} = (D_j)_{j\in I}$ to be strictly anticommuting. We expect that this definition can be relaxed to include collections $\boldsymbol{D} = (D_j)_{j\in I}$ on which there is a size constraint on the anticommutator $D_j D_k + D_k D_j$ along the lines of for instance [LM19]. For our applications to complexes, in particular Rockland complexes, we will make do with strictly anticommuting collections but in order for more general applications to Rockland sequences [DH22, GK24, Gof24] and more general Kasparov product constructions [GM15, KL13, Mes12] to fit into the framework one needs to extend the notion above to a weaker anticommutation condition. See Remark IV.1.19 for further comments on where in the proofs this is used.

Remark IV.1.10. If the operators $\boldsymbol{D} = (D_j)_{j\in I}$ have a prescribed order $\boldsymbol{m} = (m_j)_{j\in I} \in [1, \infty)^I$ in an appropriate sense, e.g. in some pseudodifferential calculus, there is an intuitive guess of bounding matrix $\boldsymbol{\epsilon}$. Similar to the intuition of a higher order spectral triple, a commutator $[D_i, a]$ should behave like one order lower than D_i and therefore be controlled by operators of order $m_i - 1$. Therefore, a natural choice that turns out to be correct in examples is the bounding matrix $\epsilon_{ij} = \frac{m_i - 1}{m_j}$, for $i, j \in I$, represented by the weighted digraph



which also matches the order of a higher order spectral triple in that $m_i = (1 - \epsilon_{ii})^{-1}$. Such an $\boldsymbol{\epsilon}$ fulfils the decreasing cycle condition since

$$\prod_{j=1}^{k} \epsilon_{\gamma_j, \gamma_{j+1}} = \frac{\prod_{j=1}^{k}(m_{\gamma_j} - 1)}{\prod_{j=1}^{k} m_{\gamma_{j+1}}} = \frac{\prod_{j=1}^{k}(m_{\gamma_j} - 1)}{\prod_{j=1}^{k} m_{\gamma_j}} = \prod_{j=1}^{k} \left(1 - \frac{1}{m_{\gamma_j}}\right) < 1 \tag{IV.1.11}$$

for any cycle $\gamma = (\gamma_1, \dots, \gamma_k)$, where we use the cycle property $\gamma_1 = \gamma_k$ in the second equality. In particular, $\Omega(\boldsymbol{\epsilon})$ contains a ray of the form

$$\boldsymbol{t}_{\boldsymbol{m}}(\tau) := \left(\frac{\tau}{m_j}\right)_{j\in I} \in \Omega(\boldsymbol{\epsilon}), \quad \tau > 0.$$

Indeed, $\boldsymbol{t}_{\boldsymbol{m}}(\tau) \in \Omega(\boldsymbol{\epsilon})$ since $\epsilon_{ij} t_i = \epsilon_{ii} t_j < t_j$. The operator

$$\overline{D}_\tau := \overline{D}_{\boldsymbol{t}_{\boldsymbol{m}}(\tau)} = \sum_{j\in I} \operatorname{sgn}(D_j)|D_j|^{\frac{\tau}{m_j}}$$

constructed from this ray should then morally be a sum of operators of order τ, which is discussed further in Remark IV.1.23 and placed in a solid mathematical foundation in Proposition IV.2.7.

Operators with prescribed orders $\boldsymbol{m}$ with this type of bounding matrix $\boldsymbol{\epsilon}$ will be considered further in §IV.2 in the context of complexes. For an ST^2 arising from a complex, we will depart from the preceding discussion by setting $\epsilon_{ij} = 0$ if the operators D_i and D_j are 'far apart' in the complex. When we consider the C*-algebras of nilpotent groups in §IV.3.1, we will see that one does not always have a natural prescription of orders. However, in the case of a Carnot group, there will be a natural way of assigning orders, related to conformal equivariance under the dilation action.

IV.1.1 Three motivating examples

Before delving into the general theory and the main examples of this Chapter, we provide some simpler examples to clarify and justify the structure underlying ST^2s. Further examples, generalising these, will be presented in §§IV.2.4, IV.3.1, and IV.3.2.

The Rumin complex on a contact manifold, which we discussed in §III.2.1.3, is an example of a Rockland complex. We will consider Rockland complexes in §IV.2 and return to explain and study Rumin complexes in more generality in §IV.2.4. Let us start with the simplest situation to explain the ideas motivating the notion of ST^2s.

Example IV.1.12. We consider the 3-dimensional Heisenberg group H_3. As a manifold, H_3 coincides with $\mathbb{R}^3$ but is equipped with the product $(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy')$. We write Γ for the cocompact subgroup defined from the integer points $\mathbb{Z}^3$. On the nilmanifold $M = \mathsf{H}_3/\Gamma$, the Rumin complex takes the form

$$\begin{tikzcd}
 & & C^\infty(M)\mathrm{d}x \arrow[r, "Z+XY"] \arrow[rdd, "Y^2" near end] & C^\infty(M)\mathrm{d}x\wedge\theta \arrow[rd, "Y"] & & \\
0 \arrow[r] & C^\infty(M) \arrow[ru, "X"] \arrow[rd, "Y"'] & & & C^\infty(M)\mathrm{d}x\wedge\mathrm{d}y\wedge\theta \arrow[r] & 0 \\
 & & C^\infty(M)\mathrm{d}y \arrow[r, "Z-YX"'] \arrow[ruu, "-X^2"' near end] & C^\infty(M)\mathrm{d}y\wedge\theta \arrow[ru, "-X"'] & &
\end{tikzcd}$$

where $X = \partial_x - y\partial_z$, $Y = \partial_y$, and $Z = \partial_z$ are the standard basis elements of the Heisenberg Lie algebra with the commutator identity $[X, Y] = Z$, here acting as vector fields on M. Here $\theta = y\mathrm{d}x + \mathrm{d}z$ denotes the contact form. We equip M with the volume density induced from the Haar measure on H_3/Γ and declare $\mathrm{d}x$, $\mathrm{d}y$ and θ to be an orthonormal frame. With these choices, the Rumin complex above is completed into a Hilbert complex, see [BL92] or §IV.2 below.

We shall shorten the notation for the operators in the Rumin complex to $\mathrm{d}^R_\bullet = (\mathrm{d}^R_0, \mathrm{d}^R_1, \mathrm{d}^R_2)$. It is a mixed order differential complex. Let

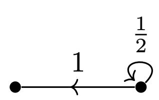
$$D_1 := \mathrm{d}^R_0 + (\mathrm{d}^R_0)^* + \mathrm{d}^R_2 + (\mathrm{d}^R_2)^* \quad \text{and} \quad D_2 := \mathrm{d}^R_1 + (\mathrm{d}^R_1)^*.$$

We view D_1 and D_2 as densely defined, self-adjoint operators on $L^2(M; \mathcal{H})$ where $\mathcal{H} \to M$ is the sum of all line bundles appearing in the Rumin complex; so $\mathcal{H} \cong M \times \mathbb{C}^6$. The differential operators D_1 and D_2 are of order $m_1 = 1$ and $m_2 = 2$ respectively. We note that $D_1 D_2 = D_2 D_1 = 0$, so D_1 and D_2 are strictly anticommuting. The Rumin Laplacian takes the form

$$\Delta^R = D_1^4 + D_2^2.$$

The data $(C^\infty(M), L^2(M; \mathcal{H}), (D_1, D_2))$ constitute an ST^2 with bounding matrix

$$\boldsymbol{\epsilon} = \begin{pmatrix} 0 & 0 \\ 1 & \frac{1}{2} \end{pmatrix}$$



.

If one squints, the bounding matrix can be guessed from the structure of the Rumin complex. The diagonal arrows, corresponding to the operators $-X^2$ and Y^2, require the weight-1/2 loop and the horizontal arrows, corresponding to the operators $Z+XY$ and $Z-YX$, require the weight-1 edge from D_2 to D_1. The other arrows, forming part of D_1 are all first order, making no contribution to the bounding matrix. Of course, this is not a rigorous argument; for that we will have to wait until §IV.2.4.

In particular, for any $\boldsymbol{t}=(t_1,t_2)\in(0,\infty)^2$ with $t_1>t_2$, we arrive at a higher order spectral triple with Dirac operator

$$\overline{D}_{\boldsymbol{t}}=D_1|D_1|^{t_1-1}+D_2|D_2|^{t_2-1}=D_1(\Delta^R)^{\frac{t_1-1}{4}}+D_2(\Delta^R)^{\frac{t_2-1}{2}}.$$

If $\boldsymbol{t}$ lies along the ray spanned by $(1,1/2)$ then $\overline{D}_{\boldsymbol{t}}$ is an H-elliptic operator in the Heisenberg calculus and if $\boldsymbol{t}=(2k_1+1,2k_2+1)$ where $k_1>k_2$ are natural numbers then $\overline{D}_{\boldsymbol{t}}$ is a differential operator; see §IV.2.4 for more details.

Below in §IV.2.4.1, we will show that the naïvely formed candidate D_1+D_2 for a noncommutative geometry on M fails to be a higher order spectral triple, motivating the need for ST^{2}s.

In the following Example, we show that the order-2 spectral triple for the C*-algebra of the Heisenberg group built in §II.4.2 and studied further in Example III.2.10 naturally arises from a strictly tangled spectral triple.

Example IV.1.13. Let H_3 be the 3-dimensional Heisenberg group. In the 3×3-matrix presentation, we can write

$$\mathsf{H}_3=\left\{g\in M_3(\mathbb{R}):g=\begin{pmatrix}1&a&c\\0&1&b\\0&0&1\end{pmatrix}\right\}$$

The group H_3 is a central extension of $\mathbb{R}^2$ by $\mathbb{R}$, fitting into the exact sequence

$$0\longrightarrow\mathbb{R}\xrightarrow{\ \iota\ }\mathsf{H}_3\xrightarrow{\ \pi\ }\mathbb{R}^2\longrightarrow0.$$

As in §II.4.2, we begin with the weights

$$\begin{aligned}\ell_{\mathbb{R}}&:\mathbb{R}\to\mathbb{C} & \qquad\qquad \ell_{\mathbb{R}^2}&:\mathbb{R}^2\to\mathscr{C}\ell_2=\operatorname{End}\mathbb{C}^2\\ c&\mapsto c & (a,b)&\mapsto a\gamma_1+b\gamma_2\end{aligned}$$

Again, let us define a weight $\widetilde{\ell_{\mathbb{R}}}:\mathsf{H}_3\to\mathbb{C}$ by $\widetilde{\ell_{\mathbb{R}}}(g)=c$. As we saw in §II.4.2, $\widetilde{\ell_{\mathbb{R}}}$ exhibits the 'parabolic' feature that

$$\left\|\Big(\widetilde{\ell_{\mathbb{R}}}(gh)-\widetilde{\ell_{\mathbb{R}}}(h)\Big)(1+|\pi^*(\ell_{\mathbb{R}^2})(h)|)^{-1}\right\|=|c+ab'|(1+({a'}^2+{b'}^2)^{1/2})^{-1}\le\|\widetilde{\ell_{\mathbb{R}}}(g)\|+\|\pi^*(\ell_{\mathbb{R}^2})(g)\|$$

so that

$$\sup_h\left\|\Big(\widetilde{\ell_{\mathbb{R}}}(gh)-\widetilde{\ell_{\mathbb{R}}}(h)\Big)(1+|\pi^*(\ell_{\mathbb{R}^2})(h)|)^{-1}\right\|<\infty.$$

Further,

$$\sup_h\|\pi^*(\ell_{\mathbb{R}^2})(gh)-\pi^*(\ell_{\mathbb{R}^2})(h)\|=\|\pi^*(\ell_{\mathbb{R}^2})(g)\|<\infty.$$

We therefore have a strictly tangled spectral triple

$$\Big(C^*(\mathsf{H}_3),L^2(\mathsf{H}_3,\mathbb{C}^2),(M_{\pi^*(\ell_{\mathbb{R}^2})},M_{\widetilde{\ell_{\mathbb{R}}}})\Big)$$

with bounding matrix

$$\boldsymbol{\epsilon}=\begin{pmatrix}0&0\\1&0\end{pmatrix}\qquad \bullet\xleftarrow{\ 1\ }\bullet\ .$$

The set $\Omega(\boldsymbol{\epsilon})$ consists of $t_1,t_2\in(0,\infty)$ such that $t_1>t_2$. We recover the order-2 spectral triple of §II.4.2 by taking $t_1=2$ and $t_2=1$.

In §IV.3.1, Example IV.1.13 will be generalized to all simply connected nilpotent Lie groups (and their closed subgroups) where there are as many Dirac operators as the step length in the group.

In §IV.3.2, we will see that ST^{2}s allow us to generalise the construction of spectral triples for elliptic dynamical systems by Bellissard, Marcolli, and Reihani [BMR10] to parabolic dynamical systems, including nilflows, horocycle flows, and large diffeomorphisms of tori, classical and noncommutative. We here give a simple instance of this latter family of examples.

Example IV.1.14. Consider the group of diffeomorphisms $(\phi_n)_{n\in\mathbb{Z}}$ of $\mathbb{T}^2$ given by

$$\phi_n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{Z}).$$

This family of diffeomorphisms is *large*, in the sense that each ϕ_n is in a distinct connected component of the diffeomorphism group of $\mathbb{T}^2$. This induces a $\mathbb{Z}$-action α on $C(\mathbb{T}^2)$ given by $\alpha_n(a) := \phi^*_{-n}(a)$ for $a \in C(\mathbb{T}^2)$, preserving $C^\infty(\mathbb{T}^2)$. Let $(C^\infty(\mathbb{T}^2), L^2(\mathbb{T}^2,\mathbb{C}^2), D)$ be the Dirac spectral triple on the torus. With N the number operator on $\ell^2(\mathbb{Z})$, we write $(C^\infty(\mathbb{T}^2) \rtimes \mathbb{Z}, \ell^2(\mathbb{Z}) \otimes C(\mathbb{T}^2)_{C(\mathbb{T}^2)}, N \otimes 1)$ for the unbounded Kasparov module associated with the crossed product.

In attempting to form the Kasparov product, we encounter the pointwise-boundedness condition of [Pat14, §1], reproduced (and generalised) in Definition IV.3.12 below. For $a \in C^\infty(\mathbb{T}^2) \rtimes \mathbb{Z}$, we require uniform boundedness of $\|[D, \alpha_n(a)]\|$ in n. Let us see how $\|[D, \alpha_n(a)]\|$ behaves as $|n| \to \infty$. For $a \in C^\infty(\mathbb{T}^2)$

$$\alpha_n(a)(x, y) = a(x - ny, y),$$

and

$$D = \gamma_1\partial_x + \gamma_2\partial_y,$$

so

$$[D, \alpha_n(a)] = \gamma_1\phi^*_{-n}(\partial_x a) + \gamma_2\left(\phi^*_{-n}(\partial_y a) - n\phi^*_{-n}(\partial_x a)\right).$$

We conclude that there is a constant $C > 0$ such that, for any $a \in C^\infty(\mathbb{T}^2)$ and $n \in \mathbb{Z}$,

$$|n|\|\partial_x a\|_{L^\infty} - C\|\nabla a\|_{L^\infty} \le \|[D, \alpha_n(a)]\| \le |n|\|\partial_x a\|_{L^\infty} + C\|\nabla a\|_{L^\infty}.$$

We see that the pointwise-boundedness condition is not satisfied, rather we have the growth behaviour $\|[D, \alpha_n(a)]\| \sim |n|\|\partial_x a\|_{L^\infty}$ as $|n| \to \infty$. Hence

$$[1 \otimes D, \pi(a)](1 + |N|)^{-1} \otimes 1$$

is bounded. In particular, the collection

$$(C_0(\mathbb{T}^2) \rtimes \mathbb{Z}, \ell^2(\mathbb{Z}) \otimes L^2(\mathbb{T}^2, S), (N \otimes \gamma, 1 \otimes D))$$

is a strictly tangled spectral triple with bounding matrix

$$\epsilon = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \bullet \xleftarrow{\;1\;} \bullet$$

and $\Omega(\epsilon) = \{(t_1, t_2) \in (0,\infty)^2 : t_1 > t_2\}$.

IV.1.2 Assembling a strictly tangled cycle into a higher order cycle

Let us study how to construct a higher order cycle from a strictly tangled cycle. In conjunction with the bounded transform, we will see that there is a well-defined KK-class associated with a strictly tangled cycle.

Definition IV.1.15. A strictly $\boldsymbol{\epsilon}$-tangled cycle $(A, E_B, \boldsymbol{D})$ is *$\boldsymbol{\rho}$-preserving* for $\boldsymbol{\rho} \in [1,\infty]^I$ if A is contained in the closure of $\mathscr{Q}$, the set of $a \in \operatorname{End}^*(E)$ such that, for all $i \in I$, $\{a, a^*\} \operatorname{dom} |D_i|^{\rho_i} \subseteq \operatorname{dom} |D_i|^{\rho_i}$ and

$$\Big(1 + \sum_{j\in I} |D_j|^{\epsilon_{ij}}\Big)^{-1} [D_i, a] \qquad [D_i, a]\Big(1 + \sum_{j\in I} |D_j|^{\epsilon_{ij}}\Big)^{-1}$$

extend to adjointable operators on $\operatorname{End}^*(E)$. If $\rho_i = \infty$, the condition $\{a, a^*\} \operatorname{dom} |D_i|^{\rho_i} \subseteq \operatorname{dom} |D_i|^{\rho_i}$ should be interpreted as requiring that $\{a, a^*\} \operatorname{dom} |D_i|^t \subseteq \operatorname{dom} |D_i|^t$ for all $t \geq 1$.

If, for a dense $*$-subalgebra $\mathscr{A} \subseteq A$, $\mathscr{A} \subseteq \mathscr{Q}$, we will say that $(\mathscr{A}, E_B, \boldsymbol{D})$ is $\boldsymbol{\rho}$-preserving.

Every strictly tangled cycle is by definition $\boldsymbol{\rho}$-preserving for $\boldsymbol{\rho} = (1, \dots, 1)$ and, if a strictly tangled cycle is $\boldsymbol{\rho}$-preserving, it is $\boldsymbol{\sigma}$-preserving for all $\boldsymbol{\sigma} \leq \boldsymbol{\rho}$ by Theorem A.3.4. Recall that

$$\Omega(\boldsymbol{\epsilon}) = \{\boldsymbol{t} = (t_j) \in (0,\infty)^I : \epsilon_{ij} t_i < t_j \ \forall i, j\},$$

For $\boldsymbol{\rho} \in [1,\infty]^I$, we will define the subset

$$\Omega(\boldsymbol{\epsilon}, \boldsymbol{\rho}) = \Omega(\boldsymbol{\epsilon}) \cap \prod_{j\in I} (0,1] \cup (1, \rho_j).$$

Here, the interval $(0,1] \cup (1,\rho_j) = (0,\rho_j) \cup \{1\}$ is simply the half-open interval $(0,1]$ if $\rho_j = 1$ and the open interval $(0, \rho_j)$ if $\rho_j > 1$. We remark that $\Omega(\boldsymbol{\epsilon}, \boldsymbol{\rho})$ is a convex set.

Theorem IV.1.16. *Let $(A, E_B, \boldsymbol{D} = \boldsymbol{D} = (D_j)_{j\in I})$ be a strictly $\boldsymbol{\epsilon}$-tangled cycle which is $\boldsymbol{\rho}$-preserving. For $\boldsymbol{t} \in (0,\infty)^I$, we define the operator*

$$\overline{D}_{\boldsymbol{t}} = \sum_{j\in I} \operatorname{sgn}(D_j) |D_j|^{t_j}.$$

If $\boldsymbol{t} \in \Omega(\boldsymbol{\epsilon}, \boldsymbol{\rho})$, then the triple $(A, E_B, \overline{D}_{\boldsymbol{t}})$ defines an order-m cycle for any

$$m > \max_{i,j\in I} \ \max\left\{1, \frac{\rho_i - 1}{\rho_i - t_i} t_i\right\}\left(1 - \frac{\epsilon_{ij} t_i}{t_j}\right)^{-1}. \tag{IV.1.17}$$

(If $\rho_i = \infty$ for some $i \in I$, we interpret $\frac{\rho_i - 1}{\rho_i - t_i}$ as 1. If $\rho_i = t_i = 1$ for some $i \in I$, we also interpret $\frac{\rho_i-1}{\rho_i-t_i}$ as 1.)

If $(\mathscr{A}, E_B, \boldsymbol{D} = (D_j)_{j\in I})$ is a $\boldsymbol{\rho}$-preserving strictly $\boldsymbol{\epsilon}$-tangled cycle, for $\boldsymbol{t} \in \Omega(\boldsymbol{\epsilon}, \boldsymbol{\rho})$, $(\mathscr{A}, E_B, \overline{D}_{\boldsymbol{t}})$ is an order-m cycle for m as in (IV.1.17).

We remark that it is impossible for $1 \neq \rho_i = t_i$. To prove Theorem IV.1.16, we use results from §A.3 about fractional powers and interpolation.

Proof. Let $\boldsymbol{t} \in \Omega(\boldsymbol{\epsilon}, \boldsymbol{\rho})$. The local compactness of the resolvent follows immediately from Lemma IV.1.5. We now proceed to show that, for all $i \in I$ and $a \in \mathscr{Q}$ (where $\mathscr{Q}$ is as in Definition IV.1.15),

$$[\operatorname{sgn}(D_i)|D_i|^{t_i}, a]\left(1 + \sum_{j\in I} |D_j|^{t_j}\right)^{-1+\frac{1}{m}} \tag{IV.1.18}$$

is bounded. If $t_i = 1$, again using Lemma IV.1.5 we see that (IV.1.18) is bounded if $(1 - 1/m)t_j \geq \epsilon_{ij} t_i$ for all j, which is equivalent to $m \geq (1 - \epsilon_{ij} t_i / t_j)^{-1}$. In the context of Proposition A.3.8, let

$$A = D_i \qquad B = 1 + \sum_{j\in I} |D_j|^{t_j}$$

and

$$\alpha_1 = 1 \qquad \beta_1 = \max_{j\in I} \frac{\epsilon_{ij}}{t_j} \qquad \alpha_2 = t_i \qquad \beta_2 = 1 - \frac{1}{m}.$$

We see that (IV.1.18) is bounded if $1-1/m > \max_{j\in I}\epsilon_{ij}t_i/t_j$, equivalent to $m > \max_{j\in I}(1-\epsilon_{ij}t_i/t_j)^{-1}$. If $1<\rho_i<\infty$ and $t_i\in(1,\rho_i)$, still in the context of Proposition A.3.8, let $\alpha_3=\rho_i$ and $\beta_3=\rho_i/t_i$. We see that (IV.1.18) is bounded if

$$1-\frac{1}{m} > \frac{1}{\rho_i-1}\left((\rho_i-t_i)\max_{j\in I}\frac{\epsilon_{ij}}{t_j}+(t_i-1)\frac{\rho_i}{t_i}\right),$$

equivalent to

$$m > \max_{j\in I}\frac{\rho_i-1}{\rho_i-t_i}t_i\left(1-\frac{\epsilon_{ij}t_i}{t_j}\right)^{-1}.$$

If $\rho_i=\infty$ and $t_i\in(1,\rho_i)$, we see by taking the limit that (IV.1.18) is bounded if

$$m > t_i\left(1-\max_{j\in I}\frac{\epsilon_{ij}t_i}{t_j}\right)^{-1}.$$

Noting that $\frac{\rho_i-1}{\rho_i-t_i}t_i>1$ if and only if $\rho_i,t_i>1$, we thus obtain the claimed order estimate. □

Remark IV.1.19. Theorem IV.1.16 is proven under strong assumptions on the anticommutators $D_jD_k+D_kD_j$, namely that they vanish for $j\neq k$. We expect that Theorem IV.1.16 holds under much milder assumptions on the anticommutators $D_jD_k+D_kD_j$. In the proof of Theorem IV.1.16, we rely heavily on Proposition A.3.8 for $A=D_i$ and $B=\Delta^{\boldsymbol{D}}_{t/2}$. Assumptions such as those in [KL13, KL12, LM19], modified according to an $\boldsymbol{\epsilon}$-power of $\boldsymbol{D}$, may allow one to extend Theorem IV.1.16.

Let us discuss a prototypical example to which Theorem IV.1.16 extends, despite a lack of vanishing anticommutators. In [CM95, §1.1–2], an order-2 spectral triple

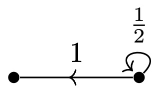
$$\left(C_c^\infty(M)\rtimes\Gamma, L^2(M,\Lambda^*V^*\otimes\Lambda^*N^*), (\mathrm{d}_L\mathrm{d}_L^*-\mathrm{d}_L^*\mathrm{d}_L)(-1)^{\partial_N}+\mathrm{d}_H+\mathrm{d}_H^*\right) \tag{IV.1.20}$$

is built from the data of a manifold M with triangular structure preserved by a group of diffeomorphisms Γ. To arrive at this higher order spectral triple, the longitudinal signature operator $\mathrm{d}_L+\mathrm{d}_L^*$ is first found to be homotopic to $\Delta_L^{-1/2}(\mathrm{d}_L\mathrm{d}_L^*-\mathrm{d}_L^*\mathrm{d}_L)$. At this point, we can consider the collection

$$\left(C_c^\infty(M)\rtimes\Gamma, L^2(M,\Lambda^*V^*\otimes\Lambda^*N^*), \left(\Delta_L^{-1/2}(\mathrm{d}_L\mathrm{d}_L^*-\mathrm{d}_L^*\mathrm{d}_L)(-1)^{\partial_N}, \mathrm{d}_H+\mathrm{d}_H^*\right)\right). \tag{IV.1.21}$$

The operators in the collection (IV.1.21) are not strictly anticommmuting but the anticommutators are of lower order in the pseudodifferential calculus of [CM95]. The pseudodifferential calculus allow us to think of (IV.1.21) as a 'tangled spectral triple' with bounding matrix

$$\boldsymbol{\epsilon}=\begin{pmatrix}0&0\\1&\frac{1}{2}\end{pmatrix}$$



so that taking $\boldsymbol{t}=(2,1)$ produces the order-2 spectral triple (IV.1.20).

Remark IV.1.22. Let us consider the consequences of Theorem IV.1.16 in the special case when the collection $\boldsymbol{D}$ has only one element. Let (A,E_B,D) be an order-m cycle which is ρ-preserving for $\rho\in[0,\infty]$. For $t\in(0,1]\cup(1,\rho)$, $(A,E_B,\mathrm{sgn}(D)|D|^t)$ is an order-m' cycle for

$$m' > m\ \max\left\{1,\frac{\rho-1}{\rho-t}t\right\}$$

If $\rho=\infty$, this means $m'>m\ \max\{1,t\}$.

In examples, it is frequently the case that the requirement in (IV.1.17) may be taken as an equality.

Remark IV.1.23. For an strictly tangled cycle $(A, E_B, \boldsymbol{D})$ such that the operators $\boldsymbol{D}$ have prescribed orders $\boldsymbol{m} \in [1,\infty)^I$ with bounding matrix on the form $\epsilon_{ij} = \frac{m_i - 1}{m_j}$, as in Remark IV.1.10, then for any $\tau > 0$ we would like the higher order cycle $(A, E_B, \overline{D}_\tau)$ to be of order τ. Abstractly, Theorem IV.1.16 guarantees the order to be at most $\tau + \delta$ for any $\delta > 0$. In the examples below coming from Rockland complexes (see Corollary IV.2.20), the pseudodifferential calculus ensures that the order can be taken to be τ on the nose.

In light of the bounded transform for higher order cycles, Theorem IV.1.16 implies the following.

Corollary IV.1.24. *Let $(A, E_B, \boldsymbol{D})$ be a $\boldsymbol{\rho}$-preserving strictly $\boldsymbol{\epsilon}$-tangled cycle. There is a well-defined class*

$$[(A, E_B, \boldsymbol{D})] := [(A, E_B, F_{\overline{D}_{\boldsymbol{t}}})] \in KK_*(A, B)$$

for any $\boldsymbol{t} \in \Omega(\boldsymbol{\epsilon}, \boldsymbol{\rho})$ with the same parity as $(A, E_B, \boldsymbol{D})$. The class $[(A, E_B, \boldsymbol{D})]$ depends only on $(A, E_B, \boldsymbol{D})$ and not on $\boldsymbol{t}$.

Proof. That a class in KK-theory is obtain for any $\boldsymbol{t} \in \Omega(\boldsymbol{\epsilon}, \boldsymbol{\rho})$ follows immediately from the bounded transform for higher order cycles, Corollary I.0.7. Consider distinct $\boldsymbol{s}, \boldsymbol{t} \in \Omega(\boldsymbol{\epsilon}, \boldsymbol{\rho})$. Since $\Omega(\boldsymbol{\epsilon}, \boldsymbol{\rho})$ is a convex set, $x\boldsymbol{s} + (1-x)\boldsymbol{t} \in \Omega(\boldsymbol{\epsilon}, \boldsymbol{\rho})$ for all $x \in [0,1]$. That $(A, E_B, F_{\overline{D}_{\boldsymbol{s}}})$ and $(A, E_B, F_{\overline{D}_{\boldsymbol{t}}})$ are equivalent can then be shown by taking the straight line homotopy. □

We note that in Corollary IV.1.24, the fact that we retain the sign of each D_j in the combined operator $\overline{D}_{\boldsymbol{t}} = \sum_{j\in I} \operatorname{sgn}(D_j)|D_j|^{t_j}$ ensures that the KK-classes can be non-trivial also when a component of $\boldsymbol{t}$ is an even integer.

Theorem IV.1.25. *Let $(A_1, E_{1,B_1}, \boldsymbol{D_1})$ and $(A_2, E_{2,B_2}, \boldsymbol{D_2})$ be two strictly tangled cycles with bounding matrices $\boldsymbol{\epsilon_1}$ and $\boldsymbol{\epsilon_2}$ respectively. Here we write $\boldsymbol{D_1} = (D_{1,j})_{j\in I_1}$ and $\boldsymbol{D_2} = (D_{2,k})_{k\in I_2}$. Then, with the collection $\boldsymbol{D_1} \tilde{\otimes} 1 \sqcup 1 \tilde{\otimes} \boldsymbol{D_2} = (\hat{D}_l)_{l\in I_1 \sqcup I_2}$, given by*

$$\hat{D}_l := \begin{cases} D_{1,l} \tilde{\otimes} 1 & l \in I_1 \\ 1 \tilde{\otimes} D_{2,l} & l \in I_2 \end{cases},$$

the data

$$(A_1 \otimes A_2, (E_1 \tilde{\otimes}_{\mathbb{C}} E_2)_{B_1\otimes B_2}, \boldsymbol{D_1} \tilde{\otimes} 1 \sqcup 1 \tilde{\otimes} \boldsymbol{D_2})$$

constitute a strictly tangled cycle with bounding matrix the direct sum $\boldsymbol{\epsilon_1} \oplus \boldsymbol{\epsilon_2}$. Moreover, the exterior Kasparov product of the associated KK-classes can be written as

$$[(A_1, E_{1,B_1}, \boldsymbol{D}_1)] \otimes_{\mathbb{C}} [(A_2, E_{2,B_2}, \boldsymbol{D}_2)] = [(A_1 \otimes A_2, (E_1 \tilde{\otimes}_{\mathbb{C}} E_2)_{B_1\otimes B_2}, \boldsymbol{D_1} \tilde{\otimes} 1 \sqcup 1 \tilde{\otimes} \boldsymbol{D_2})]$$

in $KK_*(A_1 \otimes A_2, B_1 \otimes B_2)$.

Proof. It is straightforward to verify that $(A_1 \otimes A_2, (E_1 \tilde{\otimes}_{\mathbb{C}} E_2)_{B_1\otimes B_2}, \boldsymbol{D_1} \tilde{\otimes} 1 \sqcup 1 \tilde{\otimes} \boldsymbol{D_2})$ is an ST^2 with bounding matrix $\boldsymbol{\epsilon_1} \oplus \boldsymbol{\epsilon_2}$. It is also clear that $\Omega(\boldsymbol{\epsilon_1} \oplus \boldsymbol{\epsilon_2}) = \Omega(\boldsymbol{\epsilon_1}) \times \Omega(\boldsymbol{\epsilon_2})$. For $\boldsymbol{t} = (\boldsymbol{t_1}, \boldsymbol{t_2})$ we have that

$$\overline{\hat{D}}_{\boldsymbol{t}} = \overline{D}_{\boldsymbol{t_1}} \tilde{\otimes} 1 + 1 \tilde{\otimes} \overline{D}_{\boldsymbol{t_2}},$$

which is the form of the product operator for the external product of higher order cycles. Hence, any higher order cycle assembled from $(A_1 \otimes A_2, (E_1 \tilde{\otimes}_{\mathbb{C}} E_2)_{B_1\otimes B_2}, \boldsymbol{D_1} \tilde{\otimes} 1 \sqcup 1 \tilde{\otimes} \boldsymbol{D_2})$ represents the exterior Kasparov product of the higher order spectral triples assembled from $(A_1, E_{1,B_1}, \boldsymbol{D_1})$ and $(A_2, E_{2,B_2}, \boldsymbol{D_2})$. The Theorem follows. □

In a simpler way, we obtain

Theorem IV.1.26. *Let $(A, E_{1,B}, \boldsymbol{D_1})$ and $(A, E_{2,B}, \boldsymbol{D_2})$ be two strictly tangled cycles (of the same parity) with bounding matrices $\boldsymbol{\epsilon_1}$ and $\boldsymbol{\epsilon_2}$ respectively. Here we write $\boldsymbol{D_1} = (D_{1,j})_{j\in I_1}$ and $\boldsymbol{D_2} = (D_{2,k})_{k\in I_2}$. Then, with the collection $\boldsymbol{D_1} \oplus \boldsymbol{D_2} = (\hat{D}_l)_{l\in I_1\sqcup I_2}$, given by*

$$\hat{D}_l := \begin{cases} D_{1,l} \oplus 0 & l \in I_1 \\ 0 \oplus D_{2,l} & l \in I_2 \end{cases},$$

the data

$$(A, (E_1 \oplus E_2)_B, \boldsymbol{D_1} \oplus \boldsymbol{D_2})$$

constitute a strictly tangled cycle with bounding matrix the direct sum $\boldsymbol{\epsilon_1} \oplus \boldsymbol{\epsilon_2}$. Moreover, the direct sum of the associated KK-classes can be written as

$$[(A, E_{1,B}, \boldsymbol{D_1})] \oplus [(A, E_{2,B}, \boldsymbol{D_2})] = [(A, (E_1 \oplus E_2)_B, \boldsymbol{D_1} \oplus \boldsymbol{D_2})]$$

in $KK_*(A, B)$.

It unclear whether it is an advantage or a disadvantage of the framework of ST^2s that products and sums are treated in the same way, in the sense that they have the same effect on the bounding matrix. The difference is in the support of the operators: for an external product, every operator is supported on the entire Hilbert module whereas, for the direct sum, the operators have disjoint support. In this respect, when we come to consider complexes, we will see they behave more like sums than products; on the other hand, examples coming from the constructive unbounded Kasparov product will behave more like products than sums.

IV.1.3 Finite summability of strictly tangled spectral triples

The natural notion of dimension in noncommutative geometry is determined from spectral properties in analogy with the Weyl law. We introduce a notion of summability of an ST^2 that takes into account the different directions by means of a function. To simplify the description, we restrict our discussion of summability to the Schatten ideals with exponent $p > 0$.

Definition IV.1.27. Assume that $f : (0, \infty)^n \to (0, \infty)$ is a function decreasing in each argument. An ST^2 $(\mathscr{A}, H, \boldsymbol{D})$, with $\mathscr{A}$ unital, is said to be *f-summable* if, for $\boldsymbol{t} = (t_1, \dots, t_n) \in (0, \infty)^n$, the domain inclusion

$$\cap_j \operatorname{dom}(|D_j|^{t_j}) \hookrightarrow H$$

belongs to the Schatten class $\mathcal{L}^{f(\boldsymbol{t})}(\cap_j \operatorname{dom}(|D_j|^{t_j}), H)$, where the left hand side is given the Hilbert space topology from the intersection of graph topologies.

Example IV.1.28. The notion of f-summability is for $n = 1$ compatible with the notion of summability for spectral triples or, more generally, higher order spectral triples. Indeed, if $(\mathscr{A}, H, D)$ is a p-summable higher order spectral triple then it is an f-summable ST^2 with $n = 1$ for $f(t) = p/t$. Below in §IV.2, we consider ST^2s arising from Hilbert complexes defined from mixed order operators in which case the function f plays a role of controlling different orders of summability in the different directions.

Example IV.1.29. Let us return to the exterior Kasparov product of Theorem IV.1.25. Assume that $(\mathscr{A}_1, H_1, D_1)$ and $(\mathscr{A}_2, H_2, D_2)$ are two even higher order spectral triples that are summable of order p_1 and p_2 respectively. Their external Kasparov product is represented by the ST^2 $(\mathscr{A}_1 \otimes \mathscr{A}_2, H_1 \tilde{\otimes} H_2, (D_1 \tilde{\otimes} 1, 1 \tilde{\otimes} D_2))$. The ST^2 $(\mathscr{A}_1 \otimes \mathscr{A}_2, H_1 \tilde{\otimes} H_2, (D_1 \tilde{\otimes} 1, 1 \tilde{\otimes} D_2))$ will then be f-summable for any $f : (0, \infty)^2 \to (0, \infty)$ such that

$$(1 + |D_1|^{t_1} \tilde{\otimes} 1 + 1 \tilde{\otimes} |D_2|^{t_2})^{-1} \in \mathcal{L}^{f(t_1,t_2)}(H_1 \tilde{\otimes} H_2).$$

For instance, we could take

$$f(t_1, t_2) := \frac{p_1}{t_1} + \frac{p_2}{t_2}.$$

Example IV.1.30. We return to the direct sum of Theorem IV.1.26. Let us assume that $(\mathscr{A}_1, H_1, D_1)$ and $(\mathscr{A}_2, H_2, D_2)$ are two higher order spectral triples that are summable of order p_1 and p_2 respectively. Their direct sum is represented by the ST^2 $(\mathscr{A}_1 \oplus \mathscr{A}_2, H_1 \oplus H_2, (D_1 \oplus 0, 0 \oplus D_2))$. The ST^2 $(\mathscr{A}_1 \oplus \mathscr{A}_2, H_1 \oplus H_2, (D_1 \oplus 0, 0 \oplus D_2))$ will be f-summable for any $f : (0,\infty)^2 \to (0,\infty)$ such that

$$(1 + |D_1|^{t_1})^{-1} \oplus (1 + |D_2|^{t_2})^{-1} \in \mathcal{L}^{f(t_1,t_2)}(H_1) \oplus \mathcal{L}^{f(t_1,t_2)}(H_2).$$

For instance, we could take

$$f(t_1, t_2) := \max\left\{\frac{p_1}{t_1}, \frac{p_2}{t_2}\right\}.$$

If $(\mathscr{A}, H, \boldsymbol{D})$ is f_1-summable and $f_2 \geq f_1$, then $(\mathscr{A}, H, \boldsymbol{D})$ is also f_2-summable. The reader should note that if $(\mathscr{A}, H, \boldsymbol{D})$ is f-summable then by complex interpolation it is also $\tilde{f}$-summable for any $\tilde{f} > f_0$ where f_0 is the homogeneous function of degree -1 given by

$$f_0(t) := \frac{\inf_{s>0} s f(st|t|^{-1})}{|t|}.$$

Here $|\cdot|$ is an arbitrary norm on $\mathbb{R}^n$. If the infimum is attained, $(\mathscr{A}, H, \boldsymbol{D})$ is f_0-summable.

The following is immediate from the fact that $\mathrm{dom}(D_{\boldsymbol{t}}) = \cap_{j\in I} \mathrm{dom}(|D_j|^{t_j})$ for a strictly anticommuting n-tuple $(D_j)_{j\in I}$.

Proposition IV.1.31. *Let $(\mathscr{A}, H, \boldsymbol{D})$ be an f-summable ST^2. For $\boldsymbol{t} \in \Omega(\boldsymbol{\epsilon})$, $(\mathscr{A}, H, D_{\boldsymbol{t}})$ is an $f(\boldsymbol{t})$-summable higher order spectral triple.*

IV.1.4 Equivariance of strictly tangled spectral triples

We now come to defining equivariance in strictly tangled spectral triples and, with the applications to parabolic geometry and dynamics in mind, we allow for conformal actions. In the uniform case, there are no additional technical issues arising in the equivariant setting. This follows from the same method of proof as Theorem IV.1.16 (with an application of Remark I.2.8.2 in the nondiscrete group case). We record this in a Definition and Proposition.

Definition IV.1.32. Let $(A, E_B, \boldsymbol{D})$ be an strictly $\boldsymbol{\epsilon}$-tangled A-B-cycle with E a G-equivariant A-B-correspondence. We say that $(A, E_B, \boldsymbol{D})$ is *uniformly G-equivariant* if A is contained in the closure of $\mathscr{Q}$, the set of $a \in \mathrm{End}^*(E)$ such that, for each $i \in I$, $a\,\mathrm{dom}\, D_i \subseteq U_g\,\mathrm{dom}\, D_i$ for all $g \in G$ and the maps

$$g \mapsto \overline{(U_g D_i U_g^* a - a D_i)\Big(1 + \sum_{j\in I} |D_j|^{\epsilon_{ij}}\Big)^{-1}} \qquad g \mapsto \overline{U_g\Big(1 + \sum_{j\in I} |D_j|^{\epsilon_{ij}}\Big)^{-1} U_g^*(U_g D_i U_g^* a - a D_i)}$$

are $*$-strongly continuous as maps from G into $\mathrm{End}_B^*(E)$. If $U_g D_i U_g^* = D_i$ for all $i \in I$ and $g \in G$, we say that the cycle is *isometrically equivariant.* If $\mathscr{A}$ is a dense $*$-subalgebra of A contained in $\mathscr{Q}$, we say that $(\mathscr{A}, E_B, \boldsymbol{D})$ is a uniformly G-equivariant strictly tangled $\mathscr{A}$-B-cycle.

Proposition IV.1.33. *If $(A, E_B, \boldsymbol{D})$ is a uniformly G-equivariant strictly tangled cycle, the higher order cycle $(A, E_B, \overline{D}_{\boldsymbol{t}})$ is uniformly G-equivariant for all $\boldsymbol{t} \in \Omega(\boldsymbol{\epsilon}) \cap (0,1]^I$.*

Naïvely, the right way of applying the idea of conformal equivariance to ST^2s would seem to be to have a collection of conformal factors, one for each operator in the collection $\boldsymbol{D} = (D_j)_{j\in I}$. Alas, this idea falls apart already in the simple example of the exterior product of two real line Dirac spectral triples,

$$\left(C_c^\infty(\mathbb{R}^2), L^2(\mathbb{R}^2) \otimes \mathbb{C}^2, (\partial_{x_1} \otimes \gamma_1, \partial_{x_2} \otimes \gamma_2)\right),$$

whose bounding matrix is $\boldsymbol{\epsilon} = 0$. In this simple example the action of $\mathbb{R}^2$ by dilation in each direction, $(r_1, r_2) : (x_1, x_2) \mapsto (r_1 x_1, r_2 x_2)$, makes any resulting higher order spectral triple fail to be conformally

$\mathbb{R}^2$-equivariant. The source of this problem is actually deeper, however, because the bounded transform of any resulting higher order spectral triple also cannot be $\mathbb{R}^2$-equivariant.

However, under some circumstances, it may be possible to align the conformal factors so that the resulting higher order spectral triple is conformally equivariant. We will see such a phenomenon for Carnot groups in Proposition IV.3.11. In the example above with its dilation action, this is possible by restricting to a subgroup where $r_1 = r_2^\alpha$, for some fixed $\alpha \neq 0$. If we choose $\boldsymbol{t} \in \mathbb{R}_+(1,\alpha) \subset \Omega(\boldsymbol{\epsilon}) = \mathbb{R}_+^2$, the higher order spectral triple

$$\left(C_c^\infty(\mathbb{R}^2), L^2(\mathbb{R}^2), \operatorname{sgn}(\partial_{x_1})|\partial_{x_1}|^{t_1} \otimes \gamma_1 + \operatorname{sgn}(\partial_{x_2})|\partial_{x_2}|^{t_2} \otimes \gamma_2\right)$$

is conformally equivariant, with conformal factor $r_1^{-t_1/2} = r_2^{-t_2/2}$.

Another example to consider is the direct sum of two real line Dirac spectral triples,

$$\left(C_c^\infty(\mathbb{R} \sqcup \mathbb{R}), L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), (\partial_{x_1} \oplus 0, 0 \oplus \partial_{x_2})\right) \qquad \boldsymbol{\epsilon} = 0$$

with an action of $\mathbb{R}^2$ by dilation on each corresponding copy of $\mathbb{R}$,

$$(r_1, r_2) : x_1 \mapsto r_1 x_1 \qquad x_2 \mapsto r_2 x_2 \qquad (x_1 \in \mathbb{R} \sqcup \emptyset, x_2 \in \emptyset \sqcup \mathbb{R}).$$

Here there is no restriction on $\boldsymbol{t} \in \Omega(\boldsymbol{\epsilon}) = \mathbb{R}_+^2$, as we may take the conformal factor to be $r_1^{-t_1/2} \oplus r_2^{-t_2/2}$ on the higher order spectral triple

$$\left(C_c^\infty(\mathbb{R} \sqcup \mathbb{R}), L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), \operatorname{sgn}(\partial_{x_1})|\partial_{x_1}|^{t_1} \oplus \operatorname{sgn}(\partial_{x_2})|\partial_{x_2}|^{t_2}\right).$$

Unfortunately, the development of an abstract framework for conformal equivariance of ST^{2}s seems elusive. The main technical problem is to find conditions guaranteeing that, if $UDU^* - \mu D \mu^*$ is of 'lower order', $U|D|^t U^* - \mu^t |D|^t (\mu^*)^t$ is also of 'lower order'. For natural candidate conditions, we have been able neither to prove such a result in the abstract nor to find a counterexample.

The approach we take in the examples below is to take the following Proposition as giving an ad hoc notion of a conformally equivariant ST^2. Here, we fix $\boldsymbol{t}$ and give sufficient conditions for a single conformal factor $(\mu_g)_{g\in G}$ to give rise to a conformally equivariant higher order spectral triple at $\boldsymbol{t}$. A more general statement would be possible but this will suffice for our needs. One could view this approach as similar to the 'guess-and-check' method of computing Kasparov products, discussed in §I.4.

Proposition IV.1.34. *Let $(A, H, \boldsymbol{D})$ be an ST^2 with a unitary action of G on H, implementing the action on A. Suppose there exists a family $(\mu_g)_{g\in G}$ of invertible bounded operators such that, for all $g \in G$, μ_g, μ_g^*, and U_g preserve $\operatorname{dom} D_i$ for all i, with*

$$g \mapsto [D_i, \mu_g]\left(1 + \sum_{j\in I} |D_j|^{\epsilon_{ij}}\right)^{-1} \qquad \text{and} \qquad g \mapsto [D_i, \mu_g^*]\left(1 + \sum_{j\in I} |D_j|^{\epsilon_{ij}}\right)^{-1}$$

defining $*$-strongly continuous maps from G into the space of bounded operators on H. Suppose furthermore that, for some $\boldsymbol{t} \in \Omega(\boldsymbol{\epsilon}) \cap (0,1]^I$, the maps

$$g \mapsto \left(U_g \operatorname{sgn}(D_i)|D_i|^{t_i} U_g^* - \mu_g \operatorname{sgn}(D_i)|D_i|^{t_i} \mu_g^*\right)\left(1 + \sum_{j\in I} |D_j|^{\epsilon_{ij}}\right)^{-t_i} \qquad \text{and}$$

$$g \mapsto U_g \left(1 + \sum_{j\in I} |D_j|^{\epsilon_{ij}}\right)^{-t_i} U_g^* \left(U_g \operatorname{sgn}(D_i)|D_i|^{t_i} U_g^* - \mu_g \operatorname{sgn}(D_i)|D_i|^{t_i} \mu_g^*\right)$$

are $*$-strongly continuous from G into the space of bounded operators on H. Then $(A, H, \overline{D}_{\boldsymbol{t}})$ is a conformally G-equivariant higher order spectral triple with conformal factor μ.

The proof is a straightforward extension of the proof of Theorem IV.1.16. We give in Remark IV.2.8 a statement in the context of Hilbert complexes. There, we will naturally begin with a collection of conformal factors which will need to be cajoled into cooperating with one another and so into giving a single conformal factor μ for the higher order spectral triple.

IV.2 Examples arising from differential complexes

The main application of strictly tangled spectral triples that we study in this Chapter comes from Hilbert complexes and, more concretely, Rockland complexes on filtered manifolds. We first present an abstract framework for Hilbert complexes and proceed to describe it in detail for Rockland complexes. We here work only with Hilbert spaces and ST^2 but it seems likely that our methods could be applied to Hilbert C*-modules, using [VVD25].

IV.2.1 Hilbert complexes

We first recall the notion of a Hilbert complex. We follow the presentation of [BL92] and refer the reader there for further details.

Definition IV.2.1. A Hilbert complex

$$0 \to \mathcal{H}_0 \xrightarrow{\mathrm{d}_0} \mathcal{H}_1 \xrightarrow{\mathrm{d}_1} \cdots \mathcal{H}_{n-1} \xrightarrow{\mathrm{d}_{n-1}} \mathcal{H}_n \to 0,$$

abbreviated as $(\mathcal{H}_\bullet, \mathrm{d}_\bullet)$, consists of Hilbert spaces $\mathcal{H}_0, \mathcal{H}_1, \dots, \mathcal{H}_n$ and closed densely defined maps $\mathrm{d}_i : \mathcal{H}_i \dashrightarrow \mathcal{H}_{i+1}$ with the property that

$$\mathrm{Ran}(\mathrm{d}_{i-1}) \subseteq \ker(\mathrm{d}_i).$$

We say that $(\mathcal{H}_\bullet, \mathrm{d}_\bullet)$ is Fredholm if the cohomology groups

$$H^i(\mathcal{H}_\bullet, \mathrm{d}_\bullet) = \ker(\mathrm{d}_i)/\mathrm{Ran}(\mathrm{d}_{i-1})$$

are finite-dimensional. We say that $(\mathcal{H}_\bullet, \mathrm{d}_\bullet)$ has discrete spectrum if, for each i, the self-adjoint Laplacian $\mathrm{d}_i^*\mathrm{d}_i + \mathrm{d}_{i-1}\mathrm{d}_{i-1}^*$, densely defined on $\mathcal{H}_i$, has discrete spectrum, i.e. the spectrum consists of isolated eigenvalues of finite multiplicity.

By [BL92, Theorem 2.4], $(\mathcal{H}_\bullet, \mathrm{d}_\bullet)$ is Fredholm if and only if 0 is not in the essential spectrum of all the Laplacians $\mathrm{d}_i^*\mathrm{d}_i + \mathrm{d}_{i-1}\mathrm{d}_{i-1}^*$. In particular, $(\mathcal{H}_\bullet, \mathrm{d}_\bullet)$ is Fredholm if it has discrete spectrum. We shall make use of a construction analogous to Rumin–Seshadri's construction of Laplacians in the Rumin complex [RS12]; see also [DH22]. Given parameters $\boldsymbol{m} = (m_0, \dots, m_{n-1}) \in [1,\infty)^n$ that we refer to as an order and a Hilbert complex $(\mathcal{H}_\bullet, \mathrm{d}_\bullet)$ we define the Rumin Laplacians

$$\Delta^R_{\boldsymbol{m},i} = (\mathrm{d}_i^*\mathrm{d}_i)^{a_i} + (\mathrm{d}_{i-1}\mathrm{d}_{i-1}^*)^{a_{i-1}},$$

where $a_i = \prod_{l\neq i} m_l = m/m_i$ for $m = \prod_{l=1}^n m_l$. Clearly, $(\mathcal{H}_\bullet, \mathrm{d}_\bullet)$ has discrete spectrum if and only if all the self-adjoint operators $\Delta^R_{\boldsymbol{m},i}$ have compact resolvent. We also introduce, for $s \geq 0$, the abstract Sobolev spaces

$$\mathcal{H}^s_{i,\boldsymbol{m}} = \mathrm{dom}((\Delta^R_{\boldsymbol{m},i})^{s/2m}) \subseteq \mathcal{H}_i.$$

Definition IV.2.2. Let $\mathscr{A}$ be a $*$-algebra. A Hilbert complex *over* $\mathscr{A}$ *of order* $\boldsymbol{m} = (m_0, \dots, m_{n-1}) \in [1,\infty)^n$ is a Hilbert complex

$$0 \to \mathcal{H}_0 \xrightarrow{\mathrm{d}_0} \mathcal{H}_1 \xrightarrow{\mathrm{d}_1} \cdots \xrightarrow{\mathrm{d}_{n-2}} \mathcal{H}_{n-1} \xrightarrow{\mathrm{d}_{n-1}} \mathcal{H}_n \to 0,$$

where each $\mathcal{H}_i$ is a left $\mathscr{A}$-module under $*$-representations

$$\pi_i : \mathscr{A} \to \mathbb{B}(\mathcal{H}_i)$$

such that, for all $a \in \mathscr{A}$, $\pi_i(a)$ preserves $\mathrm{dom}(\mathrm{d}_i)$ and the densely defined operators

$$\left(\mathrm{d}_i\pi_i(a) - \pi_{i+1}(a)\mathrm{d}_i\right)(1+\Delta_{\boldsymbol{m},i})^{\frac{1-m_i}{2m}} \quad \text{and}$$
$$(1+\Delta_{\boldsymbol{m},i+1})^{\frac{1-m_i}{2m}}\left(\mathrm{d}_i\pi_i(a) - \pi_{i+1}(a)\mathrm{d}_i\right)$$

are norm bounded.

If, for all $s \geq 0$, $\pi_i(a)$ preserves the domain of d_i as an operator on the Sobolev spaces $\mathcal{H}^s_{i,\boldsymbol{m}}$ and the densely defined operator $\left(\mathrm{d}_i\pi_i(a) - \pi_{i+1}(a)\mathrm{d}_i\right)(1+\Delta_{\boldsymbol{m},i})^{\frac{1-m_i}{2m}}$ is continuous in norm $\mathcal{H}^s_{i,\boldsymbol{m}} \to \mathcal{H}^s_{i+1,\boldsymbol{m}}$ then we say that $(\mathcal{H}_\bullet, \mathrm{d}_\bullet)$ is a regular Hilbert complex over $\mathscr{A}$ of order $\boldsymbol{m}$.

To ease the notation, we drop the representations π_i when they are clear from the context, writing $[\mathrm{d}_i, a]$ instead of $\mathrm{d}_i\pi_i(a) - \pi_{i+1}(a)\mathrm{d}_i$ for $a \in \mathscr{A}$.

Lemma IV.2.3. *Let $(\mathcal{H}_\bullet, \mathrm{d}_\bullet)$ be a Hilbert complex which is Fredholm and of order $\boldsymbol{m} = (m_j)_{j=0}^{n-1} \in [1,\infty)^n$. With $a_i = \prod_{l\neq i} m_l = m/m_i$ for $m = \prod_{l=0}^{n-1} m_l$. Then, setting $H = \bigoplus_i \mathcal{H}_i$ and $D_i = \mathrm{d}_i + \mathrm{d}_i^*$, the collection $\boldsymbol{D} = (D_i)_{i=0}^{n-1}$ is a strictly anticommuting collection of selfadjoint operators on H. Morever, for any α we have that*

$$D_i|D_i|^\alpha = D_i((\Delta^R_{\boldsymbol{m},i})^{\alpha/2a_{i-1}} + (\Delta^R_{\boldsymbol{m},i-1})^{\alpha/2a_{i-1}}).$$

Proof. We remark that $a_i\frac{1-m_i}{2m} = \frac{1-m_i}{2m_i} = \frac{1}{2}(-1+\frac{1}{m_i})$. Since the Hilbert complex is Fredholm,

$$(\Delta^R_{\boldsymbol{m},i})^\beta = (\mathrm{d}_i^*\mathrm{d}_i)^{\beta a_i} + (\mathrm{d}_{i-1}\mathrm{d}_{i-1}^*)^{\beta a_{i-1}}.$$

In particular,

$$D_i(\Delta^R_{\boldsymbol{m},i})^\beta = \mathrm{d}_{i-1}^*(\mathrm{d}_{i-1}\mathrm{d}_{i-1}^*)^{\beta a_{i-1}} \quad \text{and} \quad D_i(\Delta^R_{\boldsymbol{m},i-1})^\beta = \mathrm{d}_{i-1}(\mathrm{d}_{i-1}^*\mathrm{d}_{i-1})^{\beta a_{i-1}}.$$

On the other hand,

$$|D_i|^\alpha = (\mathrm{d}_{i-1}^*\mathrm{d}_{i-1})^{\alpha/2} + (\mathrm{d}_{i-1}\mathrm{d}_{i-1}^*)^{\alpha/2}$$

so

$$D_i|D_i|^\alpha = \mathrm{d}_{i-1}(\mathrm{d}_{i-1}^*\mathrm{d}_{i-1})^{\alpha/2} + \mathrm{d}_{i-1}^*(\mathrm{d}_{i-1}\mathrm{d}_{i-1}^*)^{\alpha/2}$$

and the Lemma follows. □

Theorem IV.2.4. *Let $(\mathcal{H}_\bullet, \mathrm{d}_\bullet)$ be a Hilbert complex over $\mathscr{A}$ of order $\boldsymbol{m}$ with discrete spectrum. We set $H = \bigoplus_i \mathcal{H}_i$ and write $\boldsymbol{D} = (D_i)_{i=0}^{n-1}$ for the collection $D_i = \mathrm{d}_i + \mathrm{d}_i^*$. It then holds that the collection $(\mathscr{A}, H, D)$ is a ST^2 with bounding matrix $\boldsymbol{\epsilon} = (\epsilon_{ij})_{i,j=0}^{n-1}$ where*

$$\epsilon_{ij} = \begin{cases} \frac{m_i-1}{m_j} & j = i-1, i, i+1 \\ 0 & \textit{otherwise}. \end{cases} \tag{IV.2.5}$$

Furthermore, if $(\mathcal{H}_\bullet, \mathrm{d}_\bullet)$ is regular, $(\mathscr{A}, H, \boldsymbol{D})$ is $\boldsymbol{\rho}$-preserving for $\boldsymbol{\rho} = (\infty, \dots, \infty)$.

Proof. We have that the bounding matrix (IV.2.5) satisfies the decreasing cocycle condition by the same argument as in (IV.1.11) (with the first equality of (IV.1.11) replaced by an upper bound). Since

$(\mathscr{H}_\bullet, \mathrm{d}_\bullet)$ has discrete spectrum, what remains to prove is the commutator condition. And $(\mathscr{H}_\bullet, \mathrm{d}_\bullet)$ is a Hilbert complex over $\mathscr{A}$ of order $\boldsymbol{m}$, so

$$[\mathrm{d}_i, a](1+\Delta_{\boldsymbol{m},i})^{\frac{1-m_i}{2m}} = [\mathrm{d}_i, a](1+\Delta_{\boldsymbol{m},i})^{\frac{1}{2m}-\frac{1}{2a_i}} \quad \text{and}$$
$$(1+\Delta_{\boldsymbol{m},i+1})^{\frac{1-m_i}{2m}}[\mathrm{d}_i, a] = (1+\Delta_{\boldsymbol{m},i+1})^{\frac{1}{2m}-\frac{1}{2a_i}}[\mathrm{d}_i, a]$$

are bounded. Since $\Delta^R_{\boldsymbol{m},i} = (\mathrm{d}_i^*\mathrm{d}_i)^{a_i} + (\mathrm{d}_{i-1}\mathrm{d}_{i-1}^*)^{a_{i-1}}$, we conclude from the boundedness of the first operator that

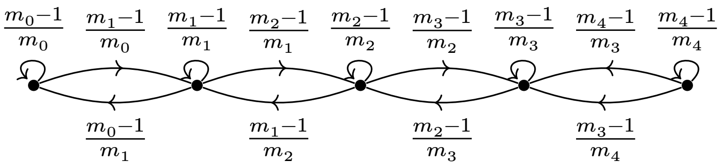
$$[\mathrm{d}_i, a]\left(1+|D_i|^{1-\frac{1}{m_{i-1}}} + |D_{i+1}|^{\frac{1-m_{i-1}}{m_i}}\right)^{-1}$$

is bounded and from the boundedness of the second operator that

$$[\mathrm{d}_i^*, a]\left(1+|D_i|^{1-\frac{1}{m_{i-1}}} + |D_{i-1}|^{\frac{1-m_{i-1}}{m_{i-2}}}\right)^{-1}$$

is bounded. □

For instance, for a complex with $n = 5$, the graph corresponding to the bounding matrix would be



.

Remark IV.2.6. If $(\mathscr{H}_\bullet, \mathrm{d}_\bullet)$ is a Hilbert complex with discrete spectrum over $\mathscr{A}$, there are multiple ways of grading the ST^2 $(\mathscr{A}, H, \boldsymbol{D})$. The first option is to use the grading coming from the complex in which

$$H_+ = \bigoplus_i \mathscr{H}_{2i} \qquad H_- = \bigoplus_i \mathscr{H}_{2i+1}.$$

Another option arises if $(\mathscr{H}_\bullet, \mathrm{d}_\bullet)$ satisfies a mild strengthening of Poincaré duality; see [BL92, Lemma 2.16]. Assume that we have $\mathscr{A}$-linear unitaries $\gamma_i : \mathscr{H}_i \to \mathscr{H}_{n-i}$ such that

$$\mathrm{d}^*_{n-i-1}\gamma_i = -\gamma_{i+1}\mathrm{d}_i \quad \text{and} \quad \gamma_{n-i}\gamma_i = 1_{\mathscr{H}_i}.$$

We can then define a symmetry $\gamma = \bigoplus \gamma_j$ on H that anticommutes with D_j, for $j = 1, \dots, n$. In particular, γ grades H in such a way that the ST^2 constructed in Theorem IV.2.4 forms an even ST^2. This construction is analogous to the grading induced from the Hodge star on differential forms defining the signature operator from the Hodge–de Rham operator.

Proposition IV.2.7. *Assume that $(\mathscr{A}, H, \boldsymbol{D})$ is an ST^2 defined from a Hilbert complex with discrete spectrum $(\mathscr{H}_\bullet, \mathrm{d}_\bullet)$ over $\mathscr{A}$ of order $\boldsymbol{m}$ and bounding matrix $\boldsymbol{\epsilon}$ as in* (IV.2.5)*. Then, for any $\tau > 0$,*

$$\boldsymbol{t}_{\boldsymbol{m}}(\tau) := \left(\frac{\tau}{m_0}, \frac{\tau}{m_1}, \dots, \frac{\tau}{m_{n-1}}\right) \in \Omega(\boldsymbol{\epsilon})$$

and

$$D_{\boldsymbol{t}_{\boldsymbol{m}}(\tau)} = \sum_{i=0}^{n-1} \mathrm{d}_i(\Delta^R_{\boldsymbol{m},i})^{\frac{\tau-m_i}{2m}} + \mathrm{d}_i^*(\Delta^R_{\boldsymbol{m},i+1})^{\frac{\tau-m_i}{2m}}.$$

Proof. We see that $\boldsymbol{t}_{\boldsymbol{m}}(\tau) \in \Omega(\boldsymbol{\epsilon})$ since $\frac{\tau}{m_j} > \frac{m_i-1}{m_j}\frac{\tau}{m_i}$ and the expression for $D_{\boldsymbol{t}_{\boldsymbol{m}}(\tau)}$ follows from Lemma IV.2.3. □

Recall the weak Hodge decomposition of [BL92, Lemma 2.1],

$$\mathcal{H}_i = \mathscr{H}_i \oplus \overline{\operatorname{im} \mathrm{d}_{i-1}} \oplus \overline{\operatorname{im} \mathrm{d}_i^*}$$

where $\mathscr{H}_i = \ker \mathrm{d}_i \cap \ker \mathrm{d}_{i-1}^* = \ker \Delta^R_{\boldsymbol{m},i}$. Note that if $(\mathcal{H}_\bullet, \mathrm{d}_\bullet)$ is Fredholm, e.g. if it has discrete spectrum, the ranges are automatically closed with

$$\operatorname{im} \mathrm{d}_{i-1} = (\ker \mathrm{d}_{i-1}^*)^\perp \quad \text{and} \quad \operatorname{im} \mathrm{d}_i^* = (\ker \mathrm{d}_i)^\perp.$$

We can build up a conformally equivariant higher order spectral triple by specifying conformal factors on each part of the decomposition.

Remark IV.2.8. Let $\mathscr{A}$ be a unital $*$-algebra with an action of a locally compact group G. Let

$$0 \to \mathcal{H}_0 \xrightarrow{\mathrm{d}_0} \mathcal{H}_1 \xrightarrow{\mathrm{d}_1} \cdots \xrightarrow{\mathrm{d}_{n-2}} \mathcal{H}_{n-1} \xrightarrow{\mathrm{d}_{n-1}} \mathcal{H}_n \to 0$$

be a Hilbert complex over $\mathscr{A}$ of order $\boldsymbol{m}$ with a unitary action U_i of G on each $\mathcal{H}_i$ intertwining the representation of $\mathscr{A}$ and preserving the domains of $\mathrm{d}_\bullet$.

Let $(\nu_{i,g})_{g\in G} \subset \mathbb{B}(\overline{\operatorname{im} \mathrm{d}_i^*})$ and $(\tilde{\nu}_i)_{g\in G} \subset \mathbb{B}(\overline{\operatorname{im} \mathrm{d}_{i-1}})$ be families of invertible operators, all of them and their adjoints preserving the domains of $\mathrm{d}_\bullet$, such that the densely defined operators

$$\left(\tilde{\nu}_{i+1,g}\mathrm{d}_i - \mathrm{d}_i\nu_{i,g}\right)(1+\Delta_{\boldsymbol{m},i})^{\frac{1-m_i}{2m}} \quad \text{and} \quad (1+\Delta_{\boldsymbol{m},i+1})^{\frac{1-m_i}{2m}}\left(\tilde{\nu}_{i+1,g}\mathrm{d}_i - \mathrm{d}_i\nu_{i,g}\right)$$

are in fact bounded and define $*$-strongly continuous functions $G \to \mathbb{B}(\mathcal{H}_i, \mathcal{H}_{i+1})$. Suppose that, for some $\boldsymbol{t} \in \Omega(\boldsymbol{\epsilon})$, the densely defined operators

$$\begin{aligned}\Big(U_{i+1,g}\mathrm{d}_i(\Delta^R_{\boldsymbol{m},i})^{\frac{-1+t_i}{2m}m_i}U^*_{i,g} - \tilde{\nu}_{i+1,g}\mathrm{d}_i(\Delta^R_{\boldsymbol{m},i})^{\frac{-1+t_i}{2m}m_i}\nu_{i,g}\Big)(1+\Delta_{\boldsymbol{m},i})^{\frac{1-m_i}{2m}t_i} \quad \text{and}& \\ (1+\Delta_{\boldsymbol{m},i})^{\frac{1-m_i}{2m}t_i}\Big(U_{i+1,g}\mathrm{d}_i(\Delta^R_{\boldsymbol{m},i})^{\frac{-1+t_i}{2m}m_i}U^*_{i,g} - \tilde{\nu}_{i+1,g}\mathrm{d}_i(\Delta^R_{\boldsymbol{m},i})^{\frac{-1+t_i}{2m}m_i}\nu_{i,g}\Big)&\end{aligned}$$

are in fact bounded and define $*$-strongly continuous functions $G \to \mathbb{B}(\mathcal{H}_i, \mathcal{H}_{i+1})$. Then $(\mathscr{A}, H, \overline{D}_{\boldsymbol{t}})$ is conformally equivariant with conformal factor

$$\mu = \bigoplus_j \nu_j + \tilde{\nu}_j + P_{\mathscr{H}_j}$$

by Proposition IV.1.34.

IV.2.2 The Heisenberg calculus

In §IV.2.3, we shall study Rockland sequences on filtered manifolds. Rockland sequences were studied in detail in Dave and Haller's work [DH22]. The associated analysis relies heavily on van Erp and Yuncken's Heisenberg calculus [EY17a] on a filtered manifold. Filtered manifolds are known also as Carnot manifolds, and relate to the equiregular differential systems of sub-Riemannian geometry.

Let us therefore outline the geometry of filtered manifolds and their Heisenberg calculus. We refer the details to the literature [DH22, GK24, EY17a]. A filtered manifold is a manifold X whose tangent bundle is equipped with a filtering

$$TX = T^{-r}X \supsetneq T^{-r+1}X \supsetneq \dots \supsetneq T^{-2}X \supsetneq T^{-1}X \supsetneq 0$$

of subbundles such that $[T^{-j}X, T^{-k}X] \subseteq T^{-j-k}X$ for any j, k. We call r the depth of X. We write

$$\mathfrak{t}_H X = \bigoplus_j T^{-j}X/T^{-j+1}X$$

for the associated graded bundle. Taking commutators of vector fields induces a fibrewise Lie bracket on $\mathfrak{t}_H X$, making $\mathfrak{t}_H X \to X$ a Lie algebroid. The fibres are nilpotent of step length at most r, so the Baker–Campbell–Hausdorff formula implies that $\mathfrak{t}_H X$ integrates to a Lie groupoid $T_H X \rightrightarrows X$ (with the same range and source map). Concretely, as a fibre bundle, $T_H X = \mathfrak{t}_H X$. However, $T_H X$ carries a fibrewise polynomial group operation defined from the Baker–Campbell–Hausdorff formula and the commutator of vector fields modulo lower order terms in the filtration. We call $T_H X \rightrightarrows X$ the osculating Lie groupoid. The osculating Lie groupoid carries an $\mathbb{R}_+$-action δ defined from integrating the $\mathbb{R}_+$-action on $\mathfrak{t}_H X$ defined from its grading.

The Heisenberg calculus on a filtered manifold introduced by van Erp and Yuncken [EY17a] is built from operators whose Schwartz kernels in appropriate exponential coordinates are defined from r-fibred distributions on $T_H X$ that expand asymptotically into a sum of almost homogeneous fibrewise convolution operators. A way to formalize this statement uses van Erp and Yuncken's parabolic tangent groupoid [EY17b], a Lie groupoid $\mathbb{T}_H X \rightrightarrows X \times [0,\infty)$. As a set,

$$\mathbb{T}_H X = T_H X \times \{0\} \sqcup X \times X \times (0,\infty),$$

with the groupoid structure of $T_H X$ on the first component and the pair groupoid structure on the second component. The Lie groupoid structure on $\mathbb{T}_H X \rightrightarrows X \times [0,\infty)$ is defined using a blowup in exponential coordinates defined from a graded connection. The parabolic tangent groupoid carries an $\mathbb{R}_+^\times$-action called the *zoom* action, which by an abuse of notation we also denote by δ, acting by

$$\delta_\lambda : (x, v, 0) \mapsto (x, \delta_\lambda(v), 0) \qquad (x, y, t) \mapsto (x, y, \lambda^{-1} t).$$

A Heisenberg pseudodifferential operator T of order m is defined to be an operator on $C^\infty(X)$ whose Schwartz kernel $k_T \in \mathcal{D}'(X \times X)$ can be written as the evaluation at $t = 1$ of a properly supported, r-fibred distribution $K \in \mathcal{D}'_r(\mathbb{T}_H X)$ which is homogeneous of order m modulo properly supported elements under the zoom action. In exponential coordinates, we can Taylor expand such a K at $t = 0$ and arrive at an asymptotic sum

$$K(x, v, t) \sim \sum_{j=0}^{\infty} t^j k_j(x, v), \tag{IV.2.9}$$

where $k_j \in \mathcal{E}'_r(T_H X)$ is homogenenous modulo $C_c^\infty(T_H X)$ of degree $m - j$. Here K and the collection $(k_j)_{j=0}^\infty$ are uniquely determined by k_T modulo respectively properly and compactly supported smooth elements. Writing $\Psi_H^m(X)$ for the space of Heisenberg pseudodifferential operators of order m, we arrive at a short exact sequence

$$0 \to \Psi_H^{m-1}(X) \to \Psi_H^m(X) \xrightarrow{\sigma_H^m} \Sigma_H^m(X) \to 0,$$

where $\Sigma_H^m(X) \subseteq \mathcal{E}'_r(T_H X)/C_c^\infty(T_H X)$ consists of elements homogenenous of degree m. The map σ_H^m is called the principal symbol and is defined by $\sigma_H^m(T) := [k_0]$ for k_0 the leading term in (IV.2.9). A composition of Heisenberg pseudodifferential operators of order m and m' respectively as operators on $C^\infty(X)$ is again a Heisenberg pseudodifferential operator but of order $m + m'$. The principal symbol respects products in the sense that

$$\sigma_H^{m+m'}(TT') = \sigma_H^m(T) * \sigma_H^{m'}(T'), \quad T \in \Psi_H^m(X),\ T' \in \Psi_H^{m'}(X)$$

where $*$ denotes groupoid convolution on $T_H X$.

We can realize the principal symbol algebra in a more concrete way. Write $\mathcal{S}(T_H X) \subseteq C^\infty(T_H X)$ for the space of fibrewise Schwarz functions: functions that together with their derivatives decay faster than the reciprocal of any polynomial in the fibre. We define $\mathcal{S}_0(T_H X)$ to consist of those functions $f \in \mathcal{S}(T_H X)$ such that for any $x \in X$ and any polynomial p on $T_x X$ we have

$$\int_{T_x X} p(v) f(x, v) \mathrm{d}v = 0.$$

The space $\mathcal{S}_0(T_HX)$ is closed under convolution and is dense in the ideal of $C^*(T_HX)$ of elements vanishing in the fibrewise trivial representations. We embed $\Sigma^m_H(X)$ in the multipliers of $\mathcal{S}_0(T_HX)$ as follows. Any element $k \in \Sigma^m_H(X)$ can be represented near the zero section $X \subseteq T_HX$ by an r-fibred distribution $\hat{k} \in \mathcal{D}'_r(T_HX)$ of the form

$$\hat{k} = \hat{k}_0 + p \log|\cdot|,$$

where $\hat{k}_0$ is homogeneous of degree m, p is fibrewise polynomial and where $|\cdot|$ is a fibrewise gauge (smooth outside the zero section and homogeneous of degree 1). Upon fixing $|\cdot|$, the distribution $\hat{k}$ is unique up to a fibrewise polynomial. In particular, the muliplier on $\mathcal{S}_0(T_HX)$ defined by convolution by $\hat{k}$ depends only on $k \in \Sigma^m_H(X)$.

To understand further the principal symbol, we study its action in localizations of $\mathcal{S}_0(T_HX)$ in its $*$-representations. Whenever $(\pi, \mathcal{H})$ is a unitary representation of a nilpotent group G, we write $\mathcal{S}_0(\pi) = \pi(\mathcal{S}_0(G))\mathcal{H}$. If π does not weakly contain the trivial representation, $\mathcal{S}_0(\pi) = \pi(\mathcal{S}(G))\mathcal{H}$ and is dense in $\mathcal{H}$. Moreover, any multiplier k of $\mathcal{S}_0(G)$ localizes to an operator $\pi(k)$ on $\mathcal{H}$ with domain $\mathcal{S}_0(\pi)$ defined by $\pi(k)(\pi(a)\xi) = \pi(k * a)\xi$ for $a \in \mathcal{S}_0(G)$ and $\xi \in \mathcal{H}$. We can therefore for a Heisenberg pseudodifferential operator T of order m, $x \in X$ and a unitary representation π of $(T_HX)_x$, define the represented symbol

$$\sigma^m_H(T, \pi) = \pi(\sigma^m_H(T)) : \mathcal{S}_0(\pi) \to \mathcal{S}_0(\pi).$$

The discussion above readily extends to operators on vector bundles. We denote the space of Heisenberg pseudodifferential operators of order m from the vector bundle E_1 to E_2 by $\Psi^m_H(X; E_1, E_2)$. We recall the following important definition.

Definition IV.2.10. Let X be a filtered manifold and $E_1, E_2 \to X$ two vector bundles. Assume that $T : C^\infty(X, E_1) \to C^\infty(X, E_2)$ is a Heisenberg pseudodifferential operator of order m. We say that T satisfies the *Rockland condition* if, for any $x \in X$ and any irreducible, non-trivial, unitary representation π of $(T_HX)_x$, the represented symbol

$$\sigma^m_H(T, \pi) = \pi(\sigma^m_H(T)) : \mathcal{S}_0(\pi) \otimes E_{1,x} \to \mathcal{S}_0(\pi) \otimes E_{2,x}$$

is injective. If the represented symbol in all points and all irreducible, non-trivial, unitary representations is bijective then we say that T is H-elliptic.

Operators in the Heisenberg calculus act continuously in a scale of Sobolev spaces adapted to the filtering. Fix a volume density on X. Following [DH19, DH22], we know that there exists a family of H-elliptic operators $(A^t)_{t\in\mathbb{R}}$ (in fact the complex powers of a single H-elliptic operator) that we can assume satisfies $A^0 = 1$. We define $W^s_H(X) = A^{-s}L^2(X) \subseteq \mathcal{D}'(X)$ with inner product defined by declaring $A^s : W^s_H(X) \to L^2(X)$ unitary. A similar definition can be made also for vector bundles. Any $T \in \Psi^m_H(X; E_1, E_2)$ extends by density to a continuous operator

$$T : W^{s_1}_H(X; E_1) \to W^{s_2}_H(X; E_2)$$

as soon as $s_1 + m \geq s_2$ and a compact operator when $s_1 + m > s_2$.

Theorem IV.2.11. *Let X be a closed filtered manifold equipped with a volume density, let $E_1, E_2 \to X$ be two hermitian vector bundles, and let $T : C^\infty(X, E_1) \to C^\infty(X, E_2)$ be a Heisenberg pseudodifferential operator of order m. Then the following are equivalent:*

1. *T and T^* satisfy the Rockland condition;*
2. *T is H-elliptic;*
3. *$T : W^s_H(X; E_1) \to W^{s-m}_H(X; E_2)$ is Fredholm for some s; and*
4. *$T : W^s_H(X; E_1) \to W^{s-m}_H(X; E_2)$ is Fredholm for all s.*

Moreover, H-elliptic operators are hypoelliptic and admit parametrices in the Heisenberg calculus.

Here it is clear that 4. implies 3. and 2. implies 1.. That 3. implies 2. is proven in [AMY22] and that 1. implies 4. is proven in [DH22].

For summability results of spectral triples and ST^{2}s on filtered manifolds, we will use Dave–Haller's Weyl law in the Heisenberg calculus [DH19]. Its statement gives a leading term in the eigenvalue of positive, even-order, H-elliptic, differential operators in the Heisenberg calculus. For a filtered manifold X, we define its homogeneous dimension as

$$\dim_h(X) = \sum_j j \operatorname{rk}(T^{-j}X/T^{-j+1}X). \tag{IV.2.12}$$

Dave and Haller's Weyl law [DH19] implies that if $T \in \Psi_H^m(X; E_1, E_2)$ for an $m < 0$ then

$$\mu_k(T) = O(k^{\dim_h(X)/m}). \tag{IV.2.13}$$

In particular, for $m < 0$,

$$\Psi_H^m(X; E_1, E_2) \subseteq \mathcal{L}^p(L^2(X, E_1), L^2(X, E_2)) \qquad (p > -\dim_h(X)/m).$$

IV.2.3 Strictly tangled spectral triples for Rockland complexes

We now turn to studying Rockland complexes in earnest. They play the role of elliptic complexes on filtered manifolds. We start by recalling the definition and proceed to place it in the context of the preceding subsection by building ST^{2}s for filtered manifolds.

Definition IV.2.14. Consider a collection $E_\bullet = (E_0, E_1, \dots, E_n)$ of hermitian vector bundles $E_j \to X$ and numbers $\boldsymbol{m} = (m_0, \dots, m_{n-1}) \in (0, \infty)^n$. We let

$$\begin{aligned} \mathrm{d}_\bullet : \quad & 0 \to C^\infty(X; E_0) \xrightarrow{\mathrm{d}_0} C^\infty(X; E_1) \xrightarrow{\mathrm{d}_1} \cdots \\ & \cdots \xrightarrow{\mathrm{d}_{n-2}} C^\infty(X; E_{n-1}) \xrightarrow{\mathrm{d}_{n-1}} C^\infty(X; E_n) \to 0 \end{aligned} \tag{IV.2.15}$$

be a complex with maps $\mathrm{d}_j \in \Psi_H^{m_j}(X; E_j, E_{j+1})$. We say that the complex $\mathrm{d}_\bullet$ in Equation (IV.2.15) is a *Rockland complex* if the symbol sequence $\sigma_H(\mathrm{d}_\bullet)$ defined by

$$\begin{aligned} \sigma_H(\mathrm{d}_\bullet) : \quad & 0 \to \mathcal{S}_0(T_H X; E_0) \xrightarrow{\sigma_H^{m_0}(\mathrm{d}_0)} \mathcal{S}_0(T_H X; E_1) \xrightarrow{\sigma_H^{m_1}(\mathrm{d}_1)} \cdots \\ & \cdots \xrightarrow{\sigma_H^{m_{n-2}}(\mathrm{d}_{n-2})} \mathcal{S}_0(T_H X; E_{n-1}) \xrightarrow{\sigma_H^{m_{n-1}}(\mathrm{d}_{n-1})} \mathcal{S}_0(T_H X; E_n) \to 0 \end{aligned} \tag{IV.2.16}$$

is localized to an exact sequence by any non-trivial, irreducible, unitary representation of the osculating Lie groupoid $T_H X$. We say that $\boldsymbol{m}$ is the order of $\mathrm{d}_\bullet$.

There are many interesting examples of Rockland sequences. As shown in [DH22], and further discussed in [Gof24], there is a general procedure for producing (graded) Rockland complexes via Čap, Slovák, and Souček's [ČSS01] (curved) BGG complexes. The notion of a graded Rockland complex is more general than that of a Rockland complex and arises from internal gradings in the bundles E_j. For a curved BGG complex to be Rockland, and not just graded Rockland, all the bundles $E_0, E_1, \dots, E_n$ need to be constantly graded, corresponding to $\mathfrak{t}_H X$ having pure cohomology groups [Hal22, §3.7]. This is known to hold for trivially filtered manifolds (where Rockland means elliptic), for contact manifolds, for generic rank-two distributions in dimension five, and for parabolic geometries of the same type as the full complex flag manifold of $SL(3, \mathbb{C})$ (as implicitly used in [Yun11]). We discuss contact manifolds in more detail below in §IV.2.4 and generic rank two distributions in dimension five in Example IV.2.19.

For the purpose of completing a Rockland complex into a Hilbert complex, we will henceforth fix a volume density on X and hermitian metrics on all the vector bundles $E_0, E_1, \dots, E_n \to X$, giving us Hilbert spaces $L^2(X; E_0), \dots, L^2(X; E_n)$. By an abuse of notation, we write also d_j for the closure of d_j as a densely defined operator

$$\mathrm{d}_j : L^2(X; E_j) \dashrightarrow L^2(X; E_{j+1}).$$

The Hilbert complex associated with a Rockland complex $(C^\infty(X; E_\bullet), \mathrm{d}_\bullet)$ is given by

$$0 \to L^2(X; E_0) \xrightarrow{\mathrm{d}_0} L^2(X; E_1) \xrightarrow{\mathrm{d}_1} \cdots \xrightarrow{\mathrm{d}_{n-1}} L^2(X; E_n) \to 0.$$

Theorem IV.2.17. *Assume that $(C^\infty(X; E_\bullet), \mathrm{d}_\bullet)$ is a Rockland complex where all differentials are differential operators and X is compact. The Hilbert complex associated with a Rockland complex $(C^\infty(X; E_\bullet), \mathrm{d}_\bullet)$ of order $\boldsymbol{m} = (m_1, \dots, m_n)$ is a regular Hilbert complex with discrete spectrum of order $\boldsymbol{m}$ over $C^\infty(X)$. In particular, with a Rockland complex we can associate the f-summable ST^2 $(C^\infty(X), L^2(X; \oplus_j E_j), \boldsymbol{D})$ where $\boldsymbol{D} = (\mathrm{d}_j + \mathrm{d}_j^*)_{j=0}^{n-1}$ and*

$$f(\boldsymbol{t}) > \min_j \frac{m_j \dim_h(X)}{t_j}.$$

Proof. The result will, upon checking the definition, follow from Theorem IV.2.4. Chasing through the definitions, we see that the Hilbert complex associated with a Rockland complex is a regular Hilbert complex over $C^\infty(X)$ as soon as the Rumin Laplacians are hypoelliptic of order $2m$. Indeed, if this is the case then, since the Rumin Laplacians additionally are even order differential operators, [DH19, Theorem 2] implies that $(\Delta^R_{\boldsymbol{m},i})^{\frac{\beta}{2m}} \in \Psi^\beta_H(X, E_i)$. The Theorem follows from order considerations in the Heisenberg calculus. The Rumin Laplacians are hypoelliptic by [DH22, Lemma 2.14].

We note the finite summability statement follows from (IV.2.13). Indeed, if we set $\delta(\boldsymbol{t}) := \min_j \frac{m_j \dim_h(X)}{t_j}$ the interpolation as in Lemma IV.1.5 and (IV.2.13) implies that $\mu_k((1 + \Delta^{\boldsymbol{D}}_{\boldsymbol{t}})^{-1}) = O(k^{-\delta(\boldsymbol{t})})$ as $k \to +\infty$. In particular, $(1 + \Delta^{\boldsymbol{D}}_{\boldsymbol{t}})^{-1} \in \mathcal{L}^p$ for any $p > \delta(\boldsymbol{t})$. $\square$

Remark IV.2.18. In the construction of Theorem IV.2.17 we group together the differentials in the easiest way possible, following Theorem IV.2.4. We can in general group together the differentials more efficiently, e.g. below in §IV.2.4 when studying the Rumin complex on a contact manifold we will group together the differentials into only two self-adjoint operators. If $(C^\infty(X; E_\bullet), \mathrm{d}_\bullet)$ is a Rockland complex of order $\boldsymbol{m} = (m_0, \dots, m_{n-1})$, we can consider a partition

$$\{0, \dots, n-1\} = \bigsqcup_{l=1}^{n_0} S_l,$$

such that $m_i = m_j$ whenever i and j belong to the same set S_l. Then the collection $\tilde{\boldsymbol{D}} := (\sum_{j \in S_l} \mathrm{d}_j + \mathrm{d}_j^*)_{l=1}^{n_0}$ also fits into an ST^2 $(C^\infty(X), L^2(X; \oplus_j E_j), \tilde{\boldsymbol{D}})$. The bounding matrix $\tilde{\boldsymbol{\epsilon}} = (\tilde{\epsilon}_{lk})_{l,k=1}^{n_0}$ for $\tilde{\boldsymbol{D}}$ is similar to (IV.2.5) and is given by

$$\tilde{\epsilon}_{lk} := \begin{cases} \frac{m_i - 1}{m_j} & \text{if there are } i \in S_l \text{ and } j \in S_k \text{ with } |j - i| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Example IV.2.19. Let us describe the Rockland complex constructed from the BGG complex on a generic rank two distribution in dimension five, i.e. a parabolic geometry of type (G_2, P) where G_2 is the split real form of the indicated exceptional Lie group and P the maximal parabolic subgroup corresponding to the shorter simple root. We aim only at describing the overall structure and refer the details to [DH22, Example 4.21] (see also Example 4.24 in the arXiv version [DH17] of [DH22] and further computational details in its appendix). Let X be a five dimensional manifold filtered by a generic rank two distribution throughout the example. We also fix a finite-dimensional representation V of G_2. The BGG complex of X looks like

$$0 \to C^\infty(X;E_0) \xrightarrow{\mathrm{d}_0} C^\infty(X;E_1) \xrightarrow{\mathrm{d}_1} C^\infty(X;E_2)$$
$$\xrightarrow{\mathrm{d}_2} C^\infty(X;E_3) \xrightarrow{\mathrm{d}_3} C^\infty(X;E_4) \xrightarrow{\mathrm{d}_4} C^\infty(X;E_5) \to 0,$$

where $E_j \to X$ is a bundle induced from the parabolic structure and the cohomology group $H^j(\mathfrak{p}_+, V)$. The BGG complex is by [DH22] a Rockland sequence of order

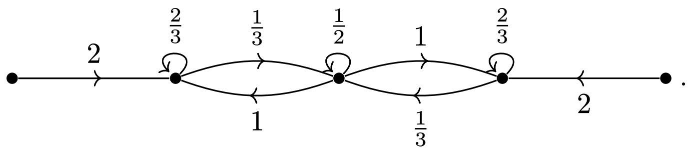
$$\boldsymbol{m} = (1,3,2,3,1).$$

To understand the principal symbol structure of the BGG complex of X, one uses the fact that X locally admits filtered charts modelled on the nilpotent chart $\overline{N} \subseteq G_2/P$ arising from the open, dense Bruhat cell $\overline{N}MAN \subseteq G_2$. In these charts, [DH22, Example 4.21] explicitly describes $\sigma_H^{m_j}(\mathrm{d}_j)$ in terms of elements of the universal enveloping Lie algebra of $\overline{N}$.

In this example, we have the bounding matrix

$$\boldsymbol{\epsilon} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & \frac{2}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the associated weighted digraph takes the form



We now turn to discussing two special cases of Theorem IV.2.17. We produce two higher order spectral triples from Rockland complexes, the first with an H-elliptic Heisenberg pseudodifferential operator and the second with a differential operator.

Corollary IV.2.20. *Let $(C^\infty(X;E_\bullet), \mathrm{d}_\bullet)$ be a Rockland complex where all differentials are differential operators. For any order $\tau > 0$, we can form the higher order spectral triple $(C^\infty(X), L^2(X; \oplus_j E_j), \overline{D}_\tau)$, p-summable for $p > \frac{\dim_h(X)}{\tau}$, from the H-elliptic Heisenberg operator*

$$\overline{D}_\tau := \overline{D}_{\boldsymbol{t}_{\boldsymbol{m}}(\tau)} \equiv \sum_{i=0}^{n-1} \mathrm{d}_i (\Delta^R_{\boldsymbol{m},i})^{\frac{\tau - m_i}{2m}} + \mathrm{d}_i^* (\Delta^R_{\boldsymbol{m},i+1})^{\frac{\tau - m_i}{2m}} \in \Psi_H^\tau(X, \oplus_j E_j).$$

Any H-elliptic Heisenberg operator of an order $\tau > 0$ defines a p-summable higher order spectral triple for $p > \frac{\dim_h(X)}{\tau}$, so Corollary IV.2.20 follows from the construction implying that $\overline{D}_\tau$ is H-elliptic of order $\tau > 0$. On the other hand, Theorem IV.1.16 together with Theorem IV.2.17 implies the next Corollary which allows us to construct from a Rockland complex a higher order spectral triple with a differential operator as its Dirac operator.

Corollary IV.2.21. *If $(C^\infty(X;E_\bullet), \mathrm{d}_\bullet)$ is a Rockland complex where all differentials are differential operators, there exist odd integers $\boldsymbol{k} = (2k_j+1)_j \in \Omega(\boldsymbol{\epsilon}) \cap (2\mathbb{N}+1)^n$ so that the differential operator*

$$\overline{D}_{\boldsymbol{k}} := \sum_{i=0}^{n-1} D_i^{2k_i+1} : C^\infty(X; \oplus_j E_j) \to C^\infty(X; \oplus_j E_j)$$

defines a higher order spectral triple $(C^\infty(X), L^2(X; \oplus_j E_j), \overline{D}_{\boldsymbol{k}})$.

We end this section by describing the K-homology class associated with a Rockland complex via Corollary IV.1.24 and Theorem IV.2.17.

Theorem IV.2.22. *Assume that $(C^\infty(X;E_\bullet),\mathrm{d}_\bullet)$ is a Rockland complex where all differentials are differential operators and write $(C^\infty(X),L^2(X;\oplus_j E_j),\boldsymbol{D})$ for its associated ST^2 graded by $L^2(X;\oplus_j E_j)=L^2(X;\oplus_j E_{2j})\oplus L^2(X;\oplus_j E_{2j+1})$. Take a $\boldsymbol{t}\in\Omega(\boldsymbol{\epsilon})$. The class of the higher order spectral triple $(C^\infty(X),L^2(X;\oplus_j E_j),\overline{D}_{\boldsymbol{t}})$ in $K_0(X)$ coincides with the class $[\mathrm{d}_\bullet]\in K_0(X)$ as defined in [GK24].*

Proof. The class $[\mathrm{d}_\bullet]\in K_0(X)$ as defined in [GK24] was defined by order reduction. If we use the Rumin–Seshadri Laplacians to define order reduction, a short algebraic manipulation shows that $|\overline{D}_{\tau=1}|$ lifts the Fredholm module defining the class $[\mathrm{d}_\bullet]\in K_0(X)$ to a bounded perturbation of $\overline{D}_{\tau=1}$. □

IV.2.3.1 Equivariance in Rockland complexes

We now turn to studying conformal equivariance of Rockland complexes.

Definition IV.2.23. Assume that $(C^\infty(X;E_\bullet),\mathrm{d}_\bullet)$ is a Rockland complex and that G is a locally compact group acting by filtered diffeomorphisms on X and that $E_0,\dots,E_n$ are G-equivariant. We say that $(C^\infty(X;E_\bullet),\mathrm{d}_\bullet)$ is a G-equivariant Rockland complex if the symbol complex $\sigma_H(\mathrm{d}_\bullet)$ (see (IV.2.16)) is G-equivariant.

If $(C^\infty(X;E_\bullet),\mathrm{d}_\bullet)$ is a G-equivariant Rockland complex with each E_j an hermitian vector bundle, we say that the G-action is a conformal G-action on $(C^\infty(X;E_\bullet),\mathrm{d}_\bullet)$ if for any j the G-representation $V_j: G\to \mathrm{GL}(L^2(X;E_j))$ defined from the G-action on $E_j\to X$ there is a function $\lambda_{j,g}\in C^\infty(X,\mathbb{R}_{>0})$ such that

$$V_{j,g}V_{j,g}^*=\lambda_{j,g}^2.$$

The associated unitary representations are

$$U_j: G\to U(L^2(X;E_j))\quad U_{j,g}=\lambda_{j,g}^{-1}V_{j,g}$$

and we observe that $U_{j+1,g}\mathrm{d}_jU_{j,g}^*=\lambda_{j+1,g}^{-1}\mathrm{d}_j\lambda_{j,g}$.

Proposition IV.2.24. *Assume that $(C^\infty(X;E_\bullet),\mathrm{d}_\bullet)$ is a Rockland complex of order $\boldsymbol{m}=(m_1,\dots,m_n)$, where all differentials are differential operators and X is compact, with a conformal action of G. For $\boldsymbol{t}\in\Omega(\boldsymbol{\epsilon})$, the higher order spectral triple*

$$(C^\infty(X),L^2(X;\oplus_j E_j),\overline{D}_{\boldsymbol{t}})$$

is conformally G-equivariant with conformal factor

$$\mu_g=\bigoplus_{j=0}^n P_{\mathscr{H}_i}+P_{\operatorname{im}\mathrm{d}_{j-1}}(\lambda_{j,g}^{-1}\lambda_{j-1,g})^{t_{j-1}/2}P_{\operatorname{im}\mathrm{d}_{j-1}}+P_{\operatorname{im}\mathrm{d}_j^*}(\lambda_{j+1,g}^{-1}\lambda_{j,g})^{t_j/2}P_{\operatorname{im}\mathrm{d}_j^*}.$$

Proof. Because $\lambda_{j,g}$ is nonvanishing, bounded, and positive, μ_g is invertible and positive. Indeed,

$$\mu_g\geq\bigoplus_{j=0}^n P_{\mathscr{H}_i}+\|\lambda_{j,g}\lambda_{j-1,g}^{-1}\|_\infty^{-t_{j-1}/2}P_{\operatorname{im}\mathrm{d}_{j-1}}+\|\lambda_{j+1,g}\lambda_{j,g}^{-1}\|^{-t_j/2}P_{\operatorname{im}\mathrm{d}_j^*}.$$

Using the notation $(\mathrm{d}_j)^{t_j}=\mathrm{d}_j(\mathrm{d}_j^*\mathrm{d}_j)^{-1+t_j}$, one can check that the difference

$$\begin{aligned}
&U_g\mathrm{d}_j(\mathrm{d}_j^*\mathrm{d}_j)^{-1+t_j}U_g^*-\mu_g\mathrm{d}_j(\mathrm{d}_j^*\mathrm{d}_j)^{-1+t_j}\mu_g^*\\
&\quad=U_g\mathrm{d}_j(\mathrm{d}_j^*\mathrm{d}_j)^{-1+t_j}U_g^*-P_{\operatorname{im}\mathrm{d}_j}(\lambda_{j+1,g}^{-1}\lambda_{j,g})^{t_j/2}\mathrm{d}_j(\mathrm{d}_j^*\mathrm{d}_j)^{-1+t_j}(\lambda_{j+1,g}^{-1}\lambda_{j,g})^{t_j/2}P_{\operatorname{im}\mathrm{d}_j^*}\\
&\quad=P_{\operatorname{im}\mathrm{d}_j}\left(U_g\mathrm{d}_j(\mathrm{d}_j^*\mathrm{d}_j)^{-1+t_j}U_g^*-(\lambda_{j+1,g}^{-1}\lambda_{j,g})^{t_j/2}\mathrm{d}_j(\mathrm{d}_j^*\mathrm{d}_j)^{-1+t_j}(\lambda_{j+1,g}^{-1}\lambda_{j,g})^{t_j/2}\right)P_{\operatorname{im}\mathrm{d}_j^*}
\end{aligned}$$

and the commutator

$$[\mathrm{d}_j(\mathrm{d}_j^*\mathrm{d}_j)^{-1+t_j}, \mu_g] = P_{\mathrm{im}\,\mathrm{d}_j}[\mathrm{d}_j(\mathrm{d}_j^*\mathrm{d}_j)^{-1+t_j}, (\lambda_{j+1,g}^{-1}\lambda_{j,g})^{t_j/2}]P_{\mathrm{im}\,\mathrm{d}_j^*}$$

are of lower order, since $\mathrm{d}_j(\mathrm{d}_j^*\mathrm{d}_j)^{-1+t}$ belongs to the Heisenberg calculus by Lemma IV.2.3, as required by the definition of conformal equivariance. □

An undesirable feature of the above construction is that the conformal factors are not functions on X. Under some circumstances, this can be remedied but then only for certain $\boldsymbol{t} \in \Omega(\boldsymbol{\epsilon})$.

Proposition IV.2.25. *Assume that $(C^\infty(X; E_\bullet), \mathrm{d}_\bullet)$ is a Rockland complex of order $\boldsymbol{m} = (m_j)_{j=0}^{n-1}$, where all differentials are differential operators and X is compact, with a conformal action of G. Suppose that, for some $\boldsymbol{s} \in \Omega(\boldsymbol{\epsilon})$,*

$$\lambda_{j-1,g}^{s_{j-1}}\lambda_{j+1,g}^{s_j} = \lambda_{j,g}^{s_j+s_{j-1}}$$

for all $j = 1, \dots, n$. Then, for all $\tau > 0$, the higher order spectral triple

$$(C^\infty(X), L^2(X; \oplus_j E_j), \overline{D}_{\tau\boldsymbol{s}})$$

is conformally G-equivariant with conformal factor

$$\mu_g = (\lambda_{1,g}^{-1}\lambda_{0,g})^{\tau s_0} = \dots = (\lambda_{n,g}^{-1}\lambda_{n-1,g})^{\tau s_{n-1}}.$$

Remark IV.2.26. If we can take $\boldsymbol{s} = \boldsymbol{m}$, in the situation of Proposition IV.2.25, i.e. if

$$\lambda_{j-1,g}^{m_j}\lambda_{j+1,g}^{m_{j-1}} = \lambda_{j,g}^{m_{j-1}+m_j},$$

the higher order spectral triple $(C^\infty(X), L^2(X; \oplus_j E_j), \overline{D}_\tau)$ of order $\tau > 0$ defined from the H-elliptic Heisenberg operator $\overline{D}_\tau$ (as in Corollary IV.2.20), is a conformally G-equivariant higher order spectral triple with conformal factor

$$\mu_g = (\lambda_{1,g}^{-1}\lambda_{0,g})^{\tau/m_0} = \dots = (\lambda_{n,g}^{-1}\lambda_{n-1,g})^{\tau/m_{n-1}}.$$

We will see that this in fact does occur for the Rumin complex on a CR-manifold, in Theorem IV.2.34.

Remark IV.2.27. In the next section we provide further context for conformally equivariant Rockland complexes by studying the Rumin complex on a contact manifold. It would be interesting to include further examples of Rockland complexes, especially in higher rank parabolic geometries. As work by Yuncken [Yun11] and Voigt–Yuncken [VY15] showcases, the interesting aspect lies in the equivariance properties. However, the approach above cannot produce conformally equivariant noncommutative geometries with nontrivial index theory, or even equivariant Fredholm modules, for a semisimple Lie group G of real rank > 1. Indeed, if G is a higher rank semisimple Lie group and and T is an H-elliptic operator on G/P (for some parabolic subgroup $P \subseteq G$) of order $m \geq 0$ commuting with G up to lower order terms then Puschnigg rigidity [Pus11] implies that $\sigma_H^m(T)$ is positive and that T defines the trivial equivariant K-homology class. For $SL(3, \mathbb{C})$, as studied in [VY15, Yun11], the BGG complex is an equivariant Rockland complex (in the sense of Definition IV.2.23) but it is not conformally equivariant. The same statement holds for the BGG complex of G_2/P, see Example IV.2.19 above.

A separate but equally serious issue at play, as discussed in [DH22, Hal22], is that a BGG complex is frequently not a Rockland complex but only a graded Rockland complex. The BGG complex of a parabolic geometry is Rockland in the usual sense only when the cohomology of the osculating nilpotent group in each fibre has pure cohomology; see [Hal22, §3.7] for more details. For index theory purposes [Gof24], the graded Rockland situation works well but it is less clear how to do spectral noncommutative geometry with graded Rockland complexes. The BGG complex arising from the quaternionic contact structure on $\mathbf{S}^{4n-1}$ [Jul95, §3] [Rum05, (66–67)] is an example which fails to be ungraded Rockland but for which the action of $Sp(n, 1)$ is conformal, in a sense made clear in [Jul19]. In particular, the two issues of conformally equivariant geometries and representing geometries by ungraded Rockland complexes are quite distinct.

IV.2.4 The Rumin complex on contact manifolds

In this section we will look at an explicit example of a Rockland complex, namely the Rumin complex on a contact manifold. We will show that the naïve way of constructing a spectral triple from the Rumin complex does not work. However, using our construction with tangled spectral triples we obtain higher order spectral triples as in Theorem IV.1.16. Lastly, we will look at conformal equivariance under CR-automorphisms when the manifold has an almost CR-structure.

We have already seen the Rumin complex in §III.2.1.3. Let X be a $(2n+1)$-dimensional contact manifold with contact structure $H \subseteq TX$. Following [Rum00] we can obtain a different description of the Rumin complex as follows. Let us fix a contact form θ and choose a Riemannian metric $\mathbf{g}$ on X. We require that these be compatible, in the sense that H is orthogonal to the Reeb field, the (unique) vector field Z such that $\theta(Z) = 1$ and $\iota_Z(d\theta) = 0$. With our choice of metric, we have an orthogonal splitting

$$T^*X = H^* \oplus H^\perp$$

defined from the contact coorientation θ spanning $H^\perp$. The exterior derivative takes the form

$$\mathrm{d} = \begin{pmatrix} \mathrm{d}_H & L \\ \mathcal{L}_Z & -\mathrm{d}_H \end{pmatrix} \text{ in the splitting } \Lambda^*T^*X = (\Lambda^*H^*) \oplus (H^\perp \otimes \Lambda^*H^*).$$

Here $\mathcal{L}_Z$ denotes the Lie derivative along the Reeb field Z and L denotes exterior multiplication with $\mathrm{d}\theta$. We note that $\mathcal{J}^{k+1} = C^\infty(X; H^\perp \otimes F_k)$ where $F_k = \ker L \cap \Lambda^k H^*$ and each element in $\Omega^k/\mathcal{J}^k$ has a unique representative in $C^\infty(X; E_k)$ where $E_k = (\operatorname{im} L)^\perp \cap \Lambda^k H^*$. With this, the Rumin complex takes the form

$$\begin{aligned} 0 \to C^\infty(X;E_0) &\xrightarrow{P_{E_1}\mathrm{d}_H} C^\infty(X;E_1) \xrightarrow{P_{E_2}\mathrm{d}_H} \cdots \\ \cdots &\xrightarrow{P_{E_n}\mathrm{d}_H} C^\infty(X;E_n) \xrightarrow{D_R} C^\infty(X;H^\perp \otimes F_n) \xrightarrow{-\mathrm{d}_H} \cdots \\ \cdots &\xrightarrow{-\mathrm{d}_H} C^\infty(X;H^\perp \otimes F_{2n-1}) \xrightarrow{-\mathrm{d}_H} C^\infty(X;H^\perp \otimes F_{2n}) \to 0 \end{aligned}$$

where the Rumin differential D_R can be expressed as the second order differential operator

$$D_R = \theta \wedge (\mathcal{L}_Z + \mathrm{d}_H L^{-1} \mathrm{d}_H). \tag{IV.2.28}$$

Note that $L\colon C^\infty(X;\Lambda^k H^*) \to C^\infty(X;\Lambda^{k+1}H^*)$ is injective for $k \leq n-1$ and surjective for $k \geq n-1$ [Rum94], which is utilized to show that D_R is well-defined.

It is well known that the Rumin complex $(C^\infty(X;E_\bullet), \mathrm{d}^R_\bullet)$ on a cooriented contact manifold X is a Rockland complex [JK95, Rum94]. A detailed discussion thereof can be found in Example 4.21 of the arXiv version of [DH22]. We shall write the Rumin complex as $\mathrm{d}^R_\bullet$. This is a mixed order differential complex.

Let us now describe the symbol complex of the Rumin complex in some more detail. We do the same procedure as for the Rockland sequences in (IV.2.16) and identify $T_H X_x \cong \mathsf{H}_{2n+1}$ with the Heisenberg group via Darboux coordinates for each point $x \in X$. Write $X_1, \dots, X_n, Y_1, \dots, Y_n, Z$ for the standard generators of $\mathfrak{h}_{2n+1}$ with $[X_i, Y_j] = \delta_{ij} Z$ corresponding to the Darboux coordinates near x. We will identify the fibres $E_{k,x}$ and $H_x^\perp \otimes F_{k,x}$ with subspaces of $\Lambda^k H_x^* = \Lambda^k \mathbb{R}^{2n}$ and $H_x^\perp \otimes \Lambda^k H_x^* = \Lambda^k \mathbb{R}^{2n}$ respectively. Consider the $\mathfrak{h}_{2n+1}$-valued vector

$$\omega_1 = \begin{pmatrix} X_1 & \dots & X_n & Y_1 & \dots & Y_n \end{pmatrix}^T \in \mathfrak{h}_{2n+1} \otimes H_x^*.$$

We can express the principal symbols of the differentials in the Rumin complex as

$$\begin{aligned} \sigma_H^1(\mathrm{d}^R_j)_x &= \omega_1 \wedge : \mathcal{S}_0(\mathsf{H}_{2n+1}, E_{j,x}) \to \mathcal{S}_0(\mathsf{H}_{2n+1}, E_{j+1,x}) && (j < n), \\ \sigma_H^2(\mathrm{d}^R_j)_x &= \sigma_H^2(D_R)_x \\ &= Z + (\omega_1 \wedge) L^{-1} (\omega_1 \wedge) : \mathcal{S}_0(\mathsf{H}_{2n+1}, E_{n,x}) \to \mathcal{S}_0(\mathsf{H}_{2n+1}, F_{n,x}) && (j = n), \\ \sigma_H^1(\mathrm{d}^R_j)_x &= -\omega_1 \wedge : \mathcal{S}_0(\mathsf{H}_{2n+1}, F_{j-1,x}) \to \mathcal{S}_0(\mathsf{H}_{2n+1}, F_{j,x}) && (j > n). \end{aligned}$$

In the case of $n = 1$, we can identify

$$E_{0,x} = \mathbb{C},\ E_{1,x} = F_{1,x} = \mathbb{C}^2,\ \text{and}\ F_{2,x} = \mathbb{C}.$$

Under these identifications, we have that $\sigma_H^2(D_R) = Z + \omega_1\omega_2^*$ where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \omega_2 = J\omega_1 = \begin{pmatrix} -Y \\ X \end{pmatrix}.$$

The symbol complex over x takes the form

$$0 \to \mathcal{S}_0(\mathsf{H}_3) \xrightarrow{\begin{pmatrix} X \\ Y \end{pmatrix}} \mathcal{S}_0(\mathsf{H}_3) \otimes \mathbb{C}^2 \xrightarrow{\begin{pmatrix} Z+XY & -X^2 \\ Y^2 & Z-YX \end{pmatrix}} \mathcal{S}_0(\mathsf{H}_3) \otimes \mathbb{C}^2 \xrightarrow{\begin{pmatrix} Y & -X \end{pmatrix}} \mathcal{S}_0(\mathsf{H}_3) \to 0.$$

The reader can compare this to Example IV.1.12 and the BGG complex for $SL(3,\mathbb{C})$ studied by Yuncken [Yun11]. In the next section, we continue to analyse this special case.

IV.2.4.1 A naïve attempt at a spectral triple for the Rumin complex

A first approach to study the noncommutative geometry of the Rumin complex is to naïvely roll up the complex as

$$\not{D}^R := \mathrm{d}_\bullet^R + (\mathrm{d}_\bullet^R)^*.$$

Rolling up a complex in this way is how one produces the Hodge–de Rham Dirac operator from the de Rham complex. We shall see that this approach fails to produce a higher order spectral triple, thereby justifying the approach of §IV.2 and the need for the decreasing cycle condition.

By discreteness of the spectrum, $\not{D}^R$ has compact resolvent. However, taking commutators with $C^\infty(X)$ does not improve the order. We will show this in the case of three-dimensional contact manifolds. In Darboux coordinates, the Rumin complex is up to lower order terms given by

$$\mathrm{d}_\bullet^R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega_1 & 0 & 0 & 0 \\ 0 & Z+\omega_1\omega_2^* & 0 & 0 \\ 0 & 0 & \omega_2^* & 0 \end{pmatrix}.$$

Therefore $\not{D}^R$ takes the form

$$\not{D}^R = \begin{pmatrix} 0 & \omega_1^* & 0 & 0 \\ \omega_1 & 0 & -Z+\omega_2\omega_1^* & 0 \\ 0 & Z+\omega_1\omega_2^* & 0 & \omega_2 \\ 0 & 0 & \omega_2^* & 0 \end{pmatrix}.$$

Proposition IV.2.29. *Let X be a compact contact manifold of dimension 3 and $a \in C^\infty(X)$. Then $[\not{D}^R, a]$ is up to a vector bundle endomorphism of the form*

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_2\omega_1^*(a) + \omega_2(a)\omega_1^* & 0 \\ 0 & \omega_1\omega_2^*(a) + \omega_1(a)\omega_2^* & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

in local Darboux coordinates.

Proof. Follows from direct computation with the Leibniz rule. □

Proposition IV.2.30. *Let X be a compact contact manifold of dimension 3. If $\alpha \in \mathbb{R}$ satisfies that $[\not{D}^R, a](1 + (\not{D}^R)^2)^{-1/2+\alpha}$ is a bounded operator on $L^2(X; \mathcal{H})$ for any $a \in C^\infty(X)$ then $\alpha \leq 0$.*

In consequence, $\not{D}^R$ does not define a higher order spectral triple for $C_\infty(X)$.

Proof. We need to show that, for all $\alpha > 0$, $[\not{D}^R, a](1 + (\not{D}^R)^2)^{-1/2+\alpha}$ fails to be bounded on $L^2(X; \mathcal{H})$ for some $a \in C^\infty(X)$. By Proposition IV.2.29 and the computations above, $[\not{D}^R, a](1 + (\not{D}^R)^2)^{-1/2+\alpha}$ is bounded if and only if $(\omega_1\omega_2^*(a) + \omega_1(a)\omega_2^*)(1 + T)^{-1/2+\alpha}$ is bounded where $T = \omega_1\omega_1^* + (-Z + \omega_2\omega_1^*)(Z + \omega_1\omega_2^*)$.

Were $(\omega_1\omega_2^*(a) + \omega_1(a)\omega_2^*)(1 + T)^{-1/2+\alpha}$ to be bounded, we could freeze coefficients in a point x and represent this operator in a non-trivial character $\xi \in \mathbb{R}^2 \subseteq \widehat{\mathsf{H}}_{2n+1}$ and obtain a uniformly bounded function in ξ. For notational simplicity, write

$$v := \omega_1(a)_x.$$

In this notation, $\omega_2(a)_x = Jv$. In a character ξ, ω_1 is represented as ξ, ω_2 is represented as $J\xi$ and Z is represented as 0. Hence, T is represented in the character $\xi \neq 0$ as the matrix valued function

$$F(\xi) = \xi\xi^* + |\xi|^2(J\xi)(J\xi)^* = |\xi|^2 e_1(\xi) + |\xi|^4 e_2(\xi),$$

where $e_1(\xi) = |\xi|^{-2}\xi\xi^*$ and $e_2(\xi) = Je_1(\xi)J$ are the orthogonal projections onto the span of ξ and $J\xi$ respectively. Since J is anti-symmetric, $e_1(\xi)$ and $e_2(\xi)$ have orthogonal ranges and $F(\xi) = |\xi|^2 e_1(\xi) + |\xi|^4 e_2(\xi)$ is the eigenvalue decomposition of $F(\xi)$. By the discussion above, we need to show that for $\alpha > 0$, boundedness fails for the matrix valued function

$$A_\alpha(\xi) := (\xi(Jv)^* + v(J\xi)^*)(1 + F(\xi))^{-1/2+\alpha}.$$

By orthogonality of $e_1(\xi)$ and $e_2(\xi)$, we compute that

$$\begin{aligned} A_\alpha(\xi) &= (1 + |\xi|^2)^{-\frac{1}{2}+\alpha}(\xi(Jv)^* + v(J\xi)^*)e_1(\xi) + (1 + |\xi|^4)^{-\frac{1}{2}+\alpha}(\xi(Jv)^* + v(J\xi)^*)e_2(\xi) \\ &= (1 + |\xi|^2)^{-\frac{1}{2}+\alpha}((Jv)^*\xi)e_1(\xi) + O(|\xi|^{-1+4\alpha}) \end{aligned}$$

and see that for $t > 0$

$$A_\alpha(tJv) = t(1 + t^2)^{-\frac{1}{2}+\alpha}e_1(Jv) + O(t^{-1+4\alpha}).$$

In particular, A_α is bounded if and only if $\alpha \leq 0$ so in particular boundedness fails for $\alpha > 0$. $\square$

IV.2.4.2 Strictly tangled spectral triples from the Rumin complex

Let us place the Rumin complex $\mathrm{d}^R_\bullet$ of a contact manifold in a spectral triple. We have a somewhat simpler structure than seen in §IV.2.1, since all but one of the differentials are order one, see Remark IV.2.18. We consider the two self-adjoint operators

$$D_1 = \sum_{j \neq n} \mathrm{d}_j^R + (\mathrm{d}_j^R)^* \qquad D_2 = D_R + (D_R)^*.$$

These are differential operators of order $m_1 = 1$ and $m_2 = 2$ respectively. We note that $D_1 D_2 = D_2 D_1 = 0$ on the common core $C^\infty(X; \oplus_j E_j)$, so D_1 and D_2 are strictly anticommuting. We compute that

$$D_1^2 = \sum_{j \neq n} \mathrm{d}_j^R(\mathrm{d}_j^R)^* + (\mathrm{d}_j^R)^*\mathrm{d}_j^R \qquad D_2^2 = D_R(D_R)^* + (D_R)^*D_R.$$

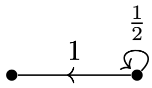
The Rumin Laplacian takes the form

$$\Delta^R = D_1^4 + D_2^2.$$

We can proceed as in §IV.2.3 to prove the following.

Proposition IV.2.31. *Consider the Rumin complex* $\mathrm{d}^R_\bullet$ *on a* $2n+1$*-dimensional compact contact manifold* X*. Then with* D_1 *and* D_2 *as in the preceding paragraph, the data* $(C^\infty(X), L^2(X;\mathcal{H}), (D_1, D_2))$ *constitute an* ST^2 *with bounding matrix*

$$\epsilon = \begin{pmatrix} 0 & 0 \\ 1 & \frac{1}{2} \end{pmatrix}$$



,

which is f*-summable for any function* f *with*

$$f(t_1, t_2) > 2\min\left(\frac{n+1}{t_1}, \frac{2(n+1)}{t_2}\right).$$

In particular, for any $\boldsymbol{t} = (t_1, t_2) \in (0,\infty)^2$ *with* $t_1 > t_2$*, we arrive at a higher order spectral triple defined from the operator*

$$\overline{D}_{\boldsymbol{t}} = D_1|D_1|^{t_1-1} + D_2|D_2|^{t_2-1} = D_1(\Delta^R)^{\frac{t_1-1}{4}} + D_2(\Delta^R)^{\frac{t_2-1}{2}}.$$

If $\boldsymbol{t}$ *lies along the ray spanned by* $(1, 1/2)$ *then* $\overline{D}_{\boldsymbol{t}}$ *is an* H*-elliptic operator in the Heisenberg calculus and if* $\boldsymbol{t} = (2k_1+1, 2k_2+1)$ *where* $k_1 > k_2$ *are natural numbers then* $\overline{D}_{\boldsymbol{t}}$ *is a differential operator.*

Remark IV.2.32. We note that

$$d(x,y) := \sup\{|a(x) - a(y)| \mid \|[D_1, a]\| \le 1\} = \sup\{|a(x) - a(y)| \mid \tfrac{1}{2}\|[[D_2, a], a]\| \le 1\}$$

and coincides with the Carnot–Carathéodory distance of X. In [Has14, §3.3], compact quantum metric spaces are built from Carnot manifolds using a 'horizontal Dirac operator' similar to D_1. Related results are found in [GG19]. There is a potential for interesting metric aspects of ST^2s to be considered. In this connection, we mention also the work [KK20, KK25] of Kaad and Kyed which uses a collection of operators for constructing quantum metric spaces.

IV.2.4.3 CR-equivariance

We discussed to some extent the equivariance of the Rumin complex in §III.2.1.3; let us see how it applies to the ST^2 of the entire complex. Recall the setup: X is a cooriented contact manifold with a fixed contact form θ and a Riemannian metric $\mathbf{g}$ in which the orthogonal complement of $H = \ker\theta$ is the Reeb field. The two-form $\mathrm{d}\theta$ defines a symplectic form on H. An almost CR-structure is the additional datum of a complex structure J on H such that $\mathbf{g}(v,w) = \mathrm{d}\theta(v, Jw)$. Note that the complex structure J and the metric $\mathbf{g}$ uniquely determine one another.

We let $\mathrm{Aut}_{CR}(X)$ denote the group of CR-automorphisms of X. That is, the group of diffeomorphisms $g : X \to X$ such that Dg preserves H (i.e. $(Dg)_x H_x \subseteq H_{g(x)}$ for all x) and acts complex linearly on H (i.e. $(Dg)_x : (H_x, J_x) \to (H_{g(x)}, J_{g(x)})$ is complex linear for all x). The group of CR-automorphisms is generically a compact subgroup, as the following result of Schoen [Sch95] proves.

Theorem IV.2.33 (Schoen's Ferrand–Obata theorem)**.** *Let* X *be a compact cooriented contact manifold with a choice of Riemannian metric as above. The group* $\mathrm{Aut}_{CR}(X)$ *can equivalently be topologized by its compact-open topology,* C^0*- or* C^∞*-topology. The group* $\mathrm{Aut}_{CR}(X)$ *is compact unless* X *is an odd-dimensional sphere with its round contact structure and metric and in this case* $X = SU(n,1)/P$ *for the standard parabolic subgroup* $P \subseteq SU(n,1)$ *and* $\mathrm{Aut}_{CR}(X) \cong SU(n,1)$*.*

As we saw in §III.2.1.3, the action of a CR-automorphism has features similar to being conformal. A contact form for a given contact structure is unique up to multiplication by a nonvanishing smooth function on X. Because the contact structure is preserved by a CR-automorphism g, the pullback $g^*(\theta)$ of the contact form must be equal to $f\theta$ for some nonvanishing smooth function f. Hence

$$g^*(\mathbf{g})(X,Y) = g^*(\mathrm{d}\theta)(X, JY) = (f\mathrm{d}\theta + \mathrm{d}f\wedge\theta)(X, JY) = f\mathrm{d}\theta(X, JY) = f\mathbf{g}(X,Y)$$

for all $X, Y \in H$. Moreover, the induced metric on TX/H is multiplied by f^2.

We conclude that, if $g \in \mathrm{Aut}_{CR}(X)$, the differential of g lifts to a graded vector-bundle action

$$v_g \colon E_\bullet \to E_\bullet,$$

with

$$v_g^* v_g = \bigoplus_{k=0}^{n} \lambda_g^{2k} \oplus \bigoplus_{k=n}^{2n} \lambda_g^{2(k+2)},$$

in accordance with the grading of $E_\bullet$. We define an action

$$V : \mathrm{Aut}_{CR}(X) \to \mathrm{GL}(L^2(X; E_\bullet)), \quad V(g)f(x) := v_g f(g^{-1}x).$$

Since the volume form belongs to $\Lambda^n H^* \otimes H^\perp$ it rescales with λ_g^{2n+2} under $g \in \mathrm{Aut}_{CR}(X)$. Therefore

$$V(g)^* V(g) = \lambda_g^{2n+2} v_g^* v_g = \bigoplus_{k=0}^{n} \lambda_g^{2(k+n+1)} \oplus \bigoplus_{k=n}^{2n} \lambda_g^{2(k+n+3)}.$$

The Rumin complex of X is defined from a quotient complex and a subcomplex of the de Rham complex spliced with the Rumin differential. As such, the Rumin complex is invariant under $\mathrm{Aut}_{CR}(X)$. In other words,

$$V(g) \mathrm{d}_\bullet^R V(g)^{-1} = \mathrm{d}_\bullet^R.$$

Set $\Lambda_g = V(g)^* V(g)$, so $V(g)^* = \Lambda_g V(g^{-1})$. We conclude that

$$V(g)^* \mathrm{d}_\bullet^R V(g) = \Lambda_g \mathrm{d}_\bullet^R.$$

If we pass to the unitarised action

$$U : \mathrm{Aut}_{CR}(X) \to \mathcal{U}(L^2(X; E_\bullet)), \quad U(g) := V(g) \Lambda_g^{-1/2},$$

we see that

$$U(g)^* \mathrm{d}_\bullet^R U(g) = \Lambda_g^{1/2} \mathrm{d}_\bullet^R \Lambda_g^{-1/2}.$$

From Proposition IV.2.25 we conclude the following.

Theorem IV.2.34. *Let $(C^\infty(X; E_\bullet), \mathrm{d}_\bullet^R)$ denote a Rumin complex on a $(2n+1)$-dimensional almost CR-manifold X with its conformal action of $G = \mathrm{Aut}_{CR}(X)$. For $\tau > 0$, the H-elliptic Heisenberg operator*

$$D_\tau = D_1 |D_1|^{\tau - 1} + D_2 |D_2|^{\frac{\tau}{2} - 1} = D_1 (\Delta^R)^{\frac{\tau-1}{4}} + D_2 (\Delta^R)^{\frac{\tau-2}{4}}$$

defines a conformally G-equivariant, $(\frac{\tau}{2n+2}, \infty)$-summable, order τ spectral triple

$$(C^\infty(X), L^2(X; \oplus_j E_j), D_\tau)$$

with conformal factor $\mu = \lambda^{-\tau}$.

IV.3 Examples arising from the Kasparov product

IV.3.1 Group C*-algebras of nilpotent groups

Let G be a simply connected, nilpotent Lie group G. As a manifold G is diffeomorphic to $\mathbb{R}^n$ for some n and its maximal compact subgroup is trivial. Hence the dual Dirac element, as defined by Kasparov [Kas88, §5], is an element of $KK_*^G(\mathbb{C}, C_0(G))$. By Baaj–Skandalis duality, there is an isomorphism of the K-groups $KK_*^G(\mathbb{C}, C_0(G))$ and $KK_*^{\hat{G}}(C^*(G), \mathbb{C})$, where $\hat{G}$ is the dual quantum group. Nilpotent groups are generally not CAT(0), and we turn to the framework of ST^2s.

Recall from Definition II.2.2 that a weight on a locally compact group G is a continuous function ℓ from G to matrices on a finite-dimensional complex vector space V. If V is $\mathbb{Z}/2\mathbb{Z}$-graded, ℓ is required to be odd.

Definition IV.3.1. Fixing a finite-dimensional complex vector space V, we will say that a finite collection of weights $\ell = (\ell_j)_{j\in I} : G \to \operatorname{End} V$ on a locally compact group G is

1. *self-adjoint* if $\ell_j^* = \ell_j$ for all j;
2. *proper* if $(\ell_j)_{j\in I}$ mutually anticommute and $\prod_j (1+|\ell_j|)^{-1} \in C_0(G, \operatorname{End} V)$; and
3. *translation bounded* with bounding matrix $\epsilon \in M_n([0,\infty))$ if, for all $g \in G$,

$$\sup_{h\in G} \left\| (\ell_i(gh) - \ell_i(h)) \Big(1 + \sum_{j\in I} |\ell_j(h)|^{\epsilon_{ij}}\Big)^{-1} \right\| < \infty$$

and there exists a neighbourhood U of the identity in G such that, for all $s \in G$,

$$\sup_{g\in U, h\in G} \left\| (\ell_i(gh) - \ell_i(h)) \Big(1 + \sum_{j\in I} |\ell_j(sh)|^{\epsilon_{ij}}\Big)^{-1} \right\| < \infty.$$

Note that the second part of the translation-boundedness condition is automatic for a discrete group. Whether this condition can be simplified in general is unclear. For the time being, we content ourselves with giving two equivalent conditions.

Lemma IV.3.2. *Let G be a locally compact group, V a finite-dimensional complex vector space, and $(\ell_j)_{j\in I} : G \to \operatorname{End} V$ a collection of weights. The following are equivalent:*

1. *For all $g \in G$,*

$$\sup_{i\in I, h\in G} \left\| (\ell_i(gh) - \ell_i(h)) \Big(1 + \sum_{j\in I} |\ell_j(h)|^{\epsilon_{ij}}\Big)^{-1} \right\| < \infty$$

and there exists a neighbourhood U of the identity in G such that, for all $s \in G$,

$$\sup_{i\in I, g\in U, h\in G} \left\| (\ell_i(gh) - \ell_i(h)) \Big(1 + \sum_{j\in I} |\ell_j(sh)|^{\epsilon_{ij}}\Big)^{-1} \right\| < \infty.$$

2. *For every compact subset $K \subseteq G$,*

$$\sup_{i\in I, g\in K, h\in G} \left\| (\ell_i(gh) - \ell_i(h)) \Big(1 + \sum_{j\in I} |\ell_j(h)|^{\epsilon_{ij}}\Big)^{-1} \right\| < \infty.$$

3. *The functions $(\zeta_i)_{i\in I} : G \to C(G, \operatorname{End} V)$ given by*

$$\zeta_i(g)(h) = (\ell_i(gh) - \ell_i(h)) \Big(1 + \sum_{j\in I} |\ell_j(h)|^{\epsilon_{ij}}\Big)^{-1}$$

are elements of $C(G, C_b(G, \operatorname{End} V)_\beta)$, where β is the strict topology.

Proof. Suppose that 1. holds and let K be a compact subset of G. The open sets $(Ug)_{g\in K}$ cover K. Let $Ug_1, \dots, Ug_k$ be a finite subcover. We have

$$\begin{aligned}
&\sup_{i\in I, g\in K, h\in G} \left\| (\ell_i(gh) - \ell_i(h))\Big(1 + \sum_{j\in I} |\ell_j(h)|^{\epsilon_{ij}}\Big)^{-1} \right\| \\
&\quad\le \sup_{1\le r\le k, i\in I, g\in Ug_r, h\in G} \left\| (\ell_i(gh) - \ell_i(h))\Big(1 + \sum_{j\in I} |\ell_j(h)|^{\epsilon_{ij}}\Big)^{-1} \right\| \\
&\quad= \sup_{1\le r\le k, i\in I, g\in U, h\in G} \left\| (\ell_i(gg_r^{-1}h) - \ell_i(h))\Big(1 + \sum_{j\in I} |\ell_j(h)|^{\epsilon_{ij}}\Big)^{-1} \right\| \\
&\quad\le \sup_{1\le r\le k, i\in I, g\in U, h\in G} \Bigg(\left\| (\ell_i(gg_r^{-1}h) - \ell_i(g_r^{-1}h))\Big(1 + \sum_{j\in I} |\ell_j(h)|^{\epsilon_{ij}}\Big)^{-1} \right\| \\
&\qquad + \sup_{1\le r\le k, i\in I, h\in G} \left\| (\ell_i(g_r^{-1}h) - \ell_i(h))\Big(1 + \sum_{j\in I} |\ell_j(h)|^{\epsilon_{ij}}\Big)^{-1} \right\| \\
&\quad\le \sup_{1\le r\le k, i\in I, g\in U, h\in G} \left\| (\ell_i(gh) - \ell_i(h))\Big(1 + \sum_{j\in I} |\ell_j(g_rh)|^{\epsilon_{ij}}\Big)^{-1} \right\| \\
&\qquad + \sup_{1\le r\le k, i\in I, h\in G} \left\| (\ell_i(g_r^{-1}h) - \ell_i(h))\Big(1 + \sum_{j\in I} |\ell_j(h)|^{\epsilon_{ij}}\Big)^{-1} \right\| \\
&\quad< \infty,
\end{aligned}$$

that is, 2. is satisfied.

Suppose that 2. holds and, by the local compactness of G, take an open neighbourhood U of the identity in G contained in a compact set K. Then

$$\begin{aligned}
&\sup_{i\in I, g\in U, h\in G} \left\| (\ell_i(gh) - \ell_i(h))\Big(1 + \sum_{j\in I} |\ell_j(sh)|^{\epsilon_{ij}}\Big)^{-1} \right\| \\
&\quad= \sup_{i\in I, g\in U, h\in G} \left\| (\ell_i(gs^{-1}h) - \ell_i(s^{-1}h))\Big(1 + \sum_{j\in I} |\ell_j(h)|^{\epsilon_{ij}}\Big)^{-1} \right\| \\
&\quad\le \sup_{i\in I, g\in U, h\in G} \left\| (\ell_i(gs^{-1}h) - \ell_i(h))\Big(1 + \sum_{j\in I} |\ell_j(h)|^{\epsilon_{ij}}\Big)^{-1} \right\| \\
&\qquad + \sup_{i\in I, h\in G} \left\| (\ell_i(s^{-1}h) - \ell_i(h))\Big(1 + \sum_{j\in I} |\ell_j(h)|^{\epsilon_{ij}}\Big)^{-1} \right\| \\
&\quad\le \sup_{i\in I, g\in Ks^{-1}, h\in G} \left\| (\ell_i(gh) - \ell_i(h))\Big(1 + \sum_{j\in I} |\ell_j(h)|^{\epsilon_{ij}}\Big)^{-1} \right\| \\
&\qquad + \sup_{i\in I, h\in G} \left\| (\ell_i(s^{-1}h) - \ell_i(h))\Big(1 + \sum_{j\in I} |\ell_j(h)|^{\epsilon_{ij}}\Big)^{-1} \right\| \\
&\quad< \infty,
\end{aligned}$$

so 1. is satisfied.

The remaining implications follow as in the Proof of Lemma II.2.4. □

Remark IV.3.3. In our construction of weights for nilpotent groups, we will have a bound of the form

$$\sup_{i\in I, h\in G} \left\| (\ell_i(gh) - \ell_i(h))\Big(1 + \sum_{j\in I} |\ell_j(h)|^{\epsilon_{ij}}\Big)^{-1} \right\| \le f((\|\ell_i(g)\|)_{i\in I}),$$

for some continuous function $f : [0,\infty)^I \to [0,\infty)$, which will imply the translation-boundedness of $(\ell_i)_{i\in I}$ for a bounding matrix $\boldsymbol{\epsilon}$.

Theorem IV.3.4. *Let G be a locally compact group and V be a finite-dimensional vector space. Let $(\ell_j)_{j\in I} : G \to \operatorname{End} V$ be a finite collection of weights which is self-adjoint, proper, and translation bounded with $\boldsymbol{\epsilon} \in M_I([0,\infty))$. Let $(M_{\ell_j})_{j\in I}$ be the operators densely defined on $L^2(G,V)$ given by multiplication by $(\ell_j)_{j\in I}$ respectively. Then, provided $\boldsymbol{\epsilon}$ satisfies the decreasing cycle condition,*

$$\left(C_r^*(G), L^2(G,V), (M_{\ell_j})_{j\in I}\right)$$

is a strictly tangled spectral triple with bounding matrix $\boldsymbol{\epsilon}$. If V is $\mathbb{Z}/2\mathbb{Z}$-graded, the ST^2 is even, otherwise it is odd.

With Lemma IV.3.2 replacing II.2.4, the proof is a fairly straightforward extension of that of Theorem II.2.24. A more general statement for fissured Fell bundles, generalising Theorem II.2.24 can easily be made.

Proof. The local compactness of the resolvent is, as in the Proof of Theorem II.2.24, a consequence of the properness of $(\ell_j)_{j\in I}$ and the isomorphism $C_0(G) \rtimes G \cong K(L^2(G))$. For the commutator bounds, fix an element $f \in C_c(G)$ and let

$$T = [M_{\ell_i}, f]\Big(1 + \sum_{j\in I} |M_{\ell_j}|^{\epsilon_{ij}}\Big)^{-1}.$$

On a vector $\xi \in C_c(G,V)$,

$$(T\xi)(h) = \int_G \left(\ell_i(h) - \ell_i(s^{-1}h)\right)\Big(1 + \sum_{j\in I} |\ell_j(s^{-1}h)|^{\epsilon_{ij}}\Big)^{-1} f(s)\xi(s^{-1}h)d\mu(s).$$

With Lemma IV.3.2, a computation similar to that in the Proof of Theorem II.2.24 shows that T is bounded. □

Let G be a simply connected nilpotent Lie group. Recall the lower central series

$$\mathfrak{g}_1 = \mathfrak{g} \qquad \mathfrak{g}_n = [\mathfrak{g}_1, \mathfrak{g}_{n-1}].$$

The successive quotients $\mathfrak{g}_n/\mathfrak{g}_{n+1}$ are abelian Lie algebras. The largest n for which $\mathfrak{g}_n$ is nonzero is the step size s of $\mathfrak{g}$. In the Baker–Campbell–Hausdorff formula for $\log(\exp X \exp Y)$, we will call the nth-order term $z_n(X,Y)$, so that, e.g.

$$z_1(X,Y) = X + Y \qquad z_2(X,Y) = \frac{1}{2}[X,Y] \qquad z_3(X,Y) = \frac{1}{12}([[X,Y],Y] + [[Y,X],X]).$$

Because $z_{s+1}(X,Y) = 0$, the exponential map from $\mathfrak{g}$ to G is a diffeomorphism.

A *Malcev basis* of $\mathfrak{g}$ through the lower central series is a basis $((e_{j,k})_{k=1}^{\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}})_{j=1}^s$ of $\mathfrak{g}$ such that $((e_{j,k})_{k=1}^{\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}})_{j=n}^s$ is a basis of $\mathfrak{g}_n$ [CG90, Theorem 1.1.13]. Remark that the span of

$$\{e_{j,k}, \dots, e_{j,\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}}, e_{j+1,1}, \dots, e_{s,\dim \mathfrak{g}_s}\}$$

(in other words, the basis with some number of elements dropped from the beginning) is automatically an ideal of $\mathfrak{g}$. Using the Malcev basis, we may write an arbitrary element of $X \in \mathfrak{g}$ as a tuple $(x_1, \dots, x_s) \in \bigoplus_{n=1}^s \mathbb{R}^{\dim \mathfrak{g}_n/\mathfrak{g}_{n+1}}$.

Proposition IV.3.5. *Let G be a simply connected s-step nilpotent Lie group and choose a Malcev basis $((e_{j,k})_{k=1}^{\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}})_{j=1}^s$ of $\mathfrak{g}$ through the lower central series. Let V be Clifford module for $\mathscr{C}\ell_{\dim \mathfrak{g}}$, whose generators we label $((\gamma_{j,k})_{k=1}^{\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}})_{j=1}^s$. Then the collection $(\ell_j)_{j=1}^s : G \to \operatorname{End} V$ of weights given by*

$$\ell_j(\exp_{\mathfrak{g}}(x_1, \dots, x_s)) = \sum_{k=1}^{\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}} x_{j,k}\gamma_{j,k}$$

is self-adjoint, proper, and translation bounded with the strictly lower triangular bounding matrix $\epsilon_{ij} = \max\{i-j, 0\}$.

Proof. Self-adjointness is by construction. For properness, observe that

$$\ell_j(\exp_{\mathfrak{g}}(x_1, \dots, x_s))^2 = \sum_{k=1}^{\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}} x_{j,k}^2.$$

For translation-boundedness, observe that ℓ_j is well-defined on the quotient G/G_{j+1} and $\epsilon_{ij} = 0$ for $j \geq i$. Without loss of generality, then, we consider only the translation-boundedness of ℓ_s. For any $1 \leq m \leq s$, the map

$$\|\cdot\|_{m+1} : X \mapsto \sqrt{\sum_{j=1}^{m} \ell_j(\exp_{\mathfrak{g}} X)^2}$$

defines a norm on the finite-dimensional vector space $\mathfrak{g}/\mathfrak{g}_{m+1}$. By the necessary continuity of the Lie bracket in this norm, there exists a constant C_{m+1} such that

$$\|[X, Y]\|_{m+1} \leq C_{m+1} \|X\|_{m+1} \|Y\|_{m+1}$$

for all $X, Y \in \mathfrak{g}/\mathfrak{g}_{m+1}$. Actually, since $[X + \mathfrak{g}_m, Y + \mathfrak{g}_m] = [X, Y]$,

$$\|[X, Y]\|_{m+1} \leq C_{m+1} \|X\|_m \|Y\|_m.$$

In the term $z_n(X, Y)$ of the Baker–Campbell–Hausdorff formula, there are no more than $n-1$ instances of X or of Y. (Actually, for even $n \geq 4$, the vanishing of the Bernoulli number B_{n-1} means that there are no more than $n-2$ instances of X or Y.) Because $\mathfrak{g}$ is s-step nilpotent,

$$z_n(X + \mathfrak{g}_{s-n+2}, Y + \mathfrak{g}_{s-n+2}) = z_n(X, Y)$$

for $X, Y \in \mathfrak{g}$, and so

$$\|z_n(X, Y)\|_{s+1} \leq C_{s-n+2}^{n-1} \|X\|_{s-n+2}^{n-1} \|Y\|_{s-n+2}^{n-1}$$

By the linearity of $\ell_s \circ \exp_{\mathfrak{g}}$,

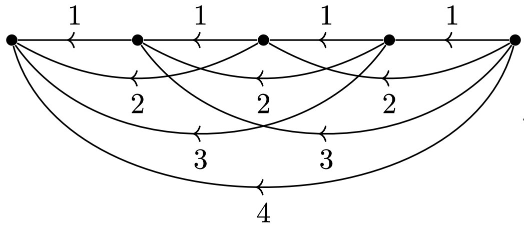
$$\ell_s(\exp_{\mathfrak{g}} X \exp_{\mathfrak{g}} Y) - \ell_s(\exp_{\mathfrak{g}} Y) = \ell_s(\exp_{\mathfrak{g}} X) + \sum_{n=2}^{s} \ell_s(\exp_{\mathfrak{g}} z_n(X, Y)).$$

We obtain a bound

$$\begin{aligned}
&\|\ell_s(\exp_{\mathfrak{g}} X \exp_{\mathfrak{g}} Y) - \ell_s(\exp_{\mathfrak{g}} Y)\| \\
&\quad \leq \|\ell_s(\exp_{\mathfrak{g}} X)\| + \sum_{n=2}^{s} \|\ell_s(\exp_{\mathfrak{g}} z_n(X, Y))\| \\
&\quad \leq \|\ell_s(\exp_{\mathfrak{g}} X)\| + \sum_{n=2}^{s} \|z_n(X, Y)\|_{s+1} \\
&\quad \leq \|\ell_s(\exp_{\mathfrak{g}} X)\| + \sum_{n=2}^{s} C_{s-n+2}^{n-1} \|X\|_{s-n+2}^{n-1} \|Y\|_{s-n+2}^{n-1} \\
&\quad = \|\ell_s(\exp_{\mathfrak{g}} X)\| + \sum_{m=1}^{s-1} C_{m+1}^{s-m} \|X\|_{m+1}^{s-m} \|Y\|_{m+1}^{s-m} \\
&\quad = \|\ell_s(\exp_{\mathfrak{g}} X)\| + \sum_{m=1}^{s-1} C_{m+1}^{s-m} \|X\|_{m+1}^{s-m} \left(\sum_{j=1}^{m} \ell_j(\exp_{\mathfrak{g}} Y)^2 \right)^{\frac{s-m}{2}} \\
&\quad \leq \|\ell_s(\exp_{\mathfrak{g}} X)\| + \sum_{m=1}^{s-1} C_{m+1}^{s-m} \|X\|_{m+1}^{s-m} m^{s-m-1} \sum_{j=1}^{m} \|\ell_j(\exp_{\mathfrak{g}} Y)\|^{s-m} \\
&\quad = \|\ell_s(\exp_{\mathfrak{g}} X)\| + \sum_{j=1}^{s-1} \sum_{m=j}^{s-1} C_{m+1}^{s-m} \|X\|_{m+1}^{s-m} m^{s-m-1} \|\ell_j(\exp_{\mathfrak{g}} Y)\|^{s-m}.
\end{aligned}$$

We conclude from this and Remark IV.3.3 that $\epsilon_{sj} = s - j$ is sufficient to give translation-boundedness. $\square$

The reader can note that a lower triangular bounding matrix automatically satisfies the decreasing cycle condition since its associated weighted directed graph has no cycles. For instance, for a 5-step nilpotent group, the bounding graph is



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Proposition IV.3.6. *Let G be a simply connected s-step nilpotent Lie group. Then, for an irreducible Clifford module V for $\mathscr{C}\ell_{\dim \mathfrak{g}}$,*

$$\left(C^*(G), L^2(G,V), (M_{\ell_n})_{n=1}^s\right) \qquad \epsilon_{ij} = \max\{i-j, 0\}$$

is an ST² with nontrivial class in $KK_{\dim \mathfrak{g}}(C^*(G), \mathbb{C})$. This ST² represents the Kasparov product

$$\begin{aligned}[(C^*(G_1), \overline{(C_c(G_1,V_1))_{C^*(G_2)}}, M_{\ell_1})] \\ \otimes_{C^*(G_2)} [(C^*(G_2), \overline{(C_c(G_2,V_2))_{C^*(G_3)}}, M_{\ell_2})] \\ \otimes_{C^*(G_3)} \cdots \otimes_{C^*(G_s)} [(C^*(G_s), L^2(G_s,V_s), M_{\ell_s})]\end{aligned}$$

where each E_j is a Clifford $\mathscr{C}\ell_{\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}}$-module with generators $(\gamma_{j,k})_{k=1}^{\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}}$ and $V = V_1 \tilde{\otimes} \cdots \tilde{\otimes} V_s$.

Proof. First, note that the maximal compact subgroup of G is the trivial subgroup, so its dual Dirac element β is in $KK^G_{\dim \mathfrak{g}}(\mathbb{C}, C_0(G))$ [Kas88, Definition 5.1]. Take $\boldsymbol{t} \in \Omega(\boldsymbol{\epsilon})$. Comparing with [Kas88, Proof of Theorem 5.7], we see from the description as an iterated Kasparov product that $(\mathbb{C}, C_0(G,V)_{C_0(G)}, \overline{M_{\ell_t}})$ represents the dual Dirac class β. By definition, $\alpha \otimes_{\mathbb{C}} \beta = 1 \in KK^G(C_0(G), C_0(G))$ for the Dirac class $\alpha \in KK^G_{\dim \mathfrak{g}}(C_0(G), \mathbb{C})$. The class of

$$\left(C^*(G), L^2(G,V), (M_{\ell_n})_{n=1}^s\right)$$

is the descent $j^G(\beta) \in KK_{\dim \mathfrak{g}}(C^*(G), C_0(G) \rtimes G) = KK_{\dim \mathfrak{g}}(C^*(G), \mathbb{C})$ of β, which is nonzero because $j^G(\alpha) \otimes_{C^*(G)} j^G(\beta) = j^G(\alpha \otimes_{\mathbb{C}} \beta) = 1$. $\square$

Proposition IV.3.7. *Let G be a simply connected s-step nilpotent Lie group and H be a cocompact, closed subgroup. Then*

$$\left(C^*(H), L^2(H,V), (M_{\ell_n})_{n=1}^s\right) \qquad \epsilon_{ij} = \max\{i-j, 0\}$$

is an ST² with nontrivial class in $KK_{\dim \mathfrak{g}}(C^*(H), \mathbb{C})$.

Proof. To show nontriviality, we argue along the lines of the Proof of Theorem II.3.8. As in Remark II.3.13, the spectral triple

$$\left(C^*(H), \ell^2(H,V), (M_{\ell_n})_{n=1}^s\right)$$

has class $\mathbf{x} = j^H(\beta \otimes_{C_0(G}[\omega]) \otimes_{C_0(H) \rtimes H} [L^2(H)]$ in $KK_{\dim \mathfrak{g}}(C^*(H), \mathbb{C})$, where $\beta \in KK^H_{\dim \mathfrak{g}}(\mathbb{C}, C_0(G))$ is the dual Dirac element and ω is the inclusion map $H \hookrightarrow G$. Using the cocompactness of $H \subseteq G$, one can construct a class $[\theta] \in KK_0(\mathbb{C}, C_0(G) \rtimes H)$, as in [Val02, §6.2], for which $[\theta] \otimes_{C_0(G)\rtimes H} j^H([\omega]) \otimes_{C_0(H)\rtimes H} [L^2(H)] = 1 \in KK_0(\mathbb{C}, \mathbb{C})$. With the Dirac element $\alpha \in KK^H_{\dim \mathfrak{g}}(C_0(G), \mathbb{C})$, we have

$$\begin{aligned}[\theta] \otimes_{C_0(G)\rtimes H} j^H(\alpha) \otimes_{C^*(H)} j^H(\beta \otimes_{C_0(G} [\omega]) \otimes_{C_0(H)\rtimes H} [L^2(H)] \\ = [\theta] \otimes_{C_0(G)\rtimes H} 1 \otimes_{C_0(G)\rtimes H} j^H([\omega]) \otimes_{C_0(H)\rtimes H} [L^2(H)] = 1,\end{aligned}$$

showing that $\mathbf{x}$ is nontrivial. □

Malcev completion says that a group Γ is isomorphic to a lattice in a simply connected nilpotent Lie group if and only if Γ is finitely generated, torsion-free, and nilpotent; see e.g. [Rag72, Theorem 2.18]. We thereby obtain

Proposition IV.3.8. *Let Γ be a finitely generated, torsion-free, nilpotent group. Let G be a simply connected nilpotent Lie group in which Γ is a lattice. Then*

$$\left(C^*(\Gamma), \ell^2(\Gamma, V), (M_{\ell_n})_{n=1}^s\right) \qquad \epsilon_{ij} = \max\{i-j, 0\}$$

is an ST^2 having a nontrivial class in $KK_{\dim \mathfrak{g}}(C^*(\Gamma), \mathbb{C})$. The ST^2 is f-summable for

$$f(\boldsymbol{t}) > \sum_{j=1}^{s} \frac{\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}}{t_j}.$$

Proof. For the statement about summability, first remark that for $\boldsymbol{t} \in (0, \infty)^s$ the map

$$(x_1, \dots, x_s) \mapsto \left(1 + \sum_{j=1}^{s} \left(\sum_{k=1}^{\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}} x_{j,k}^2\right)^{t_j/2}\right)^{-1}$$

is an element of $L^p(\mathfrak{g})$ for $p > \sum_{j=1}^s \frac{\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}}{t_j}$. The Haar measure on a simply connected nilpotent Lie group is the pushforward under the exponential of the Lebesgue measure on its Lie algebra, so $L^p(\mathfrak{g}) \cong L^p(G)$. By [CG90, Proposition 5.4.8(b)], $\log_G \Gamma$ is the union of a finite number of additive cosets of a lattice in $\mathfrak{g}$. By the integral test for convergence, then, the map

$$\exp_{\mathfrak{g}}(x_1, \dots, x_s) \mapsto \left(1 + \sum_{j=1}^{s} \left(\sum_{k=1}^{\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}} x_{j,k}^2\right)^{t_j/2}\right)^{-1}$$

is an element of $\ell^p(\Gamma)$ for $p > \sum_{j=1}^s \frac{\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}}{t_j}$. □

If one chooses the Malcev basis to be *strongly based on* Γ, as is always possible [CG90, Theorem 5.1.6], then each ℓ_j will be valued in the $\mathbb{Q}$-span of $(\gamma_{j,k})_{k=1}^{\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}}$ [CG90, Theorem 5.1.8(a)]. By rescaling by a suitable integer, one can ensure that each ℓ_j will be valued in the $\mathbb{Z}$-span of $(\gamma_{j,k})_{k=1}^{\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}}$; cf. [CG90, §5.4].

IV.3.1.1 Carnot groups and equivariance

A *Carnot group* is a simply connected nilpotent Lie group G with a stratification $\mathfrak{g} = \bigoplus_{n=1}^s \mathcal{V}_n$ of its Lie algebra $\mathfrak{g}$ such that $[\mathcal{V}_1, \mathcal{V}_n] = \mathcal{V}_{n+1}$. A basic consequence of the stratification is that $\mathfrak{g}_n = \bigoplus_{j=n}^s \mathcal{V}_j$ and so naturally $\mathcal{V}_n = \mathfrak{g}_n/\mathfrak{g}_{n+1}$; for more details see e.g. [LD17].

Proposition IV.3.9. *Let G be a Carnot group and H be a cocompact, closed subgroup (including G itself). Choose a Malcev basis $((e_{j,k})_{k=1}^{\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}})_{j=1}^{s}$ with the property that $(e_{j,k})_{k=1}^{\dim \mathfrak{g}_j/\mathfrak{g}_{j+1}} \subset \mathcal{V}_j$. Then the collection $(\ell_j)_{j=1}^{s} : G \to \operatorname{End} V$ of weights and, consequently, the ST^2*

$$\left(C^*(H), L^2(H,V), (M_{\ell_j})_{j=1}^{s}\right)$$

has the strictly lower triangular bounding matrix

$$\epsilon_{ij} = \begin{cases} \left\lfloor \frac{i-1}{j} \right\rfloor, & i > j, \\ 0, & i \le j. \end{cases}$$

The reader can note that the bounding matrix in Proposition IV.3.9 for Carnot groups improves the bounding matrix of Proposition IV.3.5 built from a general nilpotent Lie group's lower central series.

Proof. To verify the new bounding matrix, we again restrict to considering the translation-boundedness of ℓ_s. Using the stratification of $\mathfrak{g}$, for $X \in \mathcal{V}_i$ and $Y \in \mathcal{V}_j$,

$$\|\ell_{i+j}(\exp_{\mathfrak{g}}[X,Y])\| \le C_{i,j} \|\ell_i(\exp_{\mathfrak{g}} X)\| \|\ell_j(\exp_{\mathfrak{g}} Y)\|$$

for some constant $C_{i,j}$. Furthermore, for the Baker–Campbell–Hausdorff expansion $z(X,Y) = \log(\exp X \exp Y)$,

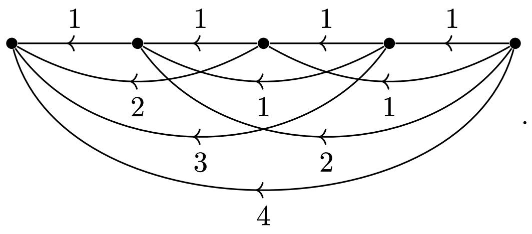
$$\|\ell_s(\exp_{\mathfrak{g}} z(X,Y))\| \le C'_{i,j,s} \|\ell_i(\exp_{\mathfrak{g}} X)\|^{\lfloor (s-j)/i \rfloor} \|\ell_j(\exp_{\mathfrak{g}} Y)\|^{\lfloor (s-i)/j \rfloor}$$

for some constant $C'_{i,j,s}$. We obtain a bound

$$\begin{aligned} &\|\ell_s(\exp_{\mathfrak{g}} X \exp_{\mathfrak{g}} Y) - \ell_s(\exp_{\mathfrak{g}} Y)\| \\ &\quad \le \|\ell_s(\exp_{\mathfrak{g}} X)\| + \|\ell_s(\exp_{\mathfrak{g}} z(X,Y))\| \\ &\quad \le \|\ell_s(\exp_{\mathfrak{g}} X)\| + C'_{i,j,s} \|\ell_i(\exp_{\mathfrak{g}} X)\|^{\lfloor (s-j)/i \rfloor} \|\ell_j(\exp_{\mathfrak{g}} Y)\|^{\lfloor (s-i)/j \rfloor}. \end{aligned}$$

Hence, remembering Remark IV.3.3, $\epsilon_{sj} = \left\lfloor \frac{s-1}{j} \right\rfloor$ is sufficient for translation-boundedness. □

For a 5-step Carnot group, the bounding graph produced by Proposition IV.3.9 is



Remark IV.3.10. It is notable that the behaviour here, in contrast to the general nilpotent Lie group case, is close to that of pseudodifferential operators. In the context of Remark IV.1.10, if we let $\boldsymbol{m} = (1, 2, \dots, s)$, we expect a bounding matrix $\epsilon'_{ij} = \frac{i-1}{j}$, which is just fractionally larger than the $\boldsymbol{\epsilon}$ given above. We therefore may think of M_{ℓ_j} as having order j. The ray

$$\boldsymbol{t}_{\boldsymbol{m}}(\tau) := \left(\frac{\tau}{j}\right)_{j=1}^{s} \qquad (\tau > 0)$$

is in the cone $\Omega(\boldsymbol{\epsilon})$ when G is Carnot, but will not be, in general, for a nilpotent Lie group and $\epsilon_{ij} = \max\{i-j, 0\}$.

The stratification provides a canonical vector space isomorphism of $\mathfrak{g}$ and $\bigoplus_{n=1}^{s} \mathfrak{g}_n/\mathfrak{g}_{n+1}$ because $\mathfrak{g}_n/\mathfrak{g}_{n+1} = \mathcal{V}_n$. We may write any element of $\mathfrak{g}$ as a tuple $(X_1, \dots, X_s) \in \bigoplus_{n=1}^{s} \mathcal{V}_n$ and any element of G as the exponential of such a tuple. The stratification induces a dilation action of $\mathbb{R}_+^\times$ as Lie algebra automorphisms on $\mathfrak{g}$, given by

$$\delta_t : (X_1, X_2, \dots, X_s) \mapsto (tX_1, t^2 X_2, \dots, t^s X_s).$$

This action exponentiates to a dilation action on G by automorphisms, given by

$$\delta_t : \exp_{\mathfrak{g}}(X_1, X_2, \dots, X_s) \mapsto \exp_{\mathfrak{g}}(tX_1, t^2 X_2, \dots, t^s X_s).$$

Let V_t be given by the pullback

$$V_t \xi(\exp_{\mathfrak{g}}(X_1, \dots, X_s)) = \xi(\exp_{\mathfrak{g}}(t^{-1} X_1, \dots, t^{-s} X_s))$$

on $\xi \in L^2(G)$. Recall that the Haar measure on a simply connected nilpotent Lie group is the pushforward under the exponential of the Lebesgue measure on its Lie algebra. We compute that

$$\begin{aligned}
\langle V_t^* \xi \mid \eta \rangle &= \int \xi(\exp_{\mathfrak{g}}(t^{-1} X_1, \dots, t^{-s} X_s)) \eta(\exp_{\mathfrak{g}}(X_1, \dots, X_s)) dX_1 \cdots dX_S \\
&= \int \xi(\exp_{\mathfrak{g}}(Y_1, \dots, Y_s)) \eta(\exp_{\mathfrak{g}}(tY_1, \dots, t^s Y_s)) t^{\dim V_1} dY_1 \cdots t^{s \dim V_s} dY_S \\
&= t^{\dim_h \mathfrak{g}} \langle \xi \mid V_{t^{-1}} \eta \rangle
\end{aligned}$$

using the notation

$$\dim_h \mathfrak{g} = \sum_{n=1}^{s} n \dim \mathcal{V}_n$$

for the homogeneous dimension of $\mathfrak{g}$ (cf. (IV.2.12)). Hence $V_t^* = t^{\dim_h \mathfrak{g}} V_{t^{-1}}$. The unitary in the polar decomposition of V_t is given by $U_t = t^{-\dim_h(\mathfrak{g})/2} V_t$. For $1 \le j \le s$,

$$\ell_j(\exp_{\mathfrak{g}}(t^{-1} X_1, \dots, t^{-s} X_s)) = t^{-j} \ell(\exp_{\mathfrak{g}}(X_1, \dots, X_s))$$

and we see that the operator M_{ℓ_j} transforms as

$$\begin{aligned}
&(U_t M_{\ell_j} U_t^* \xi)(\exp_{\mathfrak{g}}(X_1, \dots, X_s)) \\
&\quad = t^{-\dim_h(\mathfrak{g})/2} (M_{\ell_j} U_t^* \xi)(\exp_{\mathfrak{g}}(t^{-1} X_1, \dots, t^{-s} X_s)) \\
&\quad = t^{-\dim_h(\mathfrak{g})/2} \ell_j(\exp_{\mathfrak{g}}(t^{-1} X_1, \dots, t^{-s} X_s)) (U_t^* \xi)(\exp_{\mathfrak{g}}(t^{-1} X_1, \dots, t^{-s} X_s)) \\
&\quad = t^{-j} \ell_j(\exp_{\mathfrak{g}}(X_1, \dots, X_s)) \xi(\exp_{\mathfrak{g}}(X_1, \dots, X_s)) \\
&\quad = t^{-j} (M_{\ell_j} \xi)(\exp_{\mathfrak{g}}(X_1, \dots, X_s))
\end{aligned}$$

on a vector $\xi \in L^2(G, V)$. We thereby obtain, generalising Example III.2.10,

Proposition IV.3.11. *Let G be an s-step Carnot group. Then*

$$\left(C^*(G), L^2(G, V), \sum_{j=1}^{s} \operatorname{sgn}(M_{\ell_j}) |M_{\ell_j}|^{\tau/j} \right)$$

is a conformally $\mathbb{R}_+^\times$-equivariant higher order spectral triple for the dilation action δ and conformal factor $\mu_t = t^{-\tau/2}$.

IV.3.2 Spectral triples for crossed product C*-algebras and parabolic dynamics

Spectral triples for crossed products by groups of diffeomorphisms have been considered many times in the literature, first by Connes for noncommutative tori; see e.g. [GBVF01, §12.3]. We mention also [CM95, §1] in this connection, to which we shall refer again in Remark IV.1.19. The construction which we emulate and generalise below appeared first in [CMRV08, BMR10] for the group $\mathbb{Z}$ and later in [HSWZ13, Pat14] for other discrete groups.

Let α be an action of a locally compact group G by automorphisms of a C*-algebra A. The (reduced) crossed product C*-algebra $A \rtimes_\alpha G$ possesses a densely defined, completely positive map $\Phi : A \rtimes_\alpha G \dashrightarrow A$ given on $f \in C_c(G, A)$ by evaluation at the identity $e \in G$. We may complete $\mathrm{dom}(\Phi) \subseteq A \rtimes_\alpha G$ to a right Hilbert A-module under the inner product

$$\langle f_1 \mid f_2 \rangle_A = \Phi(f_1^* f_2), \qquad \text{for} \qquad f_1, f_2 \in \mathrm{dom}(\Phi).$$

There is a natural isomorphism of this Hilbert module with $L^2(G, A)_A$. The resulting representation of $A \rtimes_\alpha G$ on $L^2(G, A)_A$ is given by

$$f\xi(g) = \int_G \alpha_{g^{-1}}(f(h))\xi(h^{-1}g)d\mu(h)$$

for $f \in C_c(G, A) \subseteq A \rtimes_\alpha G$ and $\xi \in C_c(G, A) \subseteq L^2(G, A)$. Indeed, $L^2(G, A)_A$ is the Hilbert module associated with the semidirect Fell bundle; see Example II.2.12.2.

Given a self-adjoint, proper, translation-bounded weight $\ell : G \to \mathrm{End}\, V$, Theorem II.2.24 produces a vertical calculus for $A \rtimes_\alpha G$, in the form of an unbounded Kasparov $A \rtimes_\alpha G$-A-module

$$(A \rtimes_\alpha G, L^2(G, V) \otimes A_A, M_\ell \otimes 1).$$

Two weights which we shall particularly consider in later examples are the inclusions $\ell_\mathbb{Z} : \mathbb{Z} \to \mathbb{C}$ and $\ell_\mathbb{R} : \mathbb{R} \to \mathbb{C}$. The first of these gives rise to the number operator $N = M_{\ell_\mathbb{Z}}$ and the Pimsner–Voiculescu extension class and the second is related to the Connes–Thom isomorphism.

A horizontal calculus is just a spectral triple $(\mathscr{A}, H, D)$. Provided that A is represented nondegenerately on H, the internal tensor product module $L^2(G, A) \otimes_A H$ is naturally isomorphic to $L^2(G, H)$. To construct the Kasparov product of the vertical and horizontal calculi, a compatibility condition is required.

Let M be a σ-finite measure space and H a separable Hilbert space. A function f from M to bounded operators $\mathbb{B}(H)$ is *measurable* if, for every pair $\xi, \eta \in H$, the function $m \mapsto \langle \xi \mid f(m)\eta \rangle$ is measurable [RS78, §XIII.16]. It suffices to check measurability for ξ and η in a dense subspace of H (such as $\mathrm{dom}\, D$ in the context below), because of the separability of H and the fact that the pointwise limit of measurable functions is measurable. One should compare the following Definition to the fact that a Lipschitz function has a measurable weak derivative.

Definition IV.3.12. cf. [Pat14, §1] A spectral triple $(\mathscr{A}, H, D)$ is *pointwise bounded* with respect to an action α of G on A if, for all $a \in \mathscr{A}$, the function $g \mapsto [D, \alpha_g(a)]$ is measurable and

$$\sup_{g \in G} \|[D, \alpha_g(a)]\| < \infty.$$

In other words, $g \mapsto [D, \alpha_g(a)]$ is L^∞.

We remind the reader of Definition I.2.15.

Theorem IV.3.13. *cf. [CMRV08, Theorem 3.4], [BMR10, §3.4], [HSWZ13, Theorem 2.7], [Pat14, Proposition 4.1] Let $(\mathscr{A}, H, D)$ be a spectral triple. Let α be an action of a locally compact group G on*

$\mathscr{A}$. Let $\ell : G \to \operatorname{End} V$ be a self-adjoint, proper, translation-bounded weight. If the spectral triple is pointwise bounded with respect to the action of G,

$$\left(\mathscr{A} \rtimes_\alpha G, L^2(G,V) \tilde{\otimes} H, M_\ell \tilde{\otimes} 1 + 1 \tilde{\otimes} D\right)$$

is a spectral triple, representing the Kasparov product

$$(\mathscr{A} \rtimes_\alpha G, L^2(G,V) \otimes A_A, M_\ell \otimes 1) \otimes_A (\mathscr{A}, H, D).$$

Theorem IV.3.13 is known in the case of discrete groups but, to our knowledge, the generalisation to locally compact groups has not appeared in the literature, although see [Pat14, Note after Proposition 4.1].

Proof. This is an instance of the constructive unbounded Kasparov product. We note that $M_\ell \tilde{\otimes} 1$ and $1 \tilde{\otimes} D$ anticommute, so, in order to apply [LM19, Theorem 7.4] we need only checking the boundedness of commutators and the connection condition. For the latter, let $\xi \otimes a \in C_c(G,V) \otimes \mathscr{A}$ and $T_{\xi\otimes a} \in B(H, L^2(G,V) \tilde{\otimes} H)$ be given by

$$(T_{\xi\otimes a}\eta)(g) = \xi(g) \tilde{\otimes} \alpha_{g^{-1}}(a)\eta \qquad (\eta \in H).$$

Then, for $\eta \in \operatorname{dom} D$,

$$(((1 \tilde{\otimes} D)T_{\xi\otimes a} - T_{\xi\otimes a}D)\eta)(g) = \ \xi(g) \tilde{\otimes} [D, \alpha_{g^{-1}}(a)]\eta.$$

Because ξ is compactly supported and $g \mapsto [D, \alpha_g(a)]$ is measurable, $(1 \tilde{\otimes} D)T_{\xi\otimes a} - T_{\xi\otimes a}D$ is bounded.

To check bounded commutators, by [FMR14, Corollary 2.2], it suffices to show that the elements of $\mathscr{A} \rtimes_\alpha G$ take a core for $M_\ell \tilde{\otimes} 1 + 1 \tilde{\otimes} D$ to the domain and have bounded commutators on that core. Let $\eta = \eta_1 \tilde{\otimes} \eta_2 \in C_c(G,V) \tilde{\otimes} \operatorname{dom} D$, a core for $M_\ell \tilde{\otimes} 1 + 1 \tilde{\otimes} D$. (If both V and H are ungraded, then η should have an extra $\mathbb{C}^2$ tensor factor, but this detail does not change the argument below.) Then

$$(\pi(af)\eta)(g) = \int_G f(h)\eta_1(h^{-1}g)d\mu(h) \tilde{\otimes} \alpha_{g^{-1}}(a)\eta_2 \in E \tilde{\otimes} \operatorname{dom} D$$

for all $g \in G$ and

$$\begin{aligned}
&\int_G \left\| (\ell(g) \tilde{\otimes} 1 + 1 \tilde{\otimes} D) \int_G f(h)\eta_1(h^{-1}g)d\mu(h) \tilde{\otimes} \alpha_{g^{-1}}(a)\eta_2 \right\|^2 d\mu(g) \\
&\quad \leq \int_G \left\| (\ell(g) \int_G f(h)\eta_1(h^{-1}g)d\mu(h) \tilde{\otimes} \alpha_{g^{-1}}(a)\eta_2 \right\|^2 d\mu(g) \\
&\qquad + \int_G \left\| \int_G f(h)\eta_1(h^{-1}g)d\mu(h) \tilde{\otimes} \left([D, \alpha_{g^{-1}}(a)] + \alpha_{g^{-1}}(a)D\right)\eta_2 \right\|^2 d\mu(g)
\end{aligned}$$

is finite owing to the compactness of the supports of f and η_1 and pointwise-boundedness. By [RS78, Theorem XIII.85], this implies that $\pi(af)\eta$ is in the domain of $M_\ell \tilde{\otimes} 1 + 1 \tilde{\otimes} D$. It is routine to check that

$$[1 \tilde{\otimes} D, \pi(af)]\eta(g) = 1 \tilde{\otimes} [D, \alpha_{g^{-1}}(a)]\pi(f)\eta(g)$$

and

$$[M_\ell \tilde{\otimes} 1, \pi(af)]\eta(g) = \alpha_{g^{-1}}(a) \int_G \left((\ell(h^{-1}g) - \ell(g))f(h) \tilde{\otimes} 1\right) \eta(h^{-1}g)d\mu(h).$$

The commutator $[M_\ell \tilde{\otimes} 1 + 1 \tilde{\otimes} D, \pi(af)]$ is then bounded because of pointwise-boundedness and the facts that ℓ is translation bounded and that f is compactly supported. By the Leibniz rule, we are done. □

With the technology of ST^2s available, we are not so constrained as in the spectral triple case. We make the following definition.

Definition IV.3.14. A spectral triple $(\mathscr{A}, H, D)$ has *parabolic order* $s \in [0, \infty)$ with respect to an action α of G on A and a weight ℓ on G if, for all $a \in \mathscr{A}$, the function $g \mapsto [D, \alpha_g(a)]$ is measurable and, for all g, the matrix inequality

$$\|[D, \alpha_g(a)]\| \leq C_a(1 + |\ell(g)|^s)$$

holds for some constant $C_a > 0$. (If $s = 0$, we recover pointwise-boundedness.)

Remark IV.3.15. Let α be an action of a locally compact group G on $\mathscr{A}$. If β is an automorphism of A preserving $\mathscr{A}$, there is an isomorphism

$$\mathscr{A} \rtimes_{\beta\circ\alpha\circ\beta^{-1}} G \cong \mathscr{A} \rtimes_\alpha G.$$

Let $(\mathscr{A}, H, D)$ be a spectral triple which is parabolic of order s with respect to the action and a weight ℓ. Suppose that there is a constant $C' > 0$ such that, for all $a \in \mathscr{A}$,

$$\|[D, \beta(a)]\| \leq C' \|[D, a]\|.$$

Then $(\mathscr{A}, H, D)$ also is parabolic of order s with respect to $\beta \circ \alpha \circ \beta^{-1}$ and ℓ because

$$\|[D, \beta \circ \alpha_g \circ \beta^{-1}(a)]\| \leq C' \|[D, \alpha_g(\beta^{-1}(a))]\| \leq C' C_{\beta^{-1}(a)}(1 + |\ell(g)|^s).$$

Theorem IV.3.16. *Let $(\mathscr{A}, H, D)$ be a spectral triple. Let α be an action of a locally compact group G on $\mathscr{A}$. Let $\ell : G \to \operatorname{End} V$ be a self-adjoint, proper, translation-bounded weight. If the spectral triple is parabolic of order s with respect to the action and weight,*

$$\left(\mathscr{A} \rtimes_\alpha G, L^2(G, V) \tilde{\otimes} H, (M_\ell \tilde{\otimes} 1, 1 \tilde{\otimes} D)\right)$$

is an ST^2 with bounding matrix

$$\epsilon = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix} \qquad \bullet \xleftarrow{\;s\;} \bullet .$$

The ST^2 represents the Kasparov product

$$(\mathscr{A} \rtimes_\alpha G, L^2(G, V) \otimes A_A, M_\ell \otimes 1) \otimes_A (\mathscr{A}, H, D).$$

Proof. The proof of Theorem IV.3.13 carries over with the appropriate modifications for the tangled boundedness of commutators implied by the pointwise order.

For the last point, using Kucerovsky's theorem [Kuc97] (and in particular its extension to higher order spectral triples in [GM15, Theorem A.7]), we see that, e.g. for $m > s$, the higher order spectral triple

$$(\mathscr{A} \rtimes G, L^2(G, V) \tilde{\otimes} H, M_{\ell|\ell|^{-1+m}} \tilde{\otimes} 1 + 1 \tilde{\otimes} D)$$

represents the Kasparov product of $(A \rtimes_\alpha G, L^2(G, V) \otimes A_A, M_{\ell|\ell|^{-1+m}} \otimes 1)$ and $(\mathscr{A}, H, D)$. $\square$

Remark IV.3.17. cf. [HSWZ13, Theorem 2.7] In the context of Theorem IV.3.16, if G is discrete and $(1 + |\ell|)^{-1} \in \ell^{p_1}(G, \operatorname{End} V)$, so that $(C_c(G), \ell^2(G, V), M_\ell)$ is p_1-summable, and $(\mathscr{A}, H, D)$ is p_2-summable, then

$$\left(\mathscr{A} \rtimes_\alpha G, \ell^2(G, V) \tilde{\otimes} H, (M_\ell \tilde{\otimes} 1, 1 \tilde{\otimes} D)\right)$$

is f-summable for $f : (t_1, t_2) \mapsto \frac{p_1}{t_1} + \frac{p_2}{t_2}$.

To see the meaning of parabolic order, we specialise to the case of a complete Riemannian manifold $(X, \mathbf{g})$ and a spectral triple $(C_c^\infty(X), L^2(X,S), D)$, with either the Atiyah–Singer or Hodge–de Rham Dirac operator. Let φ be an action of a locally compact group G by diffeomorphisms; the resulting action on $C_c^\infty(X)$ is given by φ^{-1*}, the pullback of the inverse. For $f \in C_c^\infty(X)$, the commutator $[D, f]$ is just the one-form $\mathrm{d}f$ acting by Clifford multiplication. Hence $\|[D,f]\| = \|df\|$. Using the notation

$$(d\varphi_g)_x : T_x X \to T_{\varphi_g(x)} X$$

for the pushforward by φ_g at $x \in X$, the chain rule gives

$$d\varphi_g^*(f)_x = df_{\varphi_g(x)}(d\varphi_g)_x.$$

Hence

$$\|d\varphi_g^*(f)\|_\infty \leq \|df\|_\infty \|d\varphi_g\|_\infty$$

and the parabolic order condition reduces to the matrix inequality

$$\|d\varphi_g\|_\infty \leq C(1 + |\ell(g)|^s)$$

for a constant $C > 0$. In other words, the supremum norm of the Jacobian should be of polynomial order. To be clear, the norm of $d\varphi_g$ at $x \in X$ is

$$\|(d\varphi_g)_x\| = \sup_{u \in T_x M} \frac{\|(d\varphi_g)_x u\|}{\|u\|} = \sup_{u \in T_x M} \frac{\mathbf{g}_{\varphi(x)}((d\varphi_g)_x u, (d\varphi_g)_x u)^{1/2}}{\mathbf{g}_x(u,u)^{1/2}}.$$

Making our observation precise, we obtain:

Corollary IV.3.18. *Let $(C_c^\infty(X), L^2(X,S), D)$ be the Atiyah–Singer or Hodge–de Rham Dirac spectral triple on a complete Riemannian manifold $(X, \mathbf{g})$. Let φ be an action of a locally compact group G by diffeomorphisms on X. Let $\ell : G \to \operatorname{End} V$ be a self-adjoint, proper, translation-bounded weight. Suppose that, for some $s \geq 0$, the matrix inequality*

$$\|d\varphi_g\|_\infty \leq C(1 + |\ell(g)|^s)$$

holds for some constant $C > 0$. Then

$$(C_c^\infty(X) \rtimes G, L^2(G,V) \tilde{\otimes} L^2(X,S), (M_\ell \tilde{\otimes} 1, 1 \tilde{\otimes} D)$$

is a strictly tangled spectral triple with bounding matrix

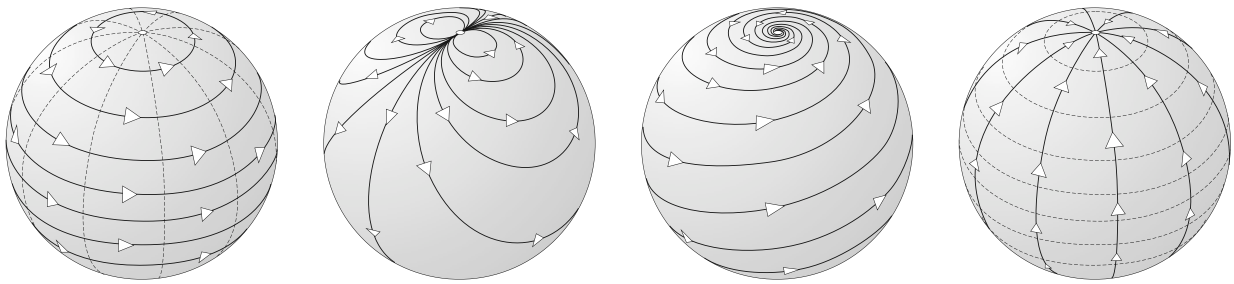
$$\epsilon = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix} \qquad \bullet \xleftarrow{\;s\;} \bullet .$$

This ST^2 represents the Kasparov product of

$$(C_c^\infty(X) \rtimes G, L^2(G,V) \otimes C_0(X)_{C_0(X)}, M_\ell \otimes 1)$$

and $(C_c^\infty(X), L^2(X,S), D)$.

The behaviour of dynamical systems can be loosely classified into three paradigms: *elliptic*, *parabolic*, and *hyperbolic* [HK02, §5.1.g]. These roughly refer to the Jacobian's having respectively constant growth, polynomial growth, or exponential growth. The classical example of the distinction is the classification of Möbius transformations, which we discuss in the following Example. The meaning of Corollary IV.3.18, then, is that ST^2s can be built for parabolic dynamical systems in addition to elliptic dynamical systems which already fall within the scope of Theorem IV.3.13. For a survey of parabolic dynamics, we refer to [HK02, Chapter 8]; see also [Frą04, AFRU21].



(a) elliptic (b) parabolic (c) loxodromic (d) hyperbolic

Figure IV.1: The classification of Möbius transformations of $\mathbf{S}^2$, taken from [Nee97, Figure 3.26].

Example IV.3.19. In terms of the complex coordinate z on the Riemann sphere $\mathbf{S}^2$, a Möbius transformation is given by

$$z \mapsto \frac{az+b}{cz+d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}).$$

The centre $\{+1,-1\}$ of $SL(2,\mathbb{C})$ acts trivially, so that the group of Möbius transformations is $PSL(2,\mathbb{C})$. Equip $\mathbf{S}^2$ with the round metric

$$ds^2 = \frac{4dzd\bar{z}}{(1+|z|^2)^2}$$

and a corresponding spectral triple $(C^\infty(\mathbf{S}^2), L^2(\mathbf{S}^2,S), D)$. We will consider the behaviour of a $\mathbb{Z}$-action generated by a single Möbius transformation, with the weight ℓ corresponding to the number operator. A Möbius transformation φ is classified by its eigenvalues λ,λ^{-1} into three types:

- If $\lambda,\lambda^{-1} \in \mathbb{T}\setminus\{-1,1\}$, φ is *elliptic*, possessing two fixed points; see Figure IV.1(a). An elliptic Möbius transformation φ is (smoothly) conjugate to a rotation $\tau : z \mapsto e^{i\theta}z$, for which $\|d\tau^*(f)\| = 1$. By Remark IV.3.15, $(C^\infty(\mathbf{S}^2), L^2(\mathbf{S}^2,S), D)$ is pointwise bounded with respect to the $\mathbb{Z}$-action generated by φ, placing it under the aegis of Theorem IV.3.13.
- If $\lambda = \lambda^{-1} = \pm 1$, φ is either the identity or it is *parabolic*, possessing one fixed point (and not diagonalisable as a matrix); see Figure IV.1(b). A parabolic Möbius transformation φ is (smoothly) conjugate to a translation $\tau : z \mapsto z+1$. We compute that

$$\|d\tau^n\|_\infty = \sup_z \frac{1+|z|^2}{1+|z+n|^2} = \frac{1}{2}\left(n^2 + |n|\sqrt{n^2+4}+2\right) \in O(n^2).$$

  Again, by Remark IV.3.15, $(C^\infty(\mathbf{S}^2), L^2(\mathbf{S}^2,S), D)$ has pointwise order 2 with respect to the $\mathbb{Z}$-action generated by φ and the number operator weight $\ell_{\mathbb{Z}}$.
- Otherwise, if $\lambda,\lambda^{-1} \in \mathbb{C}\setminus\mathbb{T}$, φ is *loxodromic*, possessing two fixed points; see Figure IV.1(c). A loxodromic Möbius transformation φ is (smoothly) conjugate to a dilation, perhaps combined with a rotation, $\tau : z \mapsto \lambda^2 z$. In this case,

$$\|d\tau^n\|_\infty = \sup_z |\lambda|^{2n}\frac{1+|z|^2}{1+|\lambda|^{4n}|z|^2} = \max\{|\lambda|^{2n}, |\lambda|^{-2n}\}$$

  which is not of polynomial order in n. In the special case that $\lambda,\lambda^{-1} \in \mathbb{R}\setminus\{-1,1\}$, φ is called *hyperbolic*; see Figure IV.1(d).

Example IV.3.20. cf. [HK02, §8.3.a] The group $SL_d(\mathbb{Z})$ acts on the torus $\mathbb{T}^d$ by large diffeomorphisms. The action is realised by identifying $\mathbb{T}^d$ with $\mathbb{R}^d/\mathbb{Z}^d$ and $SL_d(\mathbb{Z})$ acting on $\mathbb{R}^d$ in the usual way that a matrix acts on a vector. For ease of exposition, equip $\mathbb{T}^d$ with a constant Riemannian metric $\mathbf{g}$. For $A \in SL_d(\mathbb{Z})$ and the corresponding action φ_A on $\mathbb{T}^d$, $\|(d\varphi^n)_x\| = \|A^{-n}\|_{\mathbf{g}}$, which generically will be exponentially divergent. However, suppose that $A \in SL_d(\mathbb{Z})$ is a unipotent matrix, i.e. such that $(A-1)^{s+1} = 0$ for some $s \in \mathbb{N}$. By Newton's binomial series, for $n \in \mathbb{Z}$,

$$A^n = \sum_{k=0}^{s} \binom{n}{k} (A-1)^k$$

and

$$\|(d\varphi_{-n})_x\| = \|A^{-n}\|_{\mathbf{g}} \leq \sum_{k=0}^{s} \left| \binom{n}{k} \right| \|(A-1)^k\|_{\mathbf{g}} \in O(n^s).$$

Hence

$$\left(C^\infty(\mathbb{T}^d) \rtimes_{\varphi^A} \mathbb{Z}, \ell^2(\mathbb{Z}) \tilde{\otimes} L^2(H^d, S), (N \tilde{\otimes} 1, 1 \tilde{\otimes} D)\right)$$

is an ST^2 with bounding matrix

$$\epsilon = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix} \qquad \bullet \xleftarrow{\;s\;} \bullet .$$

This Example admits the following generalisation to outer automorphisms of noncommutative tori.

Example IV.3.21. Let Θ be a skew symmetric d-by-d real matrix. For $x \in \mathbb{Z}^d$, define an operator $l_\Theta(x)$ on $\ell^2(\mathbb{Z}^d)$ by

$$(l_\Theta(x)\xi)(y) = e^{\pi i \langle \Theta x, -x+y \rangle} \xi(-x+y).$$

The algebra $C^\infty(\mathbb{T}^d_\Theta)$ of smooth functions on the noncommutative torus $\mathbb{T}^d_\Theta$ is the $*$-algebra spanned by $l_\Theta(x)$ for all $x \in \mathbb{Z}^d$. We call the C*-algebra envelope $C(\mathbb{T}^d_\Theta)$. When $\Theta = 0$, we recover $C(\mathbb{T}^d)$. As in the classical case, integer matrices can act by automorphisms. Following [JL15, §2.3], let $A \in SL_d(\mathbb{Z})$ be such that $A^*\Theta A = \Theta$. Then $\alpha_A : l_\Theta(x) \mapsto l_\Theta(Ax)$ defines an automorphism of $C(\mathbb{T}^d_\Theta)$. (For $d = 2$, the condition $A^*\Theta A = \Theta$ is automatically satisfied.)

Let $(v_i)_{i=1}^d$ be a basis of $\mathbb{R}^d$. To simplify notation, we will also write $(v_i)_{i=1}^d$ for their images in $\mathscr{C}\ell_d$. Let S be a Clifford module for $\mathscr{C}\ell_d$ and define an unbounded operator

$$(D\xi)(y) = \sum_{i=1}^{d} \langle e_i, y \rangle v_i \xi(y) \qquad (y \in \mathbb{Z}^d)$$

on $\ell^2(\mathbb{Z}^d, S)$. We obtain a spectral triple $(C^\infty(\mathbb{T}^d_\Theta), \ell^2(\mathbb{Z}^d, S), D)$. We have

$$([D, l_\Theta(x)]\xi)(y) = -e^{\pi i \langle \Theta x, -x+y \rangle} \sum_{i=1}^{d} \langle e_i, x \rangle v_i \xi(-x+y)$$

so that

$$\big\|[D, l_\Theta(x)]\big\| = \left\| \sum_{i,j=1}^{d} \langle e_i, x \rangle \langle e_j, x \rangle v_i v_j \right\|^{\frac{1}{2}} = \left\| \sum_{i,j=1}^{d} \langle x, e_i \rangle \langle v_i, v_j \rangle \langle e_j, x \rangle \right\|^{\frac{1}{2}} = \|Vx\|$$

where $V : \mathbb{Z}^d \to \mathbb{R}^d$ is the linear map taking $e_i \mapsto v_i$.

If $A \in SL_d(\mathbb{Z})$ (with $A^*\Theta A = \Theta$) is unipotent, so that $(A-1)^{s+1} = 0$ for some $s \in \mathbb{N}$, then $\|A^n\| \in O(n^s)$ as in Example IV.3.20, and

$$\big\|[D, \alpha_A^n(l_\Theta(x))]\big\| = \big\|[D, l_\Theta(A^n x)]\big\| = \|VA^n x\| \leq \|V\| \|A^n\| \|x\| \in O(n^s).$$

Hence

$$\left(C^\infty(\mathbb{T}^d_\Theta) \rtimes_{\alpha_A} \mathbb{Z}, \ell^2(\mathbb{Z}) \tilde{\otimes} \ell^2(\mathbb{Z}^d, S), (N \tilde{\otimes} 1, 1 \tilde{\otimes} D)\right)$$

is an ST^2 with bounding matrix

$$\epsilon = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix} \qquad \bullet \xleftarrow{\;s\;} \bullet .$$

IV.3.2.1 Nilpotent flows on homogeneous spaces

Let G be a connected Lie group. Right-invariant Riemannian metrics on G are in bijection with inner products on $\mathfrak{g}$. If $\mathbf{g}_e$ denotes such an inner product on $\mathfrak{g} = T_e G$, we define the Riemannian metric $\mathfrak{g}$ at any other point $g \in G$ by

$$\mathbf{g}_g(u,v) = \mathbf{g}_e((dR_{g^{-1}})_g u, (dR_{g^{-1}})_g v)$$

where $R_{g^{-1}}$ is the diffeomorphism of G given by right translation by g^{-1} and $dR_{g^{-1}}$ is its pushforward. If the group is noncompact, the metric so obtained will not usually be left invariant [Mil76, §7]. The norm of the Jacobian of left translation L_g by $g \in G$, at $h \in G$ is

$$\begin{aligned}
\|(dL_g)_h\| &= \sup_{u \in T_h G} \frac{\mathbf{g}_{gh}((dL_g)_h u, (dL_g)_h u)}{\mathbf{g}_h(u,u)} \\
&= \sup_{u \in T_h G} \frac{\mathbf{g}_e((dR_{(gh)^{-1}})_{gh}(dL_g)_h u, (dR_{(gh)^{-1}})_{gh}(dL_g)_h u)}{\mathbf{g}_e((dR_{h^{-1}})_h u, (dR_{h^{-1}})_h u)} \\
&= \sup_{v \in T_e G = \mathfrak{g}} \frac{\mathbf{g}_e((d\operatorname{Ad}_g)_e v, (d\operatorname{Ad}_g)_e v)}{\mathbf{g}_e(v,v)} \\
&= \|(d\operatorname{Ad}_g)_e\| \\
&= \|\operatorname{Ad}_g\|_{\mathbf{g}_e}
\end{aligned}$$

where we have used the identity $(dR_{h^{-1}})_{gh}(dL_g)_h = (dL_g)_e(dR_{h^{-1}})_h$ resulting from the facts that left and right actions commute and that the pushforward at $e \in G$ of the adjoint action on G is the adjoint action on $\mathfrak{g}$.

If H is any closed subgroup of G then G/H is a quotient manifold. A right-invariant Riemannian metric $\mathbf{g}$ on G reduces to a Riemannian metric $\mathbf{h}$ on G/H. To construct $\mathbf{h}$, let $\pi : G \to G/H$ be the quotient map. Its pushforward at any point $g \in G$,

$$d\pi_g : T_g G \to T_{gH}(G/H),$$

restricts to an isomorphism between $T_g(gH)^\perp = (\ker d\pi_g)^\perp$ and $T_{gH}(G/H)$. Define $\mathbf{h}$ by

$$\mathbf{h}_{gH}(u,v) = \mathbf{g}_g(d\pi_g|^{-1}_{T_g(gH)^\perp} u, d\pi_g|^{-1}_{T_g(gH)^\perp} v).$$

There remains a left action of G on G/H. As a crude estimate, we have

$$\|(dL_g)_{hH}\| \leq \|(dL_g)_h\| = \|\operatorname{Ad}_g\|_{\mathbf{g}_e}$$

for the Jacobian of left translation L_g.

Recall the Campbell identity

$$\operatorname{Ad}_{\exp X}(Y) = \exp(\operatorname{ad}_X)(Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \operatorname{ad}_X^n(Y).$$

An element $X \in \mathfrak{g}$ is *nilpotent* if $\operatorname{ad}_X^{s+1} = 0$ for some step size $s \in \mathbb{N}$. In that case,

$$\operatorname{Ad}_{\exp tX}(Y) = \sum_{n=0}^{s} \frac{t^n}{n!} \operatorname{ad}_X^n(Y).$$

Consider the flow ϕ^X given by $\phi^X_t = L_{\exp tX}$ on G/H. We have

$$\|d\phi^X_t\|_\infty \leq \|\operatorname{Ad}_{\exp tX}(Y)\|_{\mathbf{g}_e} \in O(t^s)$$

so that

$$\left(C_c^\infty(X) \rtimes_{\phi^X} \mathbb{R}, L^2(\mathbb{R}) \tilde{\otimes} L^2(G/H, S), (M_{\ell_{\mathbb{R}}} \tilde{\otimes} 1, 1 \tilde{\otimes} D)\right)$$

is an ST^2 with bounding matrix

$$\epsilon = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix} \qquad \begin{tikzcd} \bullet & \bullet \arrow[l, "s"'] \end{tikzcd}.$$

Such flows ϕ^X constitute an important family of parabolic dynamical systems [HK02, §8.3.b].

Example IV.3.22. cf. [HK02, §8.3.3] Let $\Gamma \subset SL(2,\mathbb{R})$ be a cocompact lattice. A *horocycle* flow ϕ^X on $SL(2,\mathbb{R})/\Gamma$ is generated by a nilpotent element $X \in \mathfrak{sl}(2,\mathbb{R})$. Of necessity, X will be conjugate to

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R})$$

and so will be 2-step nilpotent.

Example IV.3.23. cf. [HK02, §8.3.2] [AFRU21, §2.2] A compact *nilmanifold* is a quotient G/Γ of a simply connected nilpotent Lie group G by a lattice $\Gamma \subset G$. The *nilflow* ϕ^X generated by a vector field $X \in \mathfrak{g}$ is the restriction of the left action of G to the one-parameter subgroup $(\exp tX)_{t\in\mathbb{R}}$. Every element of a nilpotent Lie algebra is nilpotent, with step size less than or equal to the step size of the Lie algebra, so the above construction may be applied.

Example IV.3.24. Let $P \subseteq SO_0(n,1)$ denote the standard parabolic subgroup. The homogeneous space $SO_0(n,1)/P$ is $\mathbf{S}^{n-1}$ and the Lorentz group $SO_0(n,1)$ acts by Möbius transformations on $\mathbf{S}^{n-1}$. We thereby recover Example IV.3.19 as a special case.

Appendix A

Matched operators and other devices on Hilbert C*-modules

In this Appendix, we present a number tools relating to Hilbert C*-modules and their operators. The main new idea is that of *matched operators*, explained in §A.1.2, and used extensively in §§Ⅲ.3 and Ⅲ.4

Let us first set some conventions and notation.

Definition A.0.1. [RW98, Definition 2.1] Let A be a C*-algebra. A *right inner product* A-module is a right A-module E with an A-valued $\mathbb{C}$-bilinear form $\langle\cdot,\cdot\rangle_A$ on E satisfying

- $\langle x \mid ya\rangle_A = \langle x \mid y\rangle_A a$,
- $\langle x \mid y\rangle_A^* = \langle y \mid x\rangle_A a$, and
- $\langle x \mid x\rangle_A \geq 0$, with equality if and only if $x = 0$.

The last property means that the expression $\|x\| = \|\langle x \mid x\rangle_A\|^{1/2}$ defines a norm on E. We call E a right *Hilbert* A-module if it is complete in this norm.

Definition A.0.2. e.g. [RW98, §2.2] Let E be a right Hilbert B-module. The C*-algebra $\mathrm{End}_B^*(E)$ is defined as the set of $\mathbb{C}$-linear maps $T : E \to E$ for which there exist a map $T^* : E \to E$ such that

$$\langle T(x) \mid y\rangle_B = \langle x \mid T^*(y)\rangle_B$$

for $x, y \in E$. These maps are automatically right B-linear, since

$$\langle T(x)b \mid y\rangle_B = b^*\langle T(x) \mid y\rangle_B = b^*\langle x \mid T^*(y)\rangle_B = \langle xb \mid T^*(y)\rangle_B = \langle T(xb) \mid y\rangle_B$$

In addition, $T^* \in \operatorname{End}^*_B(E)$, since

$$\langle T^*(x) \mid y\rangle_B = \langle y \mid T^*(x)\rangle_B^* = \langle T(y) \mid x\rangle_B^* = \langle x \mid T(y)\rangle_B$$

Let A be another C*-algebra. An *A-B-correspondence* is a right Hilbert B-module with which is made a left A-module by a $*$-homomorphism $\phi : A \to \operatorname{End}^*_B(E)$.

A.1 Hilbert C*-modules over spaces and algebras

A.1.1 Hilbert C*-modules over topological spaces

We review and extend some known facts about Hilbert modules built from functions $X \to E_B$ for a fixed Hilbert module E_B and a locally compact Hausdorff space X.

Definition A.1.1. e.g. [RW98, §B.2] Let A be a C*-algebra and X a locally compact Hausdorff space. Define $C_0(X, A)$ to be the C*-algebra of norm-continuous functions $f : X \to A$ such that $x \mapsto \|f(x)\|_A$ vanishes at infinity, equipped with the supremum norm. Let E be a right Hilbert A-module. Define $C_0(X, E)$ to be the set of continuous functions $f : X \to E$ such that $x \mapsto \|f(x)\|_E$ vanishes at infinity.

Lemma A.1.2. *cf. [RW98, Example 2.13] Let E be a right Hilbert A-module and X a locally compact Hausdorff space. Then $C_0(X, E)$ is a right Hilbert $C_0(X, A)$-module with inner product and right action defined pointwise in X.*

Proof. The algebraic conditions on a Hilbert module are satisfied for $C_0(X, E)$ since they are satisfied pointwise for E. The norm on an element $f \in C_0(X, E)$ arising from the inner product is

$$\left\|\langle f \mid f\rangle_{C_0(X,A)}\right\|^{1/2}_{C_0(X,A)} = \sup_{x\in X}\left\|\langle f \mid f\rangle_{C_0(X,A)}(x)\right\|^{1/2}_A = \sup_{x\in X}\left\|\langle f(x) \mid f(x)\rangle_A\right\|^{1/2}_A = \sup_{x\in X}\|f(x)\|_E$$

which is the supremum norm. Hence, $C_0(X, E)$ is complete as Hilbert module. □

Lemma A.1.3. *Let E be a right Hilbert B-module and X a locally compact Hausdorff space. Let $J = \overline{\operatorname{span}}\langle E \mid E\rangle_B$ be the ideal of A generated by inner products on E. There is an equality*

$$\overline{\operatorname{span}}\langle C_0(X, E) \mid C_0(X, E)\rangle_{C_0(X,B)} = C_0(X, J)$$

of ideals of $C_0(X, B)$.

Proof. Consider $f_1, f_2 \in C_0(X, E)$. Their inner product is given at $x \in X$ by

$$\langle f_1 \mid f_2\rangle_{C_0(X,B)}(x) = \langle f_1(x) \mid f_2(x)\rangle_B \in J.$$

Noting that

$$\left\|\langle f_1(x) \mid f_2(x)\rangle_B\right\|_B \leq \left\|\langle f_1(x) \mid f_1(x)\rangle_B\right\|^{1/2}_B \left\|\langle f_2(x) \mid f_2(x)\rangle_B\right\|^{1/2}_B = \|f_1(x)\|_E\|f_2(x)\|_E,$$

we see that $\langle f_1 \mid f_2\langle_{C_0(X,B)} \in C_0(X, J)$. Hence

$$\langle C_0(X, E) \mid C_0(X, E)\rangle_{C_0(X,B)} \subseteq C_0(X, J).$$

Label the ideal $I = \overline{\operatorname{span}}\langle C_0(X, E) \mid C_0(X, E)\rangle_{C_0(X,B)}$ of $C_0(X, B)$. By e.g. [Fel61, §1.2], I must have the form

$$\{s \in C_0(X, B) \mid \forall x \in X, s(x) \in I_x\}$$

where each $I_x = \{s(x) \mid s \in I\}$ is an ideal of B. We must have $I_x \subseteq J$ for every $x \in X$. Suppose that $I_{x_0} \neq J$ for some $x_0 \in X$. Since $\langle E \mid E\rangle_B$ is linearly dense in J, it is not contained in I_{x_0}, and there must be a pair $e_1, e_2 \in E$ such that $\langle e_1 \mid e_2 \rangle_B \in J \setminus I_{x_0}$. Choose a function $h \in C_0(X)$ for which $h(x_0) = 1$ and define $f_1, f_2 \in C_0(X, E)$ on $x \in X$ by $f_i(x) = e_i h(x)$. Then

$$\langle f_1 \mid f_2 \rangle_{C_0(X,B)}(x_0) = \langle f_1(x_0) \mid f_2(x_0) \rangle_B = \langle e_1 \mid e_2 \rangle_B$$

is not in I_{x_0}, so $\langle f_1 \mid f_2 \rangle_{C_0(X,B)}$ is not in I, which is a contradiction. In other words, $I_x = J$ for every $x \in X$ and $I = C_0(X, J)$. □

Lemma A.1.4. *Let E be a Morita equivalence A-B-bimodule and X a locally compact Hausdorff space. Then $C_0(X, E)$ is a Morita equivalence $C_0(X, A)$-$C_0(X, B)$-bimodule.*

Proof. The left and right norms on E agree by [RW98, Lemma 2.30], so there is no ambiguity in the continuity used to define $C_0(X, E)$. The algebraic properties of a Morita equivalence bimodule are satisfied for $C_0(X, E)$ because they are satisfied pointwise for E. The fullness of $C_0(X, E)$ as a right and left Hilbert module follows from Lemma A.1.3 and the fullness of E. □

Lemma A.1.5. *Let E be a right Hilbert B-module and X a locally compact Hausdorff space. Then*

$$\mathrm{End}^*(C_0(X, E)) = C_b(X, \mathrm{End}^*(E)_{*-s})$$

the C-algebra of $*$-strong-continuous functions $f : X \to \mathrm{End}^*(E)$ such that $\sup_{x \in X} \|f(x)\|_{\mathrm{End}^*(E)} < \infty$. Furthermore, $\mathrm{End}^0(C_0(X, E)) = C_0(X, \mathrm{End}^0(E))$.*

Proof. Let $A = \mathrm{End}^0(E)$, so that E is a Morita equivalence A-B-bimodule. By [RW98, Corollary 2.54], $\mathrm{End}^*(E) = M(A)$, the multiplier algebra of A. The equality

$$\mathrm{End}^0(C_0(X, E)) = C_0(X, \mathrm{End}^0(E)) = C_0(X, A)$$

is a consequence of Lemma A.1.4. Again by [RW98, Corollary 2.54],

$$\mathrm{End}^*(C_0(X, E)) = M(\mathrm{End}^0(C_0(X, E))) = M(C_0(X, A)).$$

Let $M(A)_\beta$ be $M(A)$ equipped with the strict topology. By [APT73, Corollary 3.4],

$$M(C_0(X, A)) = C_b(X, M(A)_\beta),$$

the C*-algebra of strictly continuous and norm-bounded functions. By [RW98, Proposition C.7], the strict topology on $M(A) = \mathrm{End}^*(E)$ agrees with the $*$-strong topology on norm-bounded subsets. Hence

$$C_b(X, M(A)_\beta) = C_b(X, \mathrm{End}^*(E)_{*-s}),$$

where the norm on both algebras is given by the operator norm on E composed with the supremum norm over X. Finally, we obtain

$$\mathrm{End}^*(C_0(X, E)) = C_b(X, \mathrm{End}^*(E)_{*-s}),$$

as required. □

Definition A.1.6. e.g. [Wil70, Definition 43.8] A topological space X is a *k-space* if a subset Y of X is open if, and only if, for every compact subset K of X, $Y \cap K$ is open in K. Conditions on X which imply that it is a k-space include local compactness and first-countability [Wil70, Theorem 43.9].

Lemma A.1.7. *e.g. [Wil70, Lemma 43.10] Let $f : X \to Y$ be a map between topological spaces with X a k-space. Then the continuity of f is equivalent to the continuity of f restricted to K for all compact subsets $K \subseteq X$.*

Lemma A.1.8. *Let E be a right Hilbert A-module and X a locally compact Hausdorff space. The norm-continuity of a function $f : X \to \operatorname{End}^0(E)$ is equivalent to the condition that $f|_K \in \operatorname{End}^0(C(K, E))$ for all compact subsets $K \subseteq X$.*

Proof. By Lemma A.1.7, the norm-continuity of a function $f : X \to \operatorname{End}^0(E)$ is equivalent to the norm-continuity of $f|_K$ for every compact subset $K \subseteq X$. By Lemma A.1.5, the norm-continuous functions from a given K to $\operatorname{End}^0(E)$ can be identified with the elements of $\operatorname{End}^0(C(K, E))$. □

Theorem A.1.9. *(Banach–Steinhaus or uniform boundedness principle) e.g. [RS80, Theorem III.9] Let V be a Banach space and W a normed linear space. Let $\mathscr{F} \subset B(V, W)$ be a family of bounded operators from V to W with $\sup_{T\in\mathscr{F}} \|Tv\|_W < \infty$ for each $v \in V$. Then $\sup_{T\in\mathscr{F}} \|T\|_{B(V,W)} < \infty$.*

Corollary A.1.10. *Let V be a Banach space and X be a compact space. Let $f : X \to B(V)$ be a strongly continuous map. Then f is bounded in operator norm; in other words, $\sup_{x\in X} \|f(x)\|_{B(V)} < \infty$.*

Proof. We have a family $\mathscr{F} = (f(x))_{x\in X} \subset B(V)$ of bounded operators. The strong continuity of f implies that $x \mapsto f(x)v$ is continuous for every $v \in V$. Since X is compact, its image $f(X)v \subseteq V$ is compact and thus bounded. Hence, for a fixed $v \in V$,

$$\sup_{T\in\mathscr{F}} \|Tv\|_V = \sup_{x\in X} \|f(x)v\|_V < \infty.$$

Applying Theorem A.1.9, we obtain that

$$\sup_{x\in X} \|f(x)\|_{B(V)} = \sup_{T\in\mathscr{F}} \|T\|_{B(V)} < \infty,$$

as required. □

Lemma A.1.11. *Let E be a right Hilbert A-module and X a compact Hausdorff space. The $*$-strong continuity of a function $f : X \to \operatorname{End}^*(E)$ is equivalent to the condition that $f \in \operatorname{End}^*(C(X, E))$.*

Proof. By Lemma A.1.5, $\operatorname{End}^*(C(X, E)) = C_b(X, \operatorname{End}^*(E)_{*-s})$, the C*-algebra of $*$-strongly continuous functions $f : X \to \operatorname{End}^*(E)$ such that $\sup_{x\in X} \|f(x)\|_{\operatorname{End}^*(E)} < \infty$. If $f \in \operatorname{End}^*(C(X, E))$, then it is $*$-strongly continuous as a function $f : X \to \operatorname{End}^*(E)$. On the other hand, if we assume $f : X \to \operatorname{End}^*(E)$ is $*$-strongly continuous, we may apply Corollary A.1.10. Thereby, $\sup_{x\in X} \|f(x)\|_{\operatorname{End}^*(E)} < \infty$ and so $f \in \operatorname{End}^*(C(X, E))$. □

Lemma A.1.12. *Let E be a right Hilbert A-module and X a locally compact Hausdorff space. The $*$-strong continuity of a function $f : X \to \operatorname{End}^*(E)$ is equivalent to the condition that $f|_K \in \operatorname{End}^*(C(K, E))$ for all compact subsets $K \subseteq X$.*

Proof. By Lemma A.1.7, the $*$-strong continuity of a function $f : X \to \operatorname{End}^*(E)$ is equivalent to the $*$-strong continuity of $f|_K$ for every compact subset $K \subseteq X$. By Lemma A.1.11, the $*$-strong continuity of $f|_K : K \to \operatorname{End}^*(E)$ for a given K is equivalent to the condition that $f|_K \in \operatorname{End}^*(C(K, E))$. □

A.1.2 Matched operators

Definition A.1.13. Let E be a Hilbert B-module and C a C*-algebra represented on the right of E by a nondegenerate C*-homomorphism $\rho : C \to M(B)$. A regular operator T on E is *C-matched* if those $c \in C$ for which

$$E\rho(c) \subseteq \operatorname{dom}(T)$$

are dense in C.

Remark A.1.14. The condition that $E\rho(c) \subseteq \mathrm{dom}(T)$ combined with Lemma A.3.2 implies that the $\mathbb{C}$-linear map

$$E \to E \qquad \xi \mapsto T\xi c$$

is bounded.

Lemma A.1.15. *Let E be a Hilbert B-module and C a C*-algebra represented on the right of E by a C*-homomorphism $\rho : C \to M(B)$. Let T be a regular operator on E. The set of $c \in C$ for which*

$$E\rho(c) \subseteq \mathrm{dom}(T)$$

form a (not necessarily closed) two-sided ideal in C.

Proof. This follows from a general statement about rings and modules. Suppose that we have $E\rho(c) \subseteq \mathrm{dom}(T)$ for some $c \in C$. If $c_1, c_2 \in C$, then

$$E\rho(c_1 c c_2) = E\rho(c_1)\rho(c)\rho(c_2) \subseteq E\rho(c)\rho(c_2) \subseteq \mathrm{dom}(T)\rho(c_2) \subseteq \mathrm{dom}(T)$$

and we are done. □

Recall that the Pedersen ideal K_C of a C*-algebra C is the minimal dense two-sided ideal of C; see e.g. [Bla06, §II.5.2].

Proposition A.1.16. *Let T be a regular operator on E_B which is C-matched. Then*

$$E\rho(c) \subseteq \mathrm{dom}(T)$$

for all $c \in K_C$, the Pedersen ideal of C. Furthermore, $E\rho(K_C)B$ is a core for T.

Proof. As those $c \in C$ for which $E\rho(c) \subseteq \mathrm{dom}(T)$ form a dense two-sided ideal, they must include the Pedersen ideal. For an element $c \in K_C$, there exists an element $d \in K_C$ such that $dc = c$. Hence

$$E\rho(c) = E\rho(d)\rho(c) \subseteq \mathrm{dom}(T)\rho(c) \subseteq E\rho(c)$$

and $E\rho(K_C) = \mathrm{dom}(T)\rho(K_C) = (1 + T^*T)^{-1/2}E\rho(K_C)$. Next, note that $\rho(K_C)$ is dense in $\rho(C)$. By the continuity of multiplication, $E\rho(K_C)B$ is dense in $E\rho(C)B$. By nondegeneracy of ρ, $B\rho(C)$ is dense in B and, again, by the continuity of multiplication, $EB\rho(C)B = E\rho(C)B$ is dense in $EB = E$. Hence $E\rho(K_C)B$ is dense in E and $E\rho(K_C)B = (1 + T^*T)^{-1/2}E\rho(K_C)B$ is consequently a core for T. □

Remark A.1.17. In [Web04], the multiplier algebra $\Gamma(K_B)$ of the Pedersen ideal of B is shown to consist of exactly those unbounded operators affiliated with B, in the sense of [Wor91], whose domains include K_B. A similar characterisation is given in [Pie06, Théorème 1.30]. The previous Proposition can be used to show that, if $\rho(C) = B$, the C-matched operators on E_B are exactly the multipliers $\Gamma(K_{\mathrm{End}^0(E)})$ of the Pedersen ideal of $\mathrm{End}^0(E)$. See [Ara01, Proposition 1.7] for the details of passing through the Morita equivalence bimodule ${}_{\mathrm{End}^0(E)}E_B$.

Lemma A.1.18. *Let E be a Hilbert B-module and C a C*-algebra represented on the right of E by a C*-homomorphism $\rho : C \to M(B)$. A regular operator T on E is C-matched if and only if, for all $c \in K_C$, the restriction $T|_{\overline{E\rho(c)}}$ of T to the Hilbert submodule $\overline{E\rho(c)}$ over the hereditary C*-subalgebra $\overline{\rho(c)^*B\rho(c)}$ of B is bounded.*

Proof. Assume that $E\rho(c) \subseteq \mathrm{dom}(T)$ for $c \in K_C$. Choose $d \in K_C$ such that $dc = c$. As $E\rho(d) \subseteq \mathrm{dom}(T)$, the $\mathbb{C}$-linear map $\xi \mapsto T\xi\rho(d)$ on E is bounded by Lemma A.3.2. On $\overline{E\rho(c)}$, $\rho(d)$ acts as the identity, meaning T restricts to a bounded operator on $\overline{E\rho(c)}$.

On the other hand, assume that $T|_{\overline{E\rho(c)}}$ is bounded for $c \in K_C$. Then $\mathrm{dom}(T) \supseteq \overline{E\rho(c)} \supseteq E\rho(c)$, as required. □

The following is well-known.

Lemma A.1.19. *Let a be an element of the multiplier algebra of a C*-algebra A. Then the closed right ideal $\overline{aA}$ is a Morita equivalence bimodule between the hereditary C*-subalgebra $\overline{aAa^*}$ of A and the (closed two-sided) ideal $\overline{\operatorname{span}}(Aa^*aA) \trianglelefteq A$.*

Proposition A.1.20. *Let E be a Hilbert B-module and C a C*-algebra represented on the right of E by a C*-homomorphism $\rho : C \to M(B)$. A regular operator T on E is C-matched if and only if, for all positive $c \in K_C$, the restriction $T|_{\overline{\operatorname{span}}(E\rho(c)B)}$ of T to the Hilbert submodule $\overline{\operatorname{span}}(E\rho(c)B)$ over the ideal $\overline{\operatorname{span}}(B\rho(c)B) \trianglelefteq B$ is bounded.*

Proof. Assume that $E\rho(c) \subseteq \operatorname{dom}(T)$ for $c \in K_C$. Then the restriction of T to $\overline{E\rho(c)}_{\overline{\rho(c)^*B\rho(c)}}$ is bounded. The closed right ideal $\overline{\rho(c)^*B}$ of B is a Morita equivalence $\overline{\rho(c)^*B\rho(c)}$-$\overline{\operatorname{span}}(B\rho(cc^*)B)$-bimodule. We have a natural isomorphism

$$\overline{\operatorname{span}}(E\rho(cc^*)B)_{\overline{\operatorname{span}}(B\rho(cc^*)B)} \cong \overline{E\rho(c)}_{\overline{\rho(c)^*B\rho(c)}} \otimes_{\overline{\rho(c)^*B\rho(c)}} \overline{\rho(c)^*B}_{\overline{\operatorname{span}}(B\rho(cc^*)B)}$$

of Hilbert $\overline{\operatorname{span}}(B\rho(cc^*)B)$-modules, under which $T|_{\overline{E\rho(cc^*)B}} \cong T|_{\overline{E\rho(c)}} \otimes_{\overline{\rho(c)^*B\rho(c)}} 1$. Hence the restriction $T|_{\overline{\operatorname{span}}(E\rho(cc^*)B)}$ is bounded. Since every positive element of K_C is of the form cc^*, we conclude this direction of the argument.

On the other hand, assume that $T|_{\overline{\operatorname{span}}(E\rho(c)B)}$ is bounded for $c \in K_C$. Recall that the product of (two-sided) closed ideals in a C*-algebra is again a closed ideal, so that $\overline{B\rho(c)B} = \overline{BM(B)\rho(c)M(B)}$. Then

$$\operatorname{dom}(T) \supseteq E\,\overline{\operatorname{span}}(B\rho(c)B) = E\,\overline{\operatorname{span}}(M(B)\rho(cc^*)M(B)) \supseteq E\rho(c),$$

as required. □

Lemma A.1.21. *cf. [LT76, Proof of Proposition 4.5] Let π be an irreducible representation of a C*-algebra A on a Hilbert space H. Then $K_AH = H$.*

Proof. Let $\xi \in H$ be a cyclic vector and choose $a \in K_A$ such that $\|\pi(a)\xi\| = 1$. (Such an $a \in K_A$ can always be found; otherwise the density of K_A in A would imply that $\xi = 0$.) Let $\eta \in H$ be any non-zero vector. The finite rank operator $|\eta\rangle\langle\pi(a)\xi|$ takes $a\xi$ to η. By [Dix77, Theorem 2.8.3(i)], there exists an element $b \in A$ such that

$$\eta = |\eta\rangle\langle\pi(a)\xi|\pi(a)\xi = \pi(b)\pi(a)\xi \in K_AH$$

as required. □

Proposition A.1.22. *The C-matched operators on E_B form a $*$-algebra $\mathrm{Mtc}^*_B(E, C)$.*

Proof. Let T be a regular operator on E_B which is C-matched. By Lemma A.1.18, T restricts to a bounded operator on $\overline{E\rho(c)}|_{\overline{\rho(c)B\rho(c)}}$ for all $c \in K_C$. The restrictions $(T|_{E\rho(c)})^* = T^*|_{E\rho(c)}$ of the adjoint T^* of T are consequently bounded, and so T^* is also C-matched, again by Lemma A.1.18.

Let T_1 and T_2 be C-matched operators. For an element $c \in K_C$, we have

$$T_2E\rho(c) = T_2\operatorname{dom}(T_2)\rho(c) \subseteq E\rho(c) \subseteq \operatorname{dom}(T_1)$$

so that T_1T_2 is well-defined on $E\rho(K_C)B$. Similarly, $T_2^*T_1^*$ is also well-defined on $E\rho(K_C)B$ so that T_1T_2 is semiregular. The localisation of $EK_B \subseteq E\rho(K_C)B$ to any irreducible $\pi \in \hat{B}$ is equal to

$$E_BK_B \otimes_\pi H_\pi = E_B \otimes_\pi \pi(K_B)H_\pi = E_B \otimes_\pi H_\pi$$

by Lemma A.1.21. Hence, $\operatorname{dom}((T_1T_2)^\pi) = E_B \otimes_\pi H_\pi$ and $(T_1T_2)^\pi$ is bounded. As the same is true for $(T_2^*T_1^*)^\pi$, we may apply the local-global principle [Pie06, Théorème 1.18(2)] to obtain that the closure of T_1T_2 is a regular operator on E. By similar reasoning, we conclude that the closure of the sum $T_1 + T_2$, defined on the common core $E\rho(K_C)B$, is a regular operator on E. □

Remark A.1.23. Combined with Proposition A.1.20, Proposition A.1.22 could be used to show that $\mathrm{Mtc}^*_B(E,C)$ is a pro-C*-algebra (or locally C*-algebra) [Phi88], [Fra05, Chapter II].

Proposition A.1.24. *Let X be a locally compact Hausdorff space and E a Hilbert B-module. Then the $C_0(X)$-matched operators on $C_0(X,E)$ are exactly the elements of $C(X,\mathrm{End}^*(E)_{*-s})$, the (not necessarily bounded) $*$-strongly continuous functions from X to $\mathrm{End}^*(E)$.*

Proof. Suppose that T is a $C_0(X)$-matched operator on $C_0(X,E)$. Because $T(1+T^*T)^{-1/2} \in \mathrm{End}^*(C_0(X,E)) = C_b(X,\mathrm{End}^*(E)_{*-s})$ uniquely determines T, we may conclude that T is given by a function from X to regular operators on E. Let K be a compact subset of X. The Pedersen ideal of $C_0(X)$ is $C_c(X)$, the compactly supported functions on X. Let f be a positive element of $C_c(X)$ which is nonzero on K. We have

$$\mathrm{dom}(T) \supseteq C_0(X,E)f = C_0(\mathrm{supp}\, f, E)$$

so that T restricts to a bounded operator on $C_0(\mathrm{supp}\, f, E)_{C_0(\mathrm{supp}\, f,B)}$. By Lemma A.1.5,

$$\mathrm{End}^*(C_0(\mathrm{supp}\, f, E)) = C_b(\mathrm{supp}\, f, \mathrm{End}^*(E)_{*-s}).$$

Furthermore, the localisation of T to $C(K,E)_{C(K,B)}$ must also be bounded and so an element of $C_b(K,\mathrm{End}^*(E)_{*-s})$. Given that T is a $*$-strongly continuous function on every compact subset K of the k-space X, by Lemma A.1.7, T is a $*$-strongly continuous function on X.

Let $T \in C(X,\mathrm{End}^*(E)_{*-s})$. Then $T(1+T^*T)^{-1/2} \in C_b(X,\mathrm{End}^*(E)_{*-s})$ and

$$(1+T^*T)^{-1/2}C_0(X,E) \supseteq C_c(X,E)$$

so that T is a regular operator on $C_0(X,E)$. (For a more detailed argument, cf. [Pal99, §4].) Furthermore, for an element $f \in K_{C_0(X)} = C_c(X)$, $C_0(X,E)f \subseteq C_c(X,E) \subseteq \mathrm{dom}(T)$ and T is $C_0(X)$-matched. □

A.1.3 Compactly supported states

Definition A.1.25. [Har23, Definition 6.11] A state ψ on a C*-algebra A is *compactly supported* if there exists an $a \in A$ such that $\psi(a) = \|a\|$. We denote the set of compactly supported states on A by $\mathcal{S}_c(A)$.

Proposition A.1.26. *For a state ψ of a C*-algebra A, the following are equivalent:*

(1) ψ is compactly supported, i.e. there exists an $a \in A$ such that $\psi(a) = \|a\|$.

(2) There exists an $a \in K_A$ such that $\psi(a) = \|a\|$.

(3) There exists a positive $a \in K_A$ such that $\psi(a) = 1 = \|a\|$ and $\psi(ab) = \psi(b)$ for all $b \in A$.

(4) ψ is given by $b \mapsto \frac{\phi(a^*ba)}{\phi(a^*a)}$ for a state ϕ of A and an $a \in K_A$.

Proof. (2) clearly implies (1). (4) implies (2) almost by definition of the Pedersen ideal. If $\psi : b \mapsto \frac{\phi(a^*ba)}{\phi(a^*a)}$ for $a \in K_A$, there exists positive $c \in A$ such that $ca = a$. Let $f \in C_c(\mathbb{R}^\times_+)$ be a compactly supported continuous function which is equal to 1 on the spectrum of c. By the continuous functional calculus, we obtain $f(c) \in K_A$ such that $f(c)a = a$ and $\|f(c)\| = 1$, and therefore

$$\psi(f(c)) = \frac{\phi(a^*f(c)a)}{\phi(a^*a)} = 1 = \|f(c)\|.$$

To see that (1) implies (3), let $a \in A$ be such that $\psi(a) = 1 = \|a\|$. By the Kadison inequality, $\psi(a^*a) \geq |\psi(a)|^2 = 1$ and since $\|a^*a\| = \|a\|^2 = 1$, we must have $\psi(a^*a) = 1$. We may assume, without loss of generality, that a is positive. Let $\tilde{A}$ be the minimal unitisation of A and $\tilde{\psi}$ the unique extension of ψ. Let $H_{\tilde{\psi}}$ be the Hilbert space of the corresponding GNS representation and $\xi_{\tilde{\psi}}$ the cyclic vector. Then

$$\|\xi_{\tilde{\psi}} - a\xi_{\tilde{\psi}}\| = \langle (1-a)^2 \xi_\psi \mid \xi_\psi \rangle = \psi(1 - 2a + a^2) = 0$$

and so $a\xi_{\tilde{\psi}} = \xi_{\tilde{\psi}}$. Let $f \in C_c(\mathbb{R}_+^\times)$ be a compactly supported continuous function such that $f(1) = 1$ and $\|f\|_\infty = 1$. By the continuous functional calculus, $f(a)$ is an element of the Pedersen ideal of A such that $f(a)\xi_{\tilde{\psi}} = \xi_{\tilde{\psi}}$ and $\psi(f(a)) = \langle f(a)\xi_\psi \mid \xi_\psi \rangle = 1 = \|f(a)\|$. Hence ψ satisfies

$$\psi(f(a)b) = \langle f(a)b\xi_\psi \mid \xi_\psi \rangle = \langle b\xi_\psi \mid f(a)\xi_\psi \rangle = \langle b\xi_\psi \mid \xi_\psi \rangle = \psi(b) \tag{A.1.27}$$

for all $b \in B$.

To see that (3) implies (4), let positive $a \in A$ be such that $\psi(a) = 1 = \|a\|$. As before, we must have $\psi(a^2) = 1$. For all $b \in A$, as in (A.1.27) we have

$$\frac{\psi(aba)}{\psi(a^2)} = \psi(aba) = \langle aba\xi_\psi, \xi_\psi \rangle = \langle b\xi_\psi, \xi_\psi \rangle = \psi(b)$$

so we may simply choose $\phi = \psi$. □

Remarks A.1.28.

1. In [LT76, Chapter 3], a topology κ on $\Gamma(K_A)$, the multipliers of the Pedersen ideal of A, is introduced. In [LT76, Proposition 6.5], condition (4) of Proposition A.1.26 is shown to be equivalent to ψ being a norm-1 positive κ-continuous functional on $\Gamma(K_A)$.
2. For a locally compact Hausdorff space X, recall that the states on $C_0(X)$ are exactly given by the Radon probability measures on X [Bla98, II.6.2.3(ii)]. The compactly supported states on $C_0(X)$ are then exactly given by the compactly supported Radon probability measures on X.

Proposition A.1.29. *cf. [Har23, Lemma 6.12] The compactly supported states $\mathcal{S}_c(A)$ on a C*-algebra A are weak-$*$-dense in $\mathcal{S}(A)$.*

Proof. Let ψ be a state on A. Using [Bla98, II.4.1.4], let $(h_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for A contained in the Pedersen ideal K_A. Consider the net of states $(\psi_\lambda)_{\lambda\in\Lambda}$ given by

$$\psi_\lambda : a \mapsto \frac{\psi(h_\lambda a h_\lambda)}{\psi(h_\lambda^2)}.$$

Each of these is compactly supported by Proposition A.1.26(4). The net $(\psi(h_\lambda^2))_{\lambda\in\Lambda}$ converges to 1 by [Bla98, II.6.2.5(i)]. To see that the net $(\psi(h_\lambda a h_\lambda))_{\lambda\in\Lambda}$ converges to $\psi(a)$, observe that

$$\begin{aligned}\|\psi(a) - \psi(h_\lambda a h_\lambda)\| &= \|\psi((1-h_\lambda)a) + \psi(h_\lambda a(1-h_\lambda))\| \\ &\leq \big(\|(1-h_\lambda)a\| + \|a(1-h_\lambda)\|\big) \\ &\to 0,\end{aligned}$$

where we have used the bounds $\|\psi\| = 1$ and $\|h_\lambda\| \leq 1$. □

Proposition A.1.30. *Let E be a Hilbert B-module and C a C*-algebra. Let T be a regular operator on $(E \otimes C)_{B\otimes C}$ which is C-matched. Then, for any compactly supported state ψ on C, $(1 \otimes \psi)(T)$ is well-defined and a bounded operator on E.*

Proof. The state ψ extends to a completely positive map $1 \otimes \psi$ from $\mathrm{End}^0(E \otimes C) = \mathrm{End}^0(E) \otimes C$ to $\mathrm{End}^0(E)$. Being nondegenerate, this completely positive map further extends to a map from $M(\mathrm{End}^0(E) \otimes C) = \mathrm{End}^*(E \otimes C)$ to $M(\mathrm{End}^0(E)) = \mathrm{End}^*(E)$ [Lan95, Corollary 5.7].

Let a be a positive element of K_C such that $\psi(a) = 1 = \|a\|$ and $\psi(c) = \psi(ac) = \psi(ca)$ for all $c \in C$. As $(E \otimes C)K_C \subseteq \operatorname{dom} T$, $1 \otimes a(E \otimes C) \subseteq \operatorname{dom} T$. By Lemma A.3.2, $T(1 \otimes a)$ is a bounded operator on $E \otimes C$. Hence we may apply $1 \otimes \psi$ to $T(1 \otimes a)$ to obtain an element of $\mathrm{End}^*(E)$. To see that the choice of a does not affect the value of $(1 \otimes \psi)(T(1 \otimes a))$, let $b \in K_C$ be another positive element such that $\psi(b) = 1 = \|b\|$ and $\psi(c) = \psi(bc) = \psi(cb)$ for all $c \in C$. We note that, because T^* is C-matched, $T^*(1 \otimes a)$ is also a bounded operator. We have a series of equalities

$$\begin{aligned}(1 \otimes \psi)(T(1 \otimes b)) &= (1 \otimes \psi)((1 \otimes a)T(1 \otimes b)) \\ &= (1 \otimes \psi)((1 \otimes b)T^*(1 \otimes a))^* \\ &= (1 \otimes \psi)(T^*(1 \otimes a))^* \\ &= (1 \otimes \psi)((1 \otimes a)T^*(1 \otimes a))^* \\ &= (1 \otimes \psi)((1 \otimes a)T(1 \otimes a)) \\ &= (1 \otimes \psi)(T(1 \otimes a))\end{aligned}$$

so that $(1 \otimes \psi)(T)$ has a unique meaning. □

Proposition A.1.31. *Let E be a Hilbert B-module and C a C^*-algebra. Then $1 \otimes \mathcal{S}_c(C)$ is dense in $1 \otimes \mathcal{S}(C)$ in the pointwise-norm topology on completely positive maps from $\mathrm{End}^0(E) \otimes C$ to $\mathrm{End}^0(E)$. That is, for $\psi \in \mathcal{S}(C)$, there exists a net $(\psi_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{S}_c(C)$ such that, for all $y \in \mathrm{End}^0(E) \otimes C$, $(1 \otimes \psi)(y) \in \mathrm{End}^0(E)$ is the norm limit of $(1 \otimes \psi_\lambda)(y)$. As a consequence, $1 \otimes \mathcal{S}_c(C)$ is dense in $1 \otimes \mathcal{S}(C)$ in the pointwise-norm topology on completely positive maps from $\mathrm{End}^*(E \otimes C)$ to $\mathrm{End}^*(E)$.*

Proof. Let $(h_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for C contained in the Pedersen ideal K_C. Let

$$\psi_\lambda : a \mapsto \frac{\psi(h_\lambda a h_\lambda)}{\psi(h_\lambda^2)}.$$

By [Fra05, Lemma 29.8], $(1 \otimes h_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for $\mathrm{End}^*(E) \otimes C$. For $y \in \mathrm{End}^0(E) \otimes C$,

$$\begin{aligned}\|(1 \otimes \psi)(y) - (1 \otimes \psi_\lambda)(y)\| &= \|(1 \otimes \psi)((1 \otimes (1 - h_\lambda))y) + (1 \otimes \psi)((1 \otimes h_\lambda)y(1 \otimes (1 - h_\lambda)))\| \\ &\leq \|1 \otimes \psi\| \left(\|(1 \otimes (1 - h_\lambda))y\| + \|y(1 \otimes (1 - h_\lambda))\|\right) \\ &\to 0,\end{aligned}$$

as required.

For the second statement, let H_ψ be the Hilbert space of the GNS representation of C corresponding to ψ. One can check that the KSGNS construction [Lan95, Chapter 5] gives

$$(\mathrm{End}^0(E) \otimes C) \otimes_{1\otimes\psi} E \cong H_\psi \otimes E.$$

Let ξ_ψ be the cyclic vector of the GNS construction. Then, by [Lan95, Theorem 5.6],

$$(1 \otimes \psi)(y) = (1 \otimes \xi_\psi)^* y(1 \otimes \xi_\psi)$$

for $y \in \mathrm{End}^0(E) \otimes C$. By [Lan95, Corollary 5.7], $1 \otimes \psi$ is extended to a completely positive map from $\mathrm{End}^*(E \otimes C)$ to $\mathrm{End}^*(E)$ by the same formula, viz.

$$(1 \otimes \psi)(y) = (1 \otimes \xi_\psi^*) y(1 \otimes \xi_\psi)$$

for $y \in \mathrm{End}^*(E \otimes C)$. We have

$$\begin{aligned}\|(1 \otimes \psi)(y) - (1 \otimes \psi_\lambda)(y)\| &= \|(1 \otimes \psi)((1 \otimes (1 - h_\lambda))y) + (1 \otimes \psi)((1 \otimes h_\lambda)y(1 \otimes (1 - h_\lambda)))\| \\ &= \|(1 \otimes \xi_\psi^*)(1 \otimes (1 - h_\lambda))y(1 \otimes \xi_\psi) \\ &\qquad + (1 \otimes \xi_\psi^*)(1 \otimes h_\lambda)y(1 \otimes (1 - h_\lambda))(1 \otimes \xi_\psi)\| \\ &\leq 2\|y\|\|(1 - h_\lambda)\xi_\psi\| \\ &\to 0,\end{aligned}$$

as required. □

A.2 Proper actions, cut-off functions, and a partial imprimitivity bimodule

In this Appendix, we recall a few details of proper actions and cut-off functions and construct a partial imprimitivity bimodule.

Definition A.2.1. [Pal61, Definition 1.2.2, Theorem 1.2.9] A *proper* action of a locally compact group G on a locally compact Hausdorff space X is one for which the map given by

$$G \times X \to X \times X$$

$$(g, x) \mapsto (g \cdot x, x)$$

is proper, meaning that the preimages of compact subsets are compact. An equivalent definition of a proper action is that, for any compact $K \subset X$, the closed subset

$$\{g \in G \mid g \cdot K \cap K \neq \emptyset\}$$

of G be compact. Some basic consequences are that

- The orbit space X/G is locally compact Hausdorff;
- The stabiliser group G_x at any point $x \in X$ is compact;
- The orbit Gx of any point $x \in X$ is locally compact Hausdorff; and
- The restriction of the action to any closed subgroup of G is also proper.

The following is presumably well-known but we provide a proof for completeness.

Proposition A.2.2. *Let G be a locally compact group acting on a metric space (X, d) (not necessarily isometrically). Picking a point $x_0 \in X$, define the function $b \in C_b(G)$ by*

$$b(g) = (1 + d(x_0, g \cdot x_0)^2)^{-1}$$

The action is proper if and only if $b \in C_0(G)$.

Proof. The continuity of b results from the continuity of each of the maps

$$\begin{array}{ccccccc} G & \longrightarrow & X & \longrightarrow & [0, \infty) & \longrightarrow & (0, 1] \\ g & \longmapsto & gx_0 = x & \longmapsto & d(x_0, x) = l & \longmapsto & (1 + l^2)^{-1} \end{array}$$

which, in turn, result from the continuity of the group action and the continuity of the metric.

Suppose that the action is proper. To show that b vanishes at infinity, we need to find for a given $\varepsilon > 0$ a compact set $S \subseteq G$ outside of which (that is, for all $g \in G \setminus S$) $b(g) < \varepsilon$. Take $0 < \varepsilon < 1$ and let $L = (\varepsilon^{-1} - 1)^{1/2}$ so that $\varepsilon = (1 + L^2)^{-1}$. Let $B(x_0, L) \subseteq X$ be the closed ball of radius L centred at x_0. By the properness of the action, the subset

$$S = \{g \in G \mid g \cdot B(x_0, L) \cap B(x_0, L) \neq \emptyset\}$$

of G is compact. For $g \in G \setminus S$,

$$g \cdot B(x_0, L) \cap B(x_0, L) = \emptyset$$

and so

$$d(x_0, g \cdot x_0) > L = (\varepsilon^{-1} - 1)^{1/2}$$

Finally,

$$b(g) = (1 + d(x_0, g \cdot x_0)^2)^{-1} < \varepsilon$$

and $b \in C_0(G)$ as required.

On the other hand, suppose that $b \in C_0(G)$. Let $K \subseteq X$ be any compact subset. The subset

$$Y = \{g \in G \mid g \cdot K \cap K \neq \emptyset\}$$

closed in G. Since K is compact, it is bounded with diameter $\operatorname{diam} K$. For a fixed $g \in Y$, pick $x \in K \cap g \cdot K$. Then

$$d(x_0, g \cdot x_0) \leq d(x_0, x) + d(x, g \cdot x) + d(g \cdot x, g \cdot x_0) = 2d(x_0, x) + d(x, g \cdot x) \leq d(x_0, K) + 3 \operatorname{diam} K$$

Hence, $b|_Y \geq (1 + (d(x_0, K) + 3 \operatorname{diam} K)^2)^{-1}$. Choosing

$$0 < \varepsilon < (1 + (d(x_0, K) + 3 \operatorname{diam} K)^2)^{-1}$$

there must be a compact set $T \subseteq X$ for which $b|_{X \setminus T} \leq \varepsilon$. In particular, $Y \subseteq T$, so Y is compact, and the action is proper. ☐

For the definition of a $C_0(X)$-algebra, we refer to [Kas88, Definition 1.5] and [Wil07, Appendix C]. For a $C_0(X)$-algebra A, denote by A_c the compactly supported elements [Kas88, §3.2].

Definition A.2.3. [Kas88, Definition 3.2, Lemma 3.2(1)] Let X be a locally compact Hausdorff space with a proper action of a locally compact group G. For a G-$C_0(X)$-algebra A, A^G is the subalgebra of G-invariant elements $a \in M(A)$ such that $C_0(X)a \subseteq A$ and $x \mapsto \|a_x\|$ gives an element of $C_0(X/G)$. In the natural way, A^G is a $C_0(X/G)$-algebra.

For a G-equivariant right Hilbert A-module E, the Hilbert A^G-module E^G is defined as the right Hilbert A^G-module consisting of G-invariant elements $\xi \in \operatorname{Hom}^*(B, E)$ such that $C_0(X)\xi \subseteq E \subseteq \operatorname{Hom}^*(B, E)$ and $x \mapsto \|\xi_x\|$ gives an element of $C_0(X/G)$. If, for another group H, A is an H-$C_0(X)$-algebra, and the actions of G and H commute, A^G is an H-$C_0(X/G)$-algebra. If H also acts on E, commuting with G, E^G is an H-equivariant Hilbert A^G-module.

In the special case of $C_0(X, A)$ for a C*-algebra A with G action α, $C_0(X, A)^G$ is the *induced algebra* and is the C*-subalgebra of $C_b(X, A)$ consisting of $f \in C_b(X, A)$ such that

$$f(gx) = \alpha_g(f(x))$$

and $x \mapsto \|f(x)\|$ gives an element of $C_0(X/G)$; see e.g. [Wil07, §3.6].

For explicit formulas involving elements of crossed product C*-algebras, we take our conventions from [Wil07, (2.16–17), (2.25–26)]. Let G, a locally compact group, act on a C*-algebra A by α and on a locally compact Hausdorff space X. For $f_1, f_2 \in C_c(G \times X, A) \subseteq C_0(X, A) \rtimes G$, their convolution product is given by

$$(f_1 f_2)(g, x) = \int_G f_1(h, x)\alpha_h(f_2(h^{-1}g, h^{-1} \cdot x))d\mu(h)$$

and, for $f \in C_c(G \times X, A) \subseteq C_0(X, A) \rtimes G$, the involution is given by

$$f^*(g, x) = \alpha_g(f(g^{-1}, g^{-1} \cdot x)^*)\Delta_G(g^{-1}).$$

The reader should keep in mind the special cases $A = \mathbb{C}$ and $X = \{\mathrm{pt}\}$.

For a C*-algebra A with an action α of a locally compact group G, we also recall the Morita equivalence $C_0(G, A) \rtimes_r G$-A-bimodule $L^2(G, A)$; cf. [EKQR06, Example A.10]. As a right Hilbert A-module, $L^2(G, A)$ is isomorphic to $L^2(G) \otimes_{\mathbb{C}} A_A$. The left action of $C_0(G, A) \rtimes_r G$ on $L^2(G, A)$ is given by

$$(f\xi)(g) = \int_G \alpha_{g^{-1}}(f(h, g))\xi(h^{-1}g)d\mu(h)$$

for $f \in C_c(G \times G, A)$ and $\xi \in C_c(G, A)$.

We also require the idea of a cut-off function.

Definition A.2.4. e.g. [EE11, §3] Let G be a locally compact group acting properly on a locally compact Hausdorff space X. A *cut-off function* is a positive function $c \in C_b(X)$, with compact support on every cocompact subset of X, with the property that

$$\int_G c(g^{-1}x)^2 d\mu(g) = 1$$

for all $x \in X$. In particular, if G acts cocompactly on X, $c \in C_c(X)$. A cut-off function always exists, provided that X/G is paracompact; for a proof see [Bou04, Proposition 7.2.8]. The cut-off function gives rise to a projection $p_c \in M(C_0(X) \rtimes_r G)$, given by

$$p_c(g,x) = c(x)c(g^{-1}x)\Delta_G(g^{-1})^{1/2} \tag{A.2.5}$$

which is an element of $C_0(X) \rtimes_r G$ if and only if X/G is compact.

Theorem A.2.6. *cf. [Kas88, Theorem 3.13], [EE11, §5] Let G act properly on a locally compact Hausdorff space X. Let A be a G-$C_0(X)$-algebra with action α of G. Let c be a cut-off function for the action of G on X and define the projection $p_c \in M(A \rtimes_r G)$ by* (A.2.5)*. Give A_c the structure of a right module over $C_c(G,A) \subseteq A \rtimes_r G$ by*

$$\xi f = \int_G \alpha_g\big(\xi f(g^{-1})\big)\Delta_G(g^{-1})^{1/2} d\mu(g) \qquad (\xi \in A_c, f \in C_c(G,A))$$

and a right $C_c(G,A)$-valued inner product by

$$\langle \xi_1 \mid \xi_2\rangle(g,x) = \xi_1^* \alpha_g(\xi_2)\Delta_G(g^{-1})^{1/2} \qquad (\xi_1, \xi_2 \in A).$$

The map $\phi : A_c \to C_c(G,A) \subseteq A \rtimes_r G$ given by

$$\phi(\xi)(g) = \alpha_g(\xi)c\Delta_G(g^{-1})^{1/2}$$

is right $C_c(G,A)$-linear, has $\phi(\xi_1)^*\phi(\xi_2) = \langle \xi_1 \mid \xi_2\rangle$, and has range dense in $p_c(A \rtimes_r G)$. Completing A_c gives a right Hilbert $A \rtimes_r G$-module, which we denote by ${}^G A$, isomorphic to $p_c(A \rtimes_r G)$. The inclusion $A^G \subseteq M(A)$ gives A_c a left A^G-module structure. The A^G-valued left inner product given by

$$_{A^G}\langle \xi_1 \mid \xi_2\rangle = \int_G \alpha_g(\xi_1\xi_2^*) d\mu(g) \qquad (\xi_1, \xi_2 \in A_c),$$

makes $Y \cong p_c(A \rtimes_r G)$ a partial imprimitivity A^G-$A \rtimes_r G$-bimodule, full on the left.

If the action of G on X is free as well as proper, [Kas88, Theorem 3.13] says that ${}^G A$ is a Morita equivalence A^G-$A \rtimes_r G$-module. This is closely related to the Symmetric Imprimitivity Theorem; see e.g. [Wil07, Chapter 4]. Otherwise, A^G is Morita equivalent to the ideal $\overline{\operatorname{span}}(A \rtimes_r G)p_c(A \rtimes_r G)$ of $A \rtimes_r G$; cf. [EE11, §3, Lemma 3.9].

Proof. Checking that the right module structure on A_c is well-defined and compatible with the inner product structure is routine. For example,

$$\begin{aligned}
(\xi f_1)f_2 &= \int_G\int_G \alpha_h\Big(\alpha_g\big(\xi f_1(g^{-1})\big)f_2(h^{-1})\Big)\Delta_G(h^{-1}g^{-1})^{1/2} d\mu(g)d\mu(h) \\
&= \int_G\int_G \alpha_g\big(\xi f_1(g^{-1}h)\big)\alpha_h(f_2(h^{-1}))\Delta_G(g^{-1})^{1/2} d\mu(g)d\mu(h) \\
&= \int_G\int_G \alpha_g\big(\xi f_1(h)\alpha_h(f_2(h^{-1}g^{-1}))\big)\Delta_G(g^{-1})^{1/2} d\mu(g)d\mu(h) \\
&= \int_G \alpha_g\big(\xi(f_1 f_2)(g^{-1})\big)\Delta_G(g^{-1})^{1/2} d\mu(g) \\
&= \xi(f_1 f_2).
\end{aligned}$$

However, it is not so easy to show that the inner product is positive definite. We will do this using the map ϕ; it will be an immediate consequence of the identity $\phi(\xi_1)^*\phi(\xi_2) = \langle \xi_1 \mid \xi_2 \rangle$.

First, we check that ϕ is linear in the right action of $C_c(G, A) \subseteq A \rtimes_r G$:

$$\begin{aligned}\phi(\xi f)(g) &= \int_G \alpha_{gh}\big(\xi f(h^{-1})\big) c\Delta_G(g^{-1}h^{-1})^{1/2} d\mu(h)\\ &= \int_G \alpha_h(\xi) c\Delta_G(h^{-1})^{1/2}\alpha_h(f(h^{-1}g)) d\mu(h)\\ &= (\phi(\xi)f)(g).\end{aligned}$$

For the inner product identity,

$$\begin{aligned}(\phi(\xi_1)^*\phi(\xi_2))(g) &= \int_G \alpha_h\big(\phi(\xi_1)(h^{-1})^*\phi(\xi_2)(h^{-1}g)\big)\Delta_G(h^{-1})d\mu(h)\\ &= \int_G \xi_1^*\alpha_h(c)\Delta_G(h)^{1/2}\alpha_g(\xi_2)\alpha_h(c)\Delta_G(g^{-1}h)^{1/2}\Delta_G(h^{-1})d\mu(h)\\ &= \xi_1^*\alpha_g(\xi_2)\Delta_G(g^{-1})^{1/2}\int_G \alpha_h(c)^2 d\mu(h)\\ &= \langle \xi_1, \xi_2\rangle(g, x).\end{aligned}$$

We will now see that the range of ϕ includes $p_c C_c(G, A)$ and so is dense in $p_c(C_0(X, A) \rtimes_r G)$. Let $\eta \in C_c(G, A) \subseteq C_0(X, A) \rtimes_r G$. We have

$$\begin{aligned}(p_c\eta)(g) &= \int_G c\alpha_h(c)\Delta_G(h^{-1})^{1/2}\alpha_h(\eta(h^{-1}g))d\mu(h)\\ &= \int_G c\alpha_{gh}(c)\Delta_G(h^{-1}g^{-1})^{1/2}\alpha_{gh}(\eta(h^{-1}))d\mu(h)\\ &= \alpha_g\Big(\int_G \alpha_h(\eta(h^{-1})c)\Delta_G(h^{-1})^{1/2}d\mu(h)\Big)c\Delta_G(g^{-1})^{1/2}\end{aligned}$$

so that $p_c\eta = \phi(\xi)$ for

$$\xi = \int_G \alpha_h(\eta(h^{-1})c)\Delta_G(h^{-1})^{1/2}d\mu(h).$$

We obtain that the completion of $C_c(X, A)$ is a right Hilbert $C_0(X, A) \rtimes_r G$-module ${}^GA$ isomorphic to $p_c(C_0(X, A) \rtimes_r G)$.

The left inner product is well-defined because of the properness of the action; cf [Bou04, Proposition VIII.27.2]. It is routine to check the linearity of the left inner product. Checking the imprimitivity condition,

$$\begin{aligned}(\xi_1\langle \xi_2 \mid \xi_3\rangle) &= \int_G \alpha_g\big(\xi_1\langle \xi_2 \mid \xi_3\rangle\big)\Delta_G(g^{-1})^{1/2}d\mu(g)\\ &= \int_G \alpha_g\big(\xi_1\xi_2^*\alpha_{g^{-1}}(\xi_3)\Delta_G(g)^{1/2}\big)\Delta_G(g^{-1})^{1/2}d\mu(g)\\ &= \int_G \alpha_g(\xi_1\xi_2^*)\xi_3 d\mu(g)\\ &= {}_{A^G}\langle \xi_1 \mid \xi_2\rangle\xi_3.\end{aligned}$$

For any $a \in A^G$, $\int_G \alpha_g(ac^2)d\mu_G(g) = \int_G a\alpha_g(c^2)d\mu_G(g) = a$. For the left inner product to be full, it then suffices for A^Gc^2 to be in the norm closure of $A_cA_c^*$. But since $A^GC_c(X/G)$ is dense in A^G, it suffices for $A^GC_c(X/G)c^2$ to be in the norm closure of $A_cA_c^*$. For $f \in C_c(X/G)$, we have $\operatorname{supp} f \cap \operatorname{supp} c$ compact by definition of the cut-off function c. So, in fact, $A^GC_c(X/G)c^2 \subseteq A_c$. □

We record two corollaries.

Corollary A.2.7. *Let A be a C*-algebra with an action α of G and let G act properly on a locally compact Hausdorff space X. Let c be a cut-off function for the action of G on X and define the projection $p_c \in M(C_0(X,A) \rtimes_r G)$ by (A.2.5). Give $C_c(X,A)$ the structure of a right module over $C_c(G \times X, A) \subseteq C_0(X,A) \rtimes_r G$ by*

$$(\xi f)(x) = \int_G \alpha_g\big(\xi(g^{-1}x) f(g^{-1}, g^{-1}x)\big)\Delta_G(g^{-1})^{1/2} d\mu(g) \qquad (\xi \in C_c(X,A), f \in C_c(G \times X, A))$$

and a right $C_c(G \times X, A)$-valued inner product by

$$\langle \xi_1 \mid \xi_2\rangle(g,x) = \xi_1(x)^* \alpha_g(\xi_2(g^{-1}x))\Delta_G(g^{-1})^{1/2} \qquad (\xi_1, \xi_2 \in C_c(X,A)).$$

The map $\phi : C_c(X,A) \to C_c(G \times X, A) \subseteq C_0(X,A) \rtimes_r G$ given by

$$\phi(\xi)(g,x) = \alpha_g(\xi(g^{-1}x))c(x)\Delta_G(g^{-1})^{1/2}$$

is right $C_c(G \times X, A)$-linear, has $\phi(\xi_1)^*\phi(\xi_2) = \langle \xi_1 \mid \xi_2\rangle$, and has range dense in $p_c(C_0(X,A) \rtimes_r G)$. Completing $C_c(X,A)$ gives a right Hilbert $C_0(X,A)\rtimes_r G$-module ${}^G C_0(X,A)$ isomorphic to $p_c(C_0(X,A)\rtimes_r G)$. There is a left module structure on $C_c(X,A)$ for the induced algebra $C_0(X,A)^G$ given by

$$(f\xi)(x) = f(x)\xi(x) \qquad (f \in C_0(X,A)^G, \xi \in C_c(X,A))$$

and $C_0(X,A)^G$-valued inner product given by

$${}_{C_0(X,A)^G}\langle \xi_1 \mid \xi_2\rangle(x) = \int_G \alpha_g(\xi_1(g^{-1}x)\xi_2(g^{-1}x)^*) d\mu(g) \qquad (\xi_1, \xi_2 \in C_c(X,A)),$$

making ${}^G C_0(X,A) \cong p_c(C_0(X,A) \rtimes_r G)$ a partial imprimitivity $C_0(X,A)^G$-$C_0(X,A) \rtimes_r G$-bimodule, full on the left.

A very special case of the above is $A = \mathbb{C}$. We obtain a partial imprimitivity $C_0(X/G)$-$C_0(X) \rtimes_r G$-bimodule ${}^G C_0(X)$. If G acts freely on X, as well as properly, ${}^G C_0(X)$ is full on the right and so a Morita equivalence bimodule. We will frequently make tacit use of the Morita equivalence of $C_0(X/G)$ and $C_0(X) \rtimes_r G$ in this case.

Corollary A.2.8. *Let G act properly on a locally compact Hausdorff space X. Let B be a G-$C_0(X)$-algebra with action β of G. Let E be a right Hilbert B-module, G-equivariant under an action U. Let c be a cut-off function for the action of G on X and define the projection $p_c \in M(\mathrm{End}^0(E)\rtimes_r G) \cong \mathrm{End}^*(E\rtimes_r G)$ by (A.2.5). Give E_c the structure of a right module over $C_c(G,B) \subseteq B \rtimes_r G$ by*

$$\xi f = \int_G U_g\big(\xi f(g^{-1})\big)\Delta_G(g^{-1})^{1/2} d\mu(g) \qquad (\xi \in E_c, f \in C_c(G,B))$$

and a right $C_c(G,B)$-valued inner product by

$$\langle \xi_1 \mid \xi_2\rangle(g,x) = \langle \xi_1 \mid U_g(\xi_2)\rangle \Delta_G(g^{-1})^{1/2} \qquad (\xi_1, \xi_2 \in E_c).$$

The map $\phi : E_c \to C_c(G,E) \subseteq E \rtimes_r G$ given by

$$\phi(\xi)(g) = U_g(\xi)c\Delta_G(g^{-1})^{1/2}$$

is right $C_c(G,B)$-linear, has $\langle \phi(\xi_1) \mid \phi(\xi_2)\rangle = \langle \xi_1 \mid \xi_2\rangle$, and has range dense in $p_c(E \rtimes_r G)$. Completing E_c gives a right Hilbert $B \rtimes_r G$-module ${}^G E$ isomorphic to $p_c(E \rtimes_r G)$. The inclusion $\mathrm{End}^0(E)^G \subseteq \mathrm{End}^*(E)$ gives E_c a left $\mathrm{End}^0(E)^G$-module structure. The $\mathrm{End}^0(E)^G$-valued inner product given by

$${}_{A^G}\langle \xi_1 \mid \xi_2\rangle = \int_G U_g|\xi_1\rangle\langle\xi_2| U_g^* d\mu(g) \qquad (\xi_1, \xi_2 \in E_c),$$

makes ${}^G E \cong p_c(E \rtimes_r G)$ a partial imprimitivity $\mathrm{End}^0(E)^G$-$B \rtimes_r G$-bimodule, full on the left.

Again, if the action of G on X is free as well as proper, ${}^G E$ is a Morita equivalence $\mathrm{End}^0(E)^G$-$B\rtimes_r G$-module.

Proof. Because B is a G-$C_0(X)$-algebra, $\mathrm{End}^0(E)$ is also a G-$C_0(X)$-algebra [Kas88, §1.5]. The structural homomorphism $C_0(X) \to M(\mathrm{End}^0(E)) = \mathrm{End}^*(E)$ is given by

$$f \cdot \xi = \lim_{n\to\infty} \xi(f\langle\xi|\xi\rangle(1/n + \langle\xi|\xi\rangle)^{-1})$$

for $f \in C_0(X)$ and $\xi \in E$. The linking algebra $\left(\begin{smallmatrix}\mathrm{End}^0(E) & E\\ E^* & B\end{smallmatrix}\right)$ of E, being the algebra of compact endomorphisms of $(E\oplus B)_B$, is also a G-$C_0(X)$-algebra. Remark also that

$$\begin{pmatrix}\mathrm{End}^0(E) & E\\ E^* & B\end{pmatrix} \rtimes_r G \cong \begin{pmatrix}\mathrm{End}^0(E)\rtimes_r G & E\rtimes_r G\\ E^*\rtimes_r G & B\rtimes_r G\end{pmatrix}$$

and

$$\begin{pmatrix}\mathrm{End}^0(E) & E\\ E^* & B\end{pmatrix}^G \cong \begin{pmatrix}\mathrm{End}^0(E)^G & E^G\\ E^{*G} & B^G\end{pmatrix}$$

(although beware that $\mathrm{End}^0(E)^G$ is not necessarily isomorphic to $\mathrm{End}^0(E^G)$ unless the action of G on X is free [Kas88, Lemma 3.2]). Putting A equal to the linking algebra of E in Theorem A.2.6 gives the required result. □

A.2.1 The unbounded assembly map

The following result has as a special case the Baum–Connes assembly map; see [Val02, §6.2] [Kuc03].

Proposition A.2.9. *Let G be a locally compact group with a proper action θ on a locally compact Hausdorff space X. Let A be a G-$C_0(X)$-algebra and B a G-C^*-algebra. Let (A, E_B, D) be an isometrically G-equivariant unbounded Kasparov module with A represented nondegenerately on E. Call the actions of G on A and E, α and U respectively.*

Let ${}^G A$ be the partial imprimitivity A^G-$A\rtimes_r G$-bimodule of Theorem A.2.6. Define a right action of $C_c(G,B) \subseteq B\rtimes_r G$ on $C_c(X)E$ by

$$(\xi f)(x) = \int_G g\cdot U_g(\xi f(g^{-1}))\Delta_G(g^{-1})^{1/2}d\mu(g) \qquad (\xi\in C_c(X)E, f\in C_c(G,B))$$

and a $C_c(G,B)$-valued inner product by

$$\langle\xi_1 \mid \xi_2\rangle(g) = \langle\xi_1 \mid U_g\xi_2\rangle\Delta_G(g^{-1})^{1/2} \qquad (\xi_1,\xi_2\in C_c(X)E).$$

The completion of $C_c(X)E$ is a Hilbert $B\rtimes_r G$-module $\overline{C_c(X)E}$.

Suppose that there exists a cut-off function for the action of G on X such that $c\,\mathrm{dom}\,D \subseteq \mathrm{dom}\,D$ and $[D,c]$ extends to an adjointable operator. Define the subspace

$$\mathcal{X} = \left\{ \int_G \theta^*_{h^{-1}}(c)f(h^{-1})\Delta_G(h^{-1})^{1/2}d\mu(h) \,\middle|\, f\in C_c(G)\right\}$$

of continuous functions on X. The Kasparov product

$$[{}^G A]\otimes_{A\rtimes_r G} j_r^G([(A,E_B,D)]) \in KK_n(A^G, B\rtimes_r G)$$

is represented by

$$(A^G, \overline{C_c(X)E}_{B\rtimes_r G}, \overline{D})$$

where we define $\overline{D}$ to be the closure of D on $\mathcal{X}\,\mathrm{dom}\,D$.

Proof. First, the descent of (A, E_B, D) is

$$(A \rtimes_r G, (E \rtimes_r G)_{B \rtimes_r G}, \tilde{D}).$$

Let $p_c \in M(C_0(X) \rtimes_r G)$ be the projection (A.2.5) associated with the cut-off function C. By Theorem A.2.6, ${}^G A \cong p(A \rtimes_r G)$. Because A is represented nondegenerately on E, and so $A \rtimes_r G$ represented nondegenerately on $E \rtimes_r G$,

$${}^G A \otimes_{A \rtimes_r G} (E \rtimes_r G) \cong p(A \rtimes_r G) \otimes_{A \rtimes_r G} (E \rtimes_r G) = p(E \rtimes_r G).$$

For $\xi \in C_c(G, E)$,

$$(p_c \xi)(g) = \int_G c\alpha_h(c)\Delta_G(h^{-1})^{1/2} U_g \xi(h^{-1}g) d\mu(h).$$

Because (A, E_B, D) is isometrically equivariant,

$$\begin{aligned}([\tilde{D}, p_c]\xi)(g) &= \int_G [D, c\alpha_h(c)\Delta_G(h^{-1})^{1/2}] U_g \xi(h^{-1}g) d\mu(h) \\ &= \int_G \big([D, c]\alpha_h(c) + c\alpha_h([D, c])\big)\Delta_G(h^{-1})^{1/2} U_g \xi(h^{-1}g) d\mu(h)\end{aligned}$$

and so $[\tilde{D}, p_c]$ extends to an adjointable operator.

There is a map $\phi : C_c(X)E \to C_c(G, E) \subseteq E \rtimes_r G$ given by

$$\phi(\xi)(g) = cU_g(\xi)\Delta_G(g^{-1})^{1/2}$$

whose range is dense in $p(E \rtimes_r G)$. By similar computations to the Proof of Theorem A.2.6, one can check that ϕ extends to a Hilbert $B \rtimes_r G$-module isomorphism $\phi : \overline{C_c(X)E} \to p(E \rtimes_r G)$. In particular, for $\eta \in C_c(G, E)$ we have

$$\phi^{-1}(p_c \eta) = \int_G \theta^*_{h^{-1}}(c) U_h(\eta(h^{-1}))\Delta_G(h^{-1})^{1/2} d\mu(h).$$

By [LRV12, §3.3], the Kasparov product

$$[{}^G A] \otimes_{A \rtimes_r G} j_r^G([(A, E_B, D)]) \in KK_n(A^G, B \rtimes_r G) \tag{A.2.10}$$

is represented by

$$(A^G, p(E \rtimes_r G)_{B \rtimes_r G}, p_c \tilde{D} p_c).$$

For $\eta f \in \operatorname{dom}(D)C_c(G) \subseteq \operatorname{dom}(\tilde{D})$,

$$\phi^{-1}(p_c(\eta f)) = \int_G \theta^*_{h^{-1}}(c) f(h^{-1})\Delta_G(h^{-1})^{1/2} d\mu(h)\eta \in \mathcal{X} \operatorname{dom} D$$

Passing through the module identification,

$$(A^G, \overline{C_c(X)E}_{B \rtimes_r G}, \overline{D})$$

also represents the product (A.2.10). □

A.3 Fractional powers of positive operators on Hilbert C*-modules

The proof of Theorem A.3.4 below can be found for the Hilbert space case in [KZPS76, Theorem 12.5]. We include a proof in the generality of Hilbert modules, beginning with a few basic Lemmas.

Lemma A.3.1. *Let A and B be closed densely defined operators on a Banach space X. If the product AB with domain $\mathrm{dom}(AB) = \{\xi \in \mathrm{dom}\, B \mid B\xi \in \mathrm{dom}\, A\}$ is densely defined then AB is closed if either*

- *A has everywhere defined and bounded inverse, or*
- *B is everywhere defined and bounded.*

Proof. Take the case that A is invertible, so that $\mathrm{dom}\, A = A^{-1}X$. Suppose that $(\xi_n)_{n\in\mathbb{N}} \subseteq \mathrm{dom}(AB) = \{x \in \mathrm{dom}\, B \mid Bx \in A^{-1}X\}$ such that $\xi_n \to \xi$ and $AB\xi_n \to \eta$ as $n \to \infty$. Because A^{-1} is bounded, $B\xi_n = A^{-1}AB\xi_n \to A^{-1}\eta$. As B is closed, $\xi \in \mathrm{dom}\, B$ and $B\xi = A^{-1}\eta$. So $\xi \in \mathrm{dom}(AB)$ and $AB\xi = AA^{-1}\eta = \eta$ and we conclude that AB is closed.

Take the case that B is bounded. Suppose that $(\xi_n)_{n\in\mathbb{N}} \subseteq \mathrm{dom}(AB) = \{x \in X \mid Bx \in \mathrm{dom}\, A\}$ such that $\xi_n \to \xi$ and $AB\xi_n \to \eta$ as $n \to \infty$. Because B is bounded, $B\xi_n \to B\xi$. As A is closed, $B\xi \in \mathrm{dom}\, A$ (meaning that $\xi \in \mathrm{dom}(AB)$) and $AB\xi = \eta$. Hence, AB is closed. □

Lemma A.3.2. *Let A and B be closed densely defined operators on Banach spaces X_1 and X_2. Let T be a bounded operator from X_2 to X_1 with $T\mathrm{dom}\, B \subseteq \mathrm{dom}\, A$. Suppose that B is invertible (so B^{-1} is everywhere-defined and bounded). Then ATB^{-1} is everywhere-defined and bounded.*

Proof. By construction, ATB^{-1} is defined everywhere. By the closed graph theorem, it is bounded if and only if it is closed, which it is by Lemma A.3.1. □

We also recall a basic fact about the norm on a Hilbert module.

Lemma A.3.3. *Let B be a C*-algebra and E a Hilbert B-module. For $\xi \in E$,*

$$\|\xi\|_E = \sup_{[\pi]\in\hat{B}} \sup_{\eta\in H_\pi} \frac{\|\xi \otimes \eta\|_{E\otimes_\pi H_\pi}}{\|\eta\|_{H_\pi}}$$

where $\hat{B}$ is the set of equivalence classes of unitary representations of B. For $T \in \mathrm{End}^*_B(E)$,

$$\|T\|_{\mathrm{End}^*(E)} = \sup_{[\pi]\in\hat{B}} \|T \otimes 1\|_{B(E\otimes_\pi H_\pi)}.$$

Proof. By e.g. [RW98, Theorem A.14],

$$\|\xi\| = \left\|\langle\xi \mid \xi\rangle\right\|^{1/2} = \left\|\langle\xi \mid \xi\rangle^{1/2}\right\| = \sup_\pi \|\pi(\langle\xi \mid \xi\rangle^{1/2})\|$$
$$= \sup_\pi \sup_{\eta\in H_\pi} \frac{\|\langle\xi \mid \xi\rangle^{1/2}\eta\|}{\|\eta\|} = \sup_\pi \sup_{\eta\in H_\pi} \frac{\langle\xi\otimes\eta \mid \xi\otimes\eta\rangle^{1/2}}{\|\eta\|} = \sup_\pi \sup_{\eta\in H_\pi} \frac{\|\xi\otimes\eta\|}{\|\eta\|}.$$

Next, note that $\|T \otimes 1\|_{B(E\otimes_\pi H_\pi)} \leq \|T\|_{\mathrm{End}^*(E)}$. On the other hand,

$$\|T\|_{\mathrm{End}^*(E)} = \sup_{\xi\in E} \frac{\|T\xi\|_E}{\|\xi\|_E} = \sup_{\xi\in E} \frac{\sup_\pi \sup_{\eta\in H_\pi} \frac{\|T\xi\otimes\eta\|}{\|\eta\|}}{\sup_\pi \sup_{\eta\in H_\pi} \frac{\|\xi\otimes\eta\|}{\|\eta\|}} \leq \sup_{\xi\in E} \sup_\pi \sup_{\eta\in H_\pi} \frac{\|T\xi\otimes\eta\|}{\|\xi\otimes\eta\|} \leq \sup_\pi \|T\otimes 1\|_{B(E\otimes_\pi H_\pi)}$$

and we obtain the required equality. □

Theorem A.3.4. *cf. [KZPS76, Theorem 12.5] Let A and B be positive regular operators on Hilbert B-modules E_1 and E_2 respectively. Let T be an adjointable operator from E_2 to E_1. If $T\mathrm{dom}(B) \subseteq \mathrm{dom}(A)$, then $T\mathrm{dom}(B^\alpha) \subseteq \mathrm{dom}(A^\alpha)$ for any $0 < \alpha \leq 1$. If, in addition, there exists an $M \geq 0$ such that, for all $\xi \in \mathrm{dom}(B)$,*

$$\|AT\xi\| \leq M\|B\xi\|, \tag{A.3.5}$$

then

$$\|A^\alpha T\xi\| \leq M^\alpha \|T\|^{1-\alpha}\|B^\alpha \xi\|.$$

In particular, if B is invertible,

$$\|A^\alpha T B^{-\alpha}\| \leq \|ATB^{-1}\|^\alpha \|T\|^{1-\alpha}.$$

Proof. By considering the direct sum $E_1 \oplus E_2$, if necessary, we can without loss of generality assume that $E_1 = E_2 =: E$.

We will begin with the case of A bounded and adjointable and B invertible. In this case, a bound of the form (A.3.5) always holds, the best available bound being given by $M = \|ATB^{-1}\|$. For any $0 < \alpha \leq 1$, A^α is adjointable and B^α is invertible. Let $\pi : B \to B(H_\pi)$ be an irreducible representation of B and let $\xi \in E_2 \otimes_\pi H_\pi$. Define the holomorphic function

$$f : z \mapsto \langle \xi \mid (B \otimes 1)^{-z}(T \otimes 1)^*(A \otimes 1)^{2z}(T \otimes 1)(B \otimes 1)^{-z}\xi\rangle \|\xi\|^{-2}$$

on the strip where $0 \leq \mathfrak{R}(z) \leq 1$. We have

$$|f(z)| \leq \|(A \otimes 1)^{\overline{z}}(T \otimes 1)(B \otimes 1)^{-\overline{z}}\|\|(A \otimes 1)^z(T \otimes 1)(B \otimes 1)^{-z}\|.$$

For $\beta \in \mathbb{R}$,

$$|f(1 + \beta i)| \leq \|(A \otimes 1)(T \otimes 1)(B \otimes 1)\|^2 \leq \|ATB^{-1}\|^2$$

and

$$|f(\beta i)| \leq \|T \otimes 1\|^2 \leq \|T\|^2.$$

By Hadamard's three-line theorem, we obtain that

$$\|(A \otimes 1)^\alpha (T \otimes 1)(B \otimes 1)^{-\alpha}\xi\|^2\|\xi\|^{-2} = |f(\alpha)| \leq \|ATB^{-1}\|^{2\alpha}\|T\|^{2-2\alpha}$$

for $0 \leq \alpha \leq 1$. Hence $\|(A \otimes 1)^\alpha(T \otimes 1)(B \otimes 1)^{-\alpha}\| \leq \|ATB^{-1}\|^\alpha \|T\|^{1-\alpha}$. Assuming further that $\alpha \neq 0$, so that A^α and $B^{-\alpha}$ are well-defined as adjointable operators on E,

$$\|A^\alpha T B^{-\alpha}\|_{\mathrm{End}^*(E)} = \sup_{[\pi] \in \hat{B}} \|A^\alpha T B^{-\alpha} \otimes 1\|_{B(E \otimes_\pi H_\pi)} \leq \|ATB^{-1}\|^\alpha \|T\|^{1-\alpha}.$$

For $\xi \in \mathrm{dom}(B^\alpha)$,

$$\|A^\alpha T\xi\| \leq \|A^\alpha T B^{-\alpha}\|\|B^\alpha \xi\| \leq \|ATB^{-1}\|^\alpha \|T\|^{1-\alpha}\|B^\alpha\xi\|$$

as required.

Now consider the case of general A and B when the bound (A.3.5) applies. As in the previous section, let $(\varphi_n)_{n\in\mathbb{N}} \subset C_c(\mathbb{R})$ be a sequence of positive functions, bounded by 1 and converging uniformly on compact subsets to the constant function 1. Let

$$A_n = A\varphi_n(A) \qquad B_n = B + \frac{1}{n} \qquad (n > 0).$$

The operators A_n are bounded and adjointable and B_n are invertible. For $\eta \in \mathrm{dom}\, A$ and $\xi \in \mathrm{dom}\, B$,

$$\|A_n\eta\| \leq \|A\eta\| \qquad \|B\xi\| \leq \|B_n\xi\|$$

and so

$$\|A_n T\xi\| \leq M\|B_n\xi\|.$$

As we have seen, for $0 < \alpha \leq 1$,

$$\|A_n^\alpha T\xi\| \leq M^\alpha \|T\|^{1-\alpha}\|B_n^\alpha \xi\| \qquad (\xi \in \operatorname{dom}(B_n^\alpha) = \operatorname{dom}(B^\alpha)).$$

The sequence $\varphi_n(A)^\alpha T\xi \to T\xi$ as $n \to \infty$ by Theorem III.1.16. The bounded functions

$$x \mapsto (x+1/n)^\alpha - x^\alpha$$

converge uniformly to zero as $n \to \infty$, hence $B_n^\alpha \xi \to B^\alpha \xi$, again by Theorem III.1.16. Then

$$\sup_n \|A^\alpha \varphi_n(A)^\alpha T\xi\| = \sup_n \|A_n^\alpha T\xi\| \leq \sup_n M^\alpha \|T\|^{1-\alpha}\|B_n^\alpha \xi\| < \infty.$$

Because A^α is a closed operator, $T\xi \in \operatorname{dom}(A^\alpha)$ and $A_n^\alpha T\xi = A^\alpha \varphi_n(A)^\alpha T\xi \to A^\alpha T\xi$ as $n \to \infty$. Taking the limit as $n \to \infty$, we find that for $\xi \in \operatorname{dom}(B^\alpha)$

$$\|A^\alpha T\xi\| \leq M^\alpha \|T\|^{1-\alpha}\|B^\alpha \xi\|.$$

For the case of general A and B with $T\operatorname{dom}(B) \subseteq \operatorname{dom}(A)$ but without the bound (A.3.5), we let $B_1 = B + 1$. As B_1 is invertible, for $\xi \in \operatorname{dom}(B)$

$$\|AT\xi\| \leq \|ATB_1^{-1}\|\|B_1\xi\|.$$

We have shown that $T\operatorname{dom}(B_1^\alpha) \subseteq \operatorname{dom}(A^\alpha)$ and, as $\operatorname{dom}(B_1) = \operatorname{dom}(B)$, we are done. □

A.3.1 A nearly convex set from relatively bounded commutators

A subset $S \subseteq \mathbb{R}^n$ is *nearly convex* if there exists a convex subset $C \subseteq \mathbb{R}^n$ such that $C \subseteq S \subseteq \overline{C}$ [MMW16, Definition 2.1]. (Remark that $\overline{S} = \overline{C}$ is convex.)

Theorem A.3.6. *Let A and B be regular operators on a Hilbert B-module E, such that A is self-adjoint, B is positive and invertible, and A and B commute on a common core. Let $T \in \operatorname{End}^*_B(E)$ and define the subset $S \subset \mathbb{R}^2$ as consisting of $(\alpha, \beta) \in (0,\infty) \times [0,\infty)$ such that T preserves $\operatorname{dom} A|A|^{-1+\alpha}$ and*

$$[A|A|^{-1+\alpha}, T]B^{-\beta}$$

extends from $\operatorname{dom} A|A|^{-1+\alpha}$ to an adjointable operator on E. The subset S is nearly convex. Provided that S is nonempty, $\overline{S}$ contains $\{0\} \times [0,\infty)$.

For the proof, we compile a couple of Lemmas.

Lemma A.3.7. *cf. [GBVF01, Lemma 10.17] Let A be a self-adjoint regular operator on a Hilbert B-module E. Let $T \in \operatorname{End}^*_B(E)$ preserve $\operatorname{dom} A$ and have $[A, T]$ extend to an adjointable operator. Then, for any $\alpha \in (0,1)$ and $y \in \mathbb{R}$, T preserves $\operatorname{dom} A|A|^{-1+\alpha} = \operatorname{dom} A|A|^{-1+\alpha+yi} = \operatorname{dom} |A|^\alpha$ and*

$$[A|A|^{-1+\alpha+yi}, T]$$

extends to an adjointable operator and

$$\sup_{y\in\mathbb{R}} |\csc \tfrac{(\alpha+yi)\pi}{2}| \|[A|A|^{-1+\alpha+yi}, T]\| < \infty.$$

Proof. Let $\langle A\rangle = (1+A^*A)^{1/2} = (1+|A|^2)^{1/2}$. First, note that

$$[\langle A\rangle^{\alpha+yi}, T] = -\langle A\rangle^{\alpha+yi}[\langle A\rangle^{-\alpha-yi}, T]\langle A\rangle^{\alpha+yi}.$$

By the integral formula (I.0.5) and using [CP98, Lemma 2.3], on $\operatorname{dom} A$,

$$\begin{aligned}-\langle A\rangle^{\alpha}[\langle A\rangle^{-\alpha-yi}, T]\langle A\rangle^{\alpha} &= \frac{\sin\frac{(\alpha+yi)\pi}{2}}{\pi}\int_0^\infty \lambda^{-\frac{\alpha+yi}{2}}\langle A\rangle^{\alpha}[T,(\lambda+1+A^2)^{-1}]\langle A\rangle^{\alpha}d\lambda \\ &= \frac{\sin\frac{(\alpha+yi)\pi}{2}}{\pi}\int_0^\infty \lambda^{-\frac{\alpha+yi}{2}}\langle A\rangle^{\alpha}\Big(A(\lambda+1+A^2)^{-1}[A,T](\lambda+1+A^2)^{-1} \\ &\quad +(\lambda+1+A^2)^{-1}[A,T]A(\lambda+1+A^2)^{-1}\Big)\langle A\rangle^{\alpha}d\lambda.\end{aligned}$$

The integral is norm-convergent and we obtain a bound

$$\begin{aligned}&\Big\|\langle A\rangle^{\alpha}[\langle A\rangle^{-\alpha-yi}, a]\langle A\rangle^{\alpha}\Big\| \\ &\le \frac{|\sin\frac{(\alpha+yi)\pi}{2}|}{\pi}\int_0^\infty \lambda^{-\frac{\alpha}{2}}\Big(\Big\|A\langle A\rangle^{\alpha}(\lambda+1+A^2)^{-1}\Big\|\|[A,T]\|\Big\|\langle A\rangle^{\alpha}(\lambda+1+A^2)^{-1}\Big\| \\ &\quad +\Big\|\langle A\rangle^{\alpha}(\lambda+1+A^2)^{-1}\Big\|\|[A,T]\|\Big\|A\langle A\rangle^{\alpha}(\lambda+1+A^2)^{-1}\Big\|\Big)d\lambda \\ &\le \frac{|\sin\frac{(\alpha+yi)\pi}{2}|}{\pi}2\|[A,T]\|\int_0^\infty \lambda^{-\frac{\alpha}{2}}(\lambda+1)^{-\frac{3}{2}+\alpha}d\lambda \\ &= \sqrt{\cosh(y\pi)-\cos(\alpha\pi)}\frac{2^{\alpha}}{\sqrt{2\pi}}\frac{\Gamma(1-\alpha)}{\Gamma(\frac{3}{2}-\alpha)}\|[A,T]\|.\end{aligned}$$

Next, with $F_A = A\langle A\rangle^{-1}$,

$$[F_A\langle A\rangle^{\alpha+yi}, T] = [F_A, T]\langle A\rangle^{\alpha+yi} + F_A\langle A\rangle^{\alpha+yi}[\langle A\rangle^{-\alpha-yi}, T]\langle A\rangle^{\alpha+yi}$$

so that

$$\begin{aligned}\Big\|[F_A\langle A\rangle^{\alpha+yi}, T]\Big\| &\le \Big\|[F_A, T]\langle A\rangle^{\alpha}\Big\| + \Big\|\langle A\rangle^{\alpha}[\langle A\rangle^{-\alpha-yi}, a]\langle A\rangle^{\alpha}\Big\| \\ &\le C'_\alpha(1+\Big|\sin\tfrac{(\alpha+yi)\pi}{2}\Big|)\Big\|[D,S]\langle D\rangle^{-\alpha}\Big\|\end{aligned}$$

for some constant C'_α, using also Theorem I.0.6. Hence $[\langle A\rangle^{\alpha+yi}, T]$ extends to an adjointable operator.

Next,

$$\begin{aligned}\Big|x|x|^{-1+\alpha+yi} - x\langle x\rangle^{-1+\alpha+yi}\Big| &= |x|\Big||x|^{-1+\alpha+yi} - \langle x\rangle^{-1+\alpha+yi}\Big| \\ &\le \Big(|x|^{\alpha}\Big|(|x|\langle x\rangle^{-1})^{yi} - 1\Big| + |x|\Big||x|^{-1+\alpha} - \langle x\rangle^{-1+\alpha}\Big|\Big)|\langle x\rangle^{yi}| \\ &\le |x|^{\alpha}\Big|1-(|x|\langle x\rangle^{-1})^{yi}\Big| + |x|\Big||x|^{-1+\alpha} - \langle x\rangle^{-1+\alpha}\Big|.\end{aligned}$$

Now

$$\begin{aligned}\Big|1-(|x|\langle x\rangle^{-1})^{yi}\Big| &= \sqrt{(1-\cos(y\log(|x|\langle x\rangle^{-1})))^2 + \sin(y\log(|x|\langle x\rangle^{-1}))^2} \\ &= \sqrt{2-2\cos(y\log(|x|\langle x\rangle^{-1}))^2} \\ &\le |y|(\log\langle x\rangle - \log|x|)\end{aligned}$$

since $|1-\cos\theta| \le \frac{1}{2}\theta^2$. One can check that there exist $c_1, c_2 > 0$ such that $|x|^{\alpha}(\log\langle x\rangle - \log|x|) \le c_1\alpha$ and $|x|\Big||x|^{-1+\alpha} - \langle x\rangle^{-1+\alpha}\Big| \le c_2\alpha$. Hence

$$\Big|x|x|^{-1+\alpha+yi} - x\langle x\rangle^{-1+\alpha+yi}\Big| \le C''\alpha(1+|y|)$$

for some $C'' > 0$ and therefore

$$\left|A|A|^{-1+\alpha+yi} - F_A\langle A\rangle^{\alpha+yi}\right| \leq C''\alpha(1+|y|).$$

Hence

$$\begin{aligned}\|[A|A|^{-1+\alpha+yi}, T]\| &\leq \left\|[F_A\langle A\rangle^{\alpha+yi}, T]\right\| + \left\|[A|A|^{-1+\alpha+yi} - F_A\langle A\rangle^{\alpha+yi}, T]\right\| \\ &\leq C'_\alpha(1 + \left|\sin \tfrac{(\alpha+yi)\pi}{2}\right|)\left\|[D,S]\langle D\rangle^{-\alpha}\right\| + 2C''\alpha(1+|y|)\|T\|\end{aligned}$$

and $[A|A|^{-1+\alpha+yi}, T]$ extends to an adjointable operator. We finally obtain that

$$\sup_{y\in\mathbb{R}} |\csc \tfrac{(\alpha+yi)\pi}{2}|\|[A|A|^{-1+\alpha+yi}, T]\| < \infty$$

as required. □

Proposition A.3.8. *Let A and B be regular operators on a Hilbert B-module E, such that A is self-adjoint, B is positive and invertible, and A and B commute on a common core. Suppose that, for some $\alpha_1 > 0$, an element $T \in \mathrm{End}^*_B(E)$ preserves* $\operatorname{dom} A|A|^{-1+\alpha_1}$ *and that, for some $\beta_1 \geq 0$,*

$$[A|A|^{-1+\alpha_1}, T]B^{-\beta_1}$$

extends from $\operatorname{dom} A|A|^{-1+\alpha_1}$ *to an adjointable operator on E. Then, for any $0 < \alpha_2 \leq \alpha_1$ and $\beta_2 > \frac{\alpha_2\beta_1}{\alpha_1}$, T preserves* $\operatorname{dom} A|A|^{-1+\alpha_2}$ *and*

$$[A|A|^{-1+\alpha_2}, T]B^{-\beta_2}$$

extends to an adjointable operator.

Suppose, further, that, for some $\alpha_3 > \alpha_1$, T preserves $\operatorname{dom} A|A|^{-1+\alpha_3}$ *and that, for some $\beta_3 \geq 0$,*

$$[A|A|^{-1+\alpha_3}, T]B^{-\beta_3}$$

extends to an adjointable operator on E. Then, for any $\alpha_1 \leq \alpha_2 \leq \alpha_3$ and

$$\beta_2 > \frac{(\alpha_3-\alpha_2)\beta_1 + (\alpha_2-\alpha_1)\beta_3}{\alpha_3-\alpha_1},$$

T preserves $\operatorname{dom} A|A|^{-1+\alpha_2}$ *and*

$$[A|A|^{-1+\alpha_2}, T]B^{-\beta_2}$$

extends to an adjointable operator.

Proof. First, noting that $\operatorname{dom} A|A|^{-1+\alpha} = \operatorname{dom} |A|^\alpha$ for all $\alpha > 0$, by Theorem A.3.4, T preserves $\operatorname{dom} A|A|^{-1+\alpha}$ for all $\alpha \leq \alpha_1$. Second,

$$[A|A|^{-1+\alpha_1}, T]B^{-\beta}$$

is bounded for all $\beta \geq \beta_1$. Third, since A and B commute on a common core,

$$[A|A|^{-1+\alpha_1}, T]B^{-\beta_1} = [A|A|^{-1+\alpha_1}, TB^{-\beta_1}]$$

extends to an adjointable operator. By Lemma A.3.7, the operator

$$[A|A|^{-1+\alpha'_1+yi}, TB^{-\beta_1}] = [A|A|^{-1+\alpha'_1+yi}, T]B^{-\beta_1}$$

is bounded for any $\alpha'_1 \in (0,1)$ and $y \in \mathbb{R}$, with

$$M_{\alpha'_1} := \sup_{y\in\mathbb{R}} |\csc \tfrac{(\alpha+yi)\pi}{2}|\|[A|A|^{-1+\alpha+yi}, T]\| < \infty.$$

Fix $\alpha_1' \in (0, \alpha_1)$. Let $\pi : B \to B(H_\pi)$ be an irreducible representation of B and let $\eta, \xi \in E \otimes_\pi H_\pi$ with $\xi \in \mathrm{dom}(A) \odot H_\pi$. Define the holomorphic function

$$f : z \mapsto \csc \tfrac{\alpha_1' z\pi}{2} \langle \eta \mid [(A \otimes 1)|A \otimes 1|^{-1+\alpha_1' z}(T \otimes 1)](B \otimes 1)^{-\beta_1 z}\xi\rangle$$

on the strip where $0 \leq \Re(z) \leq 1$. We have

$$|f(z)| \leq |\csc \tfrac{\alpha_1' z\pi}{2}| \|\eta\| \|[(A \otimes 1)|A \otimes 1|^{-1+\alpha_1' z}(T \otimes 1)](B \otimes 1)^{-\beta_1 z}\xi\|.$$

For $y \in \mathbb{R}$,

$$|f(1+yi)| \leq |\csc \tfrac{\alpha_1'(1+yi)\pi}{2}| \|\eta\| \|([A|A|^{-1+\alpha_1'+\alpha_1' yi}T]B^{-\beta_1} \otimes 1)(B \otimes 1)^{-\beta_1 yi}\xi\| \leq M_{\alpha_1'} \|\eta\| \|\xi\|$$

and

$$|f(yi)| \leq 2\|T\| \|\eta\| \|\xi\|.$$

By Hadamard's three-line theorem, we obtain, for $\alpha_2 \leq \alpha_1'$ that

$$|\csc \tfrac{\alpha_2 \pi}{2} \langle \eta \mid [(A \otimes 1)|A \otimes 1|^{-1+\alpha_2}(T \otimes 1)](B \otimes 1)^{-\beta_1 \alpha_2/\alpha_1'}\xi\rangle| = |f(\tfrac{\alpha_2}{\alpha_1'})| \leq M_{\alpha_1'}^{\alpha_2/\alpha_1'} (2\|T\|)^{1-\alpha_2/\alpha_1'} \|\eta\| \|\xi\|$$

for $0 \leq \alpha \leq 1$. Hence, putting $\eta = [(A \otimes 1)|A \otimes 1|^{-1+\alpha_2}(T \otimes 1)](B \otimes 1)^{-\beta_1 \alpha_2/\alpha_1'}\xi$,

$$\|\eta\|^2 \leq \sin \tfrac{\alpha_2 \pi}{2} M_{\alpha_1'}^{\alpha_2/\alpha_1'} \|T\|^{1-\alpha_2/\alpha_1'} \|\eta\| \|\xi\|$$

and so

$$\|\eta\| \leq \sin \tfrac{\alpha_2 \pi}{2} M_{\alpha_1'}^{\alpha_2/\alpha_1'} \|T\|^{1-\alpha_2/\alpha_1'} \|\xi\|.$$

By the density of $\mathrm{dom}(A) \odot H_\pi$ in $E \otimes_\pi H_\pi$,

$$\|[(A \otimes 1)|A \otimes 1|^{-1+\alpha_2}(T \otimes 1)](B \otimes 1)^{-\beta_1 \alpha_2/\alpha_1'}\| \leq \sin \tfrac{\alpha_2 \pi}{2} M_{\alpha_1'}^{\alpha_2/\alpha_1'} \|T\|^{1-\alpha_2/\alpha_1'}.$$

Restricting to $0 < \alpha_2 \leq \alpha_1'$ so that $|A|^{-1+\alpha_2}$ is a well-defined as adjointable operator on E,

$$\begin{aligned} \|[A|A|^{-1+\alpha_2}T]B^{-\beta_1 \alpha_2/\alpha_1'}\|_{\mathrm{End}^*_B(E)} &= \sup_{[\pi] \in \hat{B}} \|A|A|^{-1+\alpha_2}T]B^{-\beta_1 \alpha_2/\alpha_1'} \otimes 1\|_{B(E \otimes_\pi H_\pi)} \\ &\leq \sin \tfrac{\alpha_2 \pi}{2} M_{\alpha_1'}^{\alpha_2/\alpha_1'} \|T\|^{1-\alpha_2/\alpha_1'}. \end{aligned}$$

By making a suitable choice of α_1', we obtain that

$$[A|A|^{-1+\alpha_2}, T]B^{-\beta_2}$$

is bounded for any $\alpha_2 < \alpha_1$ and $\beta_2 > \beta_1 \alpha_2/\alpha_1$.

For the second part, fix $\alpha_1' \in (0, \alpha_1)$ and $\alpha_3' \in (0, \alpha_3)$. Let $\pi : B \to B(H_\pi)$ be an irreducible representation of B and let $\eta, \xi \in E \otimes_\pi H_\pi$ with $\xi \in \mathrm{dom}(A) \odot H_\pi$. Define the holomorphic function

$$f : z \mapsto \csc \tfrac{(\alpha_1'(1-z)+\alpha_3' z)\pi}{2} \langle \eta \mid [(A \otimes 1)|A \otimes 1|^{-1+\alpha_1'(1-z)+\alpha_3' z}(T \otimes 1)](B \otimes 1)^{-\beta_1(1-z)-\beta_3 z}\xi\rangle$$

on the strip where $0 \leq \Re(z) \leq 1$. By similar machinations to the ones above, we obtain that

$$[A|A|^{-1+\alpha_3}T]B^{-\frac{(\alpha_3'-\alpha_2)\beta_1+(\alpha_2-\alpha_1')\beta_3}{\alpha_3'-\alpha_1'}}$$

is bounded for $\alpha_1' \leq \alpha_2 \leq \alpha_3'$. By making suitable choices of α_1' and α_3', we obtain that

$$[A|A|^{-1+\alpha_2}, T]B^{-\beta_2}$$

is bounded for any $\alpha_1 < \alpha_2 < \alpha_3$ and $\beta_2 > \frac{(\alpha_3-\alpha_2)\beta_1+(\alpha_2-\alpha_1)\beta_3}{\alpha_3-\alpha_1}$. □

Proof of Theorem A.3.6. Define the subset C of $(\alpha, \beta) \in S$ such that, for all $y \in \mathbb{R}$,

$$[A|A|^{-1+\alpha+iy}, T]B^{-\beta}$$

extends from $\operatorname{dom}|A|^\alpha$ to an adjointable operator on E and

$$\sup_{y\in\mathbb{R}} |\csc \tfrac{(\alpha+yi)\pi}{2}| \|[A|A|^{-1+\alpha+yi}, T]\| < \infty.$$

Then Lemma A.3.7 says that $S \subseteq \overline{C}$. That C is convex follows from the Proof of Proposition A.3.8. □

A.3.2 A form condition for relatively bounded commutators on Hilbert C*-modules

Lemma A.3.9. *Let D be a self-adjoint, regular operator on a right Hilbert B-module E and $a \in \operatorname{End}^*_B(E)$. Then $a \operatorname{dom} D \subseteq \operatorname{dom} D$ if and only if*

$$X = \{\xi \in \operatorname{dom} D \mid a\xi \in \operatorname{dom} D\}$$

is a core for D and, for some constant $M > 0$,

$$\|Da\xi\| \leq M\|\langle D\rangle\xi\|$$

for all $\xi \in X$.

Proof. Suppose that $a \operatorname{dom} D \subseteq \operatorname{dom} D$. Then $X = \operatorname{dom} D$ is a core for D. For $\xi \in X = \operatorname{dom} D$,

$$\|Da\xi\| = \|F_D\langle D\rangle a\langle D\rangle^{-1}\langle D\rangle\xi\| \leq \|\langle D\rangle a\langle D\rangle^{-1}\|\|\langle D\rangle\xi\|.$$

Suppose, on the other hand, that X is a core for D, and the bound applies. Let $\xi \in \operatorname{dom} D$ and choose $(\xi_n)_{n=1}^\infty \subset X$ converging to ξ in the graph norm. This means that $(\xi_n)_{n=1}^\infty$ converges to ξ and $(D\xi_n)_{n=1}^\infty$ converges to $D\xi$ in the norm on E. Because

$$\left\|Da\xi_m - Da\xi_n\right\| \leq M\left\|\langle D\rangle(\xi_m - \xi_n)\right\|$$

and $(a\xi_n)_{n=1}^\infty$ converges to $a\xi$ in the norm on E, $(a\xi_n)_{n=1}^\infty$ is Cauchy in the graph norm, converging to $a\xi \in \operatorname{dom} D$. Hence, $a \operatorname{dom} D \subseteq \operatorname{dom} D$ as required. □

Proposition A.3.10. *Let D be a self-adjoint, regular operator on a right Hilbert B-module E and $a \in \operatorname{End}^*_B(E)$. Then $a \operatorname{dom} D \subseteq \operatorname{dom} D$ if and only if, for every irreducible representation $\pi : B \to B(H^\pi)$,*

$$a^\pi \operatorname{dom} D^\pi \subseteq \operatorname{dom} D^\pi$$

and $\sup_{[\pi]\in\hat{B}} \|D^\pi a^\pi \langle D^\pi\rangle^{-1}\| < \infty$.

Proof. By Lemma A.3.9, the subspace

$$X = \{\xi \in \operatorname{dom} D \mid a\xi \in \operatorname{dom} D\}$$

is a core for D if and only if $a \operatorname{dom} D \subseteq \operatorname{dom} D$ and for some constant $M > 0$,

$$\|Da\xi\| \leq M\|\langle D\rangle\xi\|$$

for all $\xi \in X$. By [KL12, Theorem 3.3] and [KL17, Theorem 2.1], X is a core for D if and only if, for every irreducible representation $\pi : B \to B(H^\pi)$, the algebraic tensor product $X \odot_B H^\pi$ is a core for D^π. The subspace $X \odot_B H^\pi$ is equal to

$$\begin{aligned} X \odot_B H^\pi &= \{\xi \otimes \eta \in \operatorname{dom} D \odot H^\pi \mid (a \otimes 1)\xi \otimes \eta \in \operatorname{dom} D \odot H_\pi\} \\ &= \{\xi \in \operatorname{dom} D \odot H^\pi \mid a^\pi\xi \in \operatorname{dom} D \odot H^\pi\}. \end{aligned}$$

Suppose that $a \operatorname{dom} D \subseteq \operatorname{dom} D$. Then $X \odot_B H^\pi$ is a core for D^π, for all irreducible representations π of B, and $\|Da\xi\| \leq M\|\langle D\rangle\xi\|$. By Lemma A.3.9, $a^\pi \operatorname{dom} D^\pi \subseteq \operatorname{dom} D^\pi$ and so $(Da\langle D\rangle^{-1})\pi = D^\pi a^\pi \langle D^\pi\rangle^{-1}$. Furthermore, by Lemma A.3.3,

$$\sup_{[\pi]\in\hat{B}} \|D^\pi a^\pi \langle D^\pi\rangle^{-1}\| = \|Da\langle D\rangle^{-1}\| < \infty$$

as required.

On the other hand, suppose that $a^\pi \operatorname{dom} D^\pi \subseteq \operatorname{dom} D^\pi$ and $\sup_\pi \|\langle D^\pi\rangle a^\pi \langle D^\pi\rangle^{-1}\| = M < \infty$. By [Pie06, Lemme 1.15(1)], the graph $\mathscr{G}(D^\pi)$ of D^π is equal to $\mathscr{G}(D) \otimes_B H^\pi$, where $\mathscr{G}(D)$ is the graph of D. Hence, using also the regularity of D,

$$\begin{aligned} \mathscr{G}(D^\pi) \cap ((E \odot_B H^\pi) \oplus (E \odot_B H^\pi)) &= \mathscr{G}(D) \otimes_B H^\pi \cap ((E \oplus E) \odot_B H^\pi) \\ &= \mathscr{G}(D) \otimes_B H^\pi \cap ((\mathscr{G}(D) \odot_B H^\pi) \oplus (\mathscr{G}(D)^\perp \odot_B H^\pi)) \\ &= \mathscr{G}(D) \odot_B H^\pi. \end{aligned}$$

Projecting onto the first terms of $\mathscr{G}(D^\pi)$ and $\mathscr{G}(D)$ in the direct sums $E \oplus E$ and $(E \oplus E) \otimes_B H^\pi$, we find that

$$\operatorname{dom} D^\pi \cap (E \odot_B H^\pi) = \operatorname{dom} D \odot_B H^\pi.$$

Noting that $a^\pi(E \odot_B H^\pi) = (a \otimes 1)(E \odot_B H^\pi) \subseteq E \odot_B H^\pi$, we find that

$$a^\pi(\operatorname{dom} D \odot_B H^\pi) = a(\operatorname{dom} D^\pi \cap (E \odot_B H^\pi)) \subseteq \operatorname{dom} D \odot_B H^\pi$$

and

$$X \odot_B H^\pi = \operatorname{dom} D \odot_B H^\pi,$$

which is a core for D^π. Hence, X is a core for D and so $a \operatorname{dom} D \subseteq \operatorname{dom} D$. Furthermore,

$$\begin{aligned} \|Da\xi\| &= \sup_\pi \sup_{\eta\in H^\pi} \frac{\|D^\pi a^\pi(\xi \otimes \eta)\|}{\|\eta\|} \\ &= \sup_\pi \sup_{\eta\in H^\pi} \frac{\|D^\pi a^\pi \langle D^\pi\rangle^{-1}\langle D^\pi\rangle(\xi \otimes \eta)\|}{\|\eta\|} \\ &\leq \sup_\pi \|D^\pi a^\pi \langle D^\pi\rangle^{-1}\| \sup_{\eta\in H^\pi} \frac{\|\langle D^\pi\rangle(\xi \otimes \eta)\|}{\|\eta\|} \\ &\leq M \sup_\pi \sup_{\eta\in H^\pi} \frac{\|\langle D^\pi\rangle(\xi \otimes \eta)\|}{\|\eta\|} \\ &= M\|\langle D\rangle\xi\|, \end{aligned}$$

by Lemma A.3.3. □

Proposition A.3.11. *cf. [BR87, Proposition 3.2.55] Let D be a self-adjoint, regular operator on a right Hilbert B-module E. Let $0 \leq \alpha \leq 1$. For $a \in \operatorname{End}^*_B(E)$, the following conditions are equivalent:*

1. *$a \operatorname{dom} D \subseteq \operatorname{dom} D$ and $[D,a]\langle D\rangle^{-\alpha}$ is bounded on $\operatorname{dom} D\langle D\rangle^{-\alpha} = \langle D\rangle^{-1+\alpha}E$; and*
2. *the B-sesquilinear map $\varphi : \operatorname{dom} D \times \operatorname{dom} D\langle D\rangle^{-\alpha} \to B$ given by*

$$\varphi : (\xi, \eta) \mapsto \langle D\xi \mid a\langle D\rangle^{-\alpha}\eta\rangle_B - \langle \xi \mid aD\langle D\rangle^{-\alpha}\eta\rangle_B$$

is bounded, meaning that $\sup_{\xi,\eta} \frac{\|\varphi(\xi,\eta)\|}{\|\xi\|\|\eta\|} < \infty$.

When these conditions are satisfied, $\varphi(\xi,\eta) = \langle \xi \mid [D,a]\langle D\rangle^{-\alpha}\eta\rangle$ and $\sup_{\xi,\eta} \frac{\|\varphi(\xi,\eta)\|}{\|\xi\|\|\eta\|} = \|[D,a]\langle D\rangle^{-\alpha}\|$.

Proof. That condition 1. $\Rightarrow$ 2. is a consequence of the identity

$$\langle D\xi \mid a\langle D\rangle^{-\alpha}\eta\rangle_B - \langle \xi \mid aD\langle D\rangle^{-\alpha}\eta\rangle_B = \langle \xi \mid [D,a]\langle D\rangle^{-\alpha}\eta\rangle_B$$

when $a \operatorname{dom} D \subseteq \operatorname{dom} D$. We have

$$\sup_{\xi,\eta} \frac{\|\varphi(\xi,\eta)\|}{\|\xi\|\|\eta\|} = \sup_{\xi,\eta} \frac{\|\langle \xi \mid [D,a]\langle D\rangle^{-\alpha}\eta\rangle\|}{\|\xi\|\|\eta\|} \leq \|[D,a]\langle D\rangle^{-\alpha}\|$$

by the Cauchy–Schwarz inequality.

For the other direction, 2. $\Rightarrow$ 1., let $\pi : B \to B(H_\pi)$ be an irreducible representation and consider the Hilbert space $E^\pi = E \hat{\otimes}_B H_\pi$ and the operators $D^\pi = D \otimes 1$ and $a^\pi = a \otimes 1$. There is a sesquilinear map $\varphi_\pi : \operatorname{dom} D^\pi \times \operatorname{dom} D^\pi \langle D^\pi\rangle^{-\alpha} \to \mathbb{C}$ given by

$$\varphi_\pi : (\xi,\eta) \mapsto \langle D^\pi \xi \mid a\langle D^\pi\rangle^{-\alpha}\eta\rangle - \langle \xi \mid a^\pi D^\pi \langle D^\pi\rangle^{-\alpha}\eta\rangle .$$

The sesquilinear map φ_π is bounded because of the density of $\operatorname{dom} D \odot_B H_\pi$ in $\operatorname{dom} D^\pi$ and of $\operatorname{dom} D\langle D\rangle^{-\alpha} \odot_B H_\pi$ in $\operatorname{dom} D^\pi\langle D^\pi\rangle^{-\alpha}$. There must, therefore, be an operator $b_\pi \in B(E^\pi)$ for which $\varphi_\pi(\xi,\eta) = \langle \xi \mid b_\pi \eta\rangle$ for all $\xi \in \operatorname{dom} D^\pi$ and $\eta \in \operatorname{dom} D^\pi\langle D^\pi\rangle^{-\alpha}$. Then

$$\langle D^\pi \xi \mid a^\pi \langle D^\pi\rangle^{-\alpha}\eta\rangle = \langle \xi \mid a^\pi D^\pi \langle D^\pi\rangle^{-\alpha}\eta\rangle + \langle \xi \mid b_\pi \eta\rangle$$

which demonstrates that

$$\xi \mapsto \langle D^\pi \xi \mid a^\pi \langle D^\pi\rangle^{-\alpha}\eta\rangle$$

is continuous for fixed $\eta \in \operatorname{dom} D^\pi\langle D^\pi\rangle^{-\alpha}$. We find that $a^\pi\langle D^\pi\rangle^{-\alpha}\eta \in \operatorname{dom}(D^\pi)^* = \operatorname{dom} D^\pi$. Hence,

$$a^\pi \operatorname{dom} D^\pi = a^\pi\langle D^\pi\rangle^{-\alpha} \operatorname{dom} D\langle D^\pi\rangle^{-\alpha} \subseteq \operatorname{dom} D^\pi$$

and $[D^\pi, a^\pi]\langle D^\pi\rangle^{-\alpha} = b_\pi$ is bounded by

$$\begin{aligned}
&\sup\left\{ \frac{\|\varphi_\pi(\xi,\eta)\|}{\|\xi\|\|\eta\|} \,\middle|\, \xi \in \operatorname{dom} D^\pi, \eta \in \operatorname{dom} D^\pi\langle D^\pi\rangle^{-\alpha} \right\} \\
&= \sup\left\{ \frac{\|\varphi_\pi(\xi,\eta)\|}{\|\xi\|\|\eta\|} \,\middle|\, \xi \in \operatorname{dom} D \odot_B H_\pi, \eta \in \operatorname{dom} D\langle D\rangle^{-\alpha} \otimes \odot_B H_\pi \right\} \\
&\leq \sup\left\{ \frac{\|\varphi(\xi,\eta)\|}{\|\xi\|\|\eta\|} \,\middle|\, \xi \in \operatorname{dom} D, \eta \in \operatorname{dom} D\langle D\rangle^{-\alpha} \right\}.
\end{aligned}$$

Now, taking the supremum over all irreducible representations $\pi \in \hat{B}$,

$$\begin{aligned}
\sup_{[\pi]\in\hat{B}} \|\langle D^\pi\rangle a^\pi \langle D^\pi\rangle^{-1}\| &= \sup_{[\pi]\in\hat{B}} \|[\langle D^\pi\rangle, a^\pi]\langle D^\pi\rangle^{-1} + a^\pi\| \\
&\leq \sup_{[\pi]\in\hat{B}} \left(\|[\langle D^\pi\rangle, a^\pi]\langle D^\pi\rangle^{-1}\| + \|a^\pi\|\right) \\
&\leq \|a\| + \sup_{[\pi]\in\hat{B}} \left(\|[\langle D^\pi\rangle, a^\pi]\langle D^\pi\rangle^{-\alpha}\| + \|a^\pi\|\right) \\
&\leq \|a\| + \sup\left\{ \frac{\|\varphi(\xi,\eta)\|}{\|\xi\|\|\eta\|} \,\middle|\, \xi \in \operatorname{dom} D, \eta \in \operatorname{dom} D\langle D\rangle^{-\alpha} \right\} \\
&< \infty.
\end{aligned}$$

We may, therefore, apply Proposition A.3.10 to obtain that $a \operatorname{dom} D \subseteq \operatorname{dom} D$. Then the boundedness of

$$\varphi(\xi,\eta) = \langle D\xi \mid a\langle D\rangle^{-\alpha}\eta\rangle_B - \langle \xi \mid aD\langle D\rangle^{-\alpha}\eta\rangle_B = \langle \xi \mid [D,a]\langle D\rangle^{-\alpha}\eta\rangle_B$$

makes $[D,a]\langle D\rangle^{-\alpha}$ bounded. By Lemma A.3.3,

$$\|[D,a]\langle D\rangle^{-\alpha}\| = \sup_{[\pi]\in\hat{B}} \|b_\pi\| \leq \sup_{\xi,\eta} \frac{\|\varphi(\xi,\eta)\|}{\|\xi\|\|\eta\|},$$

establishing the required equality of bounds. $\square$

A.4 Functional calculus for higher order Kasparov modules

A.4.1 Closure under the holomorphic functional calculus

We begin by recalling a few details in the abstract. For an open subset $U \subseteq \mathbb{C}$, we denote by $\mathcal{O}(U)$ the holomorphic complex valued functions on U. For any subset $S \subseteq \mathbb{C}$, we denote by $\mathcal{O}(S)$ all those functions holomorphic on some open set containing S.

Definition A.4.1. e.g. [LMN05, Definition 2.1] Let A be a unital Banach algebra. A unital subalgebra B of A is *closed under the holomorphic functional calculus of* A if, for every $b \in B$ and $f \in \mathcal{O}(\sigma_A(b))$, $f(b) \in B$.

Lemma A.4.2. *[LMN05, Remark 2.2(c)] Let A be a unital Banach algebra. Let B be a unital subalgebra of A which is closed under the holomorphic functional calculus of A. Then*

$$A^{-1} \cap B = B^{-1}$$

i.e. the invertible elements of B are exactly those elements of A which are invertible and lie in B. As a consequence, $\sigma_B(b) = \sigma_A(b)$ for all $b \in B$.

Proof. Let $b \in A^{-1} \cap B$. Since $0 \notin \sigma_A(b)$ and the spectrum is closed, the function $f : \lambda \to \lambda^{-1}$ is in $\mathcal{O}(\sigma_A(b))$. Then $f(b) = b^{-1}$ is in B and so $A^{-1} \cap B \subseteq B^{-1}$. The opposite inclusion, $B^{-1} \subseteq A^{-1} \cap B$, holds because the inclusion $B \subseteq A$ is unital. Finally, for any $b \in B$,

$$\sigma_B(b) = \{\lambda \in \mathbb{C} \mid \lambda - b \in B^{-1}\} = \{\lambda \in \mathbb{C} \mid \lambda - b \in A^{-1}\} = \sigma_A(b)$$

as required. □

Lemma A.4.3. *cf. [LMN05, Proposition 2.4(d)] Let A be a unital Banach algebra. Let B be a unital subalgebra of A which is a Banach algebra, not necessarily with the inherited norm. If*

$$A^{-1} \cap B = B^{-1}$$

then B is closed under the holomorphic functional calculus of A.

Proof. Let $b \in B$. As in the proof of the previous Lemma, $\sigma_A(b) = \sigma_B(b)$ and so $\mathcal{O}(\sigma_A(c)) = \mathcal{O}(\sigma_B(c))$. Let $f \in \mathcal{O}(\sigma_A(c)) = \mathcal{O}(\sigma_B(c))$. Since B is a Banach algebra, we can employ its functional calculus and write

$$f(b) = \frac{1}{2\pi i} \oint_\gamma f(\lambda)(\lambda - b)^{-1} d\lambda \in B$$

as required. □

Next, we see that closure under the holomorphic functional calculus is a transitive property.

Lemma A.4.4. *Let A be a unital Banach algebra. Let B be a unital subalgebra of A which is a Banach algebra, not necessarily with the inherited norm, which is closed under the holomorphic functional calculus of A. Let C be a unital subalgebra of B which is a Banach algebra, not necessarily with the inherited norm. Then C is closed under the holomorphic functional calculus of B if and only if it is closed under the holomorphic functional calculus of A.*

Proof. Let $c \in C$. Since $\sigma_A(b) = \sigma_B(b)$ for all $b \in B$, $\mathcal{O}(\sigma_A(c)) = \mathcal{O}(\sigma_B(c))$. What it means for C to be closed under the holomorphic functional calculus is the same for A and B, viz. that, for all $f \in \mathcal{O}(\sigma_A(c)) = \mathcal{O}(\sigma_B(c))$, $f(c) \in C$. □

Lemma A.4.5. *cf. [BC91, Proposition 3.12], [LMN05, Lemma 2.7] Let A be a unital Banach algebra. Let B be a unital subalgebra of A which is a Banach algebra, not necessarily with the inherited norm. Suppose that B is dense in A and*

$$r_B(b) \leq \|b\|_A$$

for all $b \in B$, where r_B denotes the spectral radius in B. Then

$$A^{-1} \cap B = B^{-1}$$

and B is closed under the holomorphic functional calculus of A.

Proof. Let $a \in A^{-1} \cap B$. By the density of B in A, we can find an element $b \in B$ such that $\|1 - ab\|_A < 1$. By the assumption, $r_B(1 - ab) < 1$, meaning that $1 \notin \sigma_B(1 - ab)$. Then $0 \notin \sigma_B(ab)$ so ab is invertible in B and

$$a^{-1} = b(ab)^{-1} \in B$$

Hence, $A^{-1} \cap B \subseteq B^{-1}$. Because the inclusion $B \subseteq A$ is unital, the opposite inclusion is also true. □

We now come to the setting of higher order Kasparov modules.

Definition A.4.6. Let D be a self-adjoint, regular operator on a right Hilbert B-module E. For $0 \leq \alpha < 1$, let

$$\operatorname{Lip}_\alpha(D) \subseteq \operatorname{End}^*_B(E)$$

be the subspace consisting of elements $a \in \operatorname{End}^*_B(E)$ for which $a \operatorname{dom} D \subseteq \operatorname{dom} D$ and $[D, a]\langle D\rangle^{-\alpha}$ is bounded on $\operatorname{dom} D\langle D\rangle^{-\alpha}$. Let $\operatorname{Lip}^*_\alpha(D) = \operatorname{Lip}_\alpha(D) \cap \operatorname{Lip}_\alpha(D)^* \subseteq \operatorname{End}^*_B(E)$.

Note that, because $\|\langle D\rangle^{-1}\| \leq 1$, $\operatorname{Lip}_\alpha(D) \subseteq \operatorname{Lip}_\beta(D)$ for any $\alpha \leq \beta$.

Proposition A.4.7. *Let D be a self-adjoint, regular operator on a right Hilbert B-module E and let $a \in \operatorname{End}^*_B(E)$ preserve* $\operatorname{dom} D$. *Suppose that*

$$[D, a]\langle D\rangle^{-\alpha}$$

is bounded for some $0 \leq \alpha < 1$. Then, for $0 \leq \gamma < 1$ and $0 \leq \beta$, such that $\alpha - \beta + \gamma < 1$,

$$[\langle D\rangle^\gamma, a]\langle D\rangle^{-\beta}$$

is bounded by

$$C \left\| [D, a]\langle D\rangle^{-\alpha} \right\|$$

for some constant C depending on $\alpha - \beta$ and γ.

Proof. First,

$$[\langle D\rangle^\gamma, a]\langle D\rangle^{-\beta} = -\langle D\rangle^\gamma [\langle D\rangle^{-\gamma}, a]\langle D\rangle^{-\beta+\gamma}.$$

By the integral formula (I.0.5), on $\operatorname{dom} D$, using [CP98, Lemma 2.3],

$$\begin{aligned}
\langle D\rangle^\gamma [\langle D\rangle^{-\gamma}, a]\langle D\rangle^{-\beta+\gamma} &= \frac{\sin\frac{\gamma\pi}{2}}{\pi} \int_0^\infty \lambda^{-\gamma/2} \langle D\rangle^\gamma [a, (\lambda + 1 + D^2)^{-1}] \langle D\rangle^{-\beta+\gamma} d\lambda \\
&= \frac{\sin\frac{\gamma\pi}{2}}{\pi} \int_0^\infty \lambda^{-\gamma/2} \langle D\rangle^\gamma \Big(D(\lambda + 1 + D^2)^{-1}[D, a](\lambda + 1 + D^2)^{-1} \\
&\qquad + (\lambda + 1 + D^2)^{-1}[D, a] D(\lambda + 1 + D^2)^{-1} \Big) \langle D\rangle^{-\beta+\gamma} d\lambda
\end{aligned}$$

The integral is norm-convergent, with a bound

$$\begin{aligned}
&\left\|\langle D\rangle^{\gamma}[\langle D\rangle^{-\gamma},a]\langle D\rangle^{-\beta+\gamma}\right\| \\
&\quad\le \frac{\sin\frac{\gamma\pi}{2}}{\pi}\int_0^\infty \lambda^{-\gamma/2} \\
&\qquad\times\Big(\left\|\langle D\rangle^{\gamma}D(\lambda+1+D^2)^{-1}\right\|\left\|[D,a]\langle D\rangle^{-\alpha}\right\|\left\|\langle D\rangle^{\alpha-\beta+\gamma}(\lambda+1+D^2)^{-1}\right\| \\
&\qquad\quad+\left\|\langle D\rangle^{\gamma}(\lambda+1+D^2)^{-1}\right\|\left\|[D,a]\langle D\rangle^{-\alpha}\right\|\left\|\langle D\rangle^{\alpha-\beta+\gamma}D(\lambda+1+D^2)^{-1}\right\|\Big)\,d\lambda \\
&\quad\le \frac{\sin\frac{\gamma\pi}{2}}{\pi}2\left\|[D,a]\langle D\rangle^{\delta}\right\|\int_0^\infty \lambda^{-\gamma/2}(\lambda+1)^{-3/2+(\alpha-\beta)/2+\gamma}d\lambda \\
&=\frac{\sin\frac{\gamma\pi}{2}}{\pi}\frac{\Gamma(\frac{2-\gamma}{2})\Gamma(\frac{1-\alpha+\beta-\gamma}{2})}{\Gamma(\frac{3-\alpha+\beta-2\gamma}{2})}2\left\|[D,a]\langle D\rangle^{-\alpha}\right\| \\
&\quad<\infty
\end{aligned}$$

so $[\langle D\rangle^{\gamma},a]\langle D\rangle^{-\beta}$ has the required bound. □

Corollary A.4.8. *Let D be a self-adjoint, regular operator on a right Hilbert B-module E. For $0\le\gamma<1$, $0\le\beta\le\gamma$, and $\alpha-\beta+\gamma<1$,*

$$\mathrm{Lip}_{\alpha}(D)\subseteq\mathrm{Lip}_{\beta/\gamma}(\langle D\rangle^{\gamma})$$

Similarly, with $0\le\gamma\le1$, $0\le\beta\le\gamma$, and $\alpha-\beta+\gamma<1$

$$\mathrm{Lip}_{\alpha}(\langle D\rangle^{\delta})\subseteq\mathrm{Lip}_{\beta/\gamma}(\langle D\rangle^{\gamma\delta})$$

Proof. Let $a\in\mathrm{Lip}_{\alpha}(D)$. Then, by Proposition A.4.7,

$$[\langle D\rangle^{\gamma},a]\langle D\rangle^{-\beta}$$

is bounded for $0\le\gamma\le1$, $0\le\beta\le\gamma$, and $\alpha-\beta+\gamma<1$. The real function

$$x\mapsto\frac{(1+x^2)^{\beta/2}}{(1+(1+x^2)^{\gamma})^{\beta/2\gamma}}=\left(\frac{1+x^2}{(1+(1+x^2)^{\gamma})^{1/\gamma}}\right)^{\beta/2}$$

is bounded by 1, so

$$\|\langle D\rangle^{\beta}\langle\langle D\rangle^{\gamma}\rangle^{-\beta/\gamma}\|\le1$$

and

$$[\langle D\rangle^{\gamma},a]\langle\langle D\rangle^{\gamma}\rangle^{-\beta/\gamma}=[\langle D\rangle^{\gamma},a]\langle D\rangle^{-\beta}\langle D\rangle^{\beta}\langle\langle D\rangle^{\gamma}\rangle^{-\beta/\gamma}$$

is bounded. □

Lemma A.4.9. *cf. [BMR10, Lemma 1], [GM15, Proposition A.5] Let D be a self-adjoint, regular operator on a right Hilbert B-module E. For $0\le\alpha<1$, $\mathrm{Lip}_{\alpha}(D)$ is closed under multiplication and can be equipped with a norm*

$$\|\cdot\|_{D,\alpha}:a\to\|a\|+K_{\alpha}\|[D,a]\langle D\rangle^{-\alpha}\|$$

for some constant $K_{\alpha}>0$, making it a unital Banach algebra.

Proof. First, it is clear that $\mathrm{Lip}_\alpha(D)$ is closed under addition and multiplication by $\mathbb{C}$ and shares a unit with $\mathrm{End}^*_B(E)$. Let $a, b \in \mathrm{Lip}_\alpha(D)$; then $ab \operatorname{dom} D \subseteq a \operatorname{dom} D \subseteq \operatorname{dom} D$. By the Leibniz rule,

$$\begin{aligned}[D, ab]\langle D\rangle^{-\alpha} &= [D, a]b\langle D\rangle^{-\alpha} + a[D, b]\langle D\rangle^{-\alpha} \\ &= -[D, a][\langle D\rangle^{-\alpha}, b] + [D, a]\langle D\rangle^{-\alpha} b + a[D, b]\langle D\rangle^{-\alpha} \\ &= [D, a]\langle D\rangle^{-\alpha}[\langle D\rangle^{\alpha}, b]\langle D\rangle^{-\alpha} + [D, a]\langle D\rangle^{-\alpha} b + a[D, b]\langle D\rangle^{-\alpha}\end{aligned}$$

By Proposition A.4.7,

$$\|[\langle D\rangle^{\alpha}, b]\langle D\rangle^{-\alpha}\| \leq C\|[D, b]\langle D\rangle^{-\alpha}\|$$

where

$$C = \frac{2}{\pi}\frac{\Gamma(\frac{2-\alpha}{2})\Gamma(\frac{3-\alpha}{2})}{\Gamma(\frac{3-2\alpha}{2})}\sin\frac{\alpha\pi}{2}.$$

Let $K_\alpha \geq C$. (In fact, for the purposes of the Proposition A.4.12, we also insist that $K_\alpha \geq 2^{\alpha/2}$.) Then

$$\|[D, ab]\langle D\rangle^{-\alpha}\| \leq \|[D, a]\langle D\rangle^{-\alpha}\|\|b\| + \|a\|\|[D, b]\langle D\rangle^{-\alpha}\| + C\|[D, a]\langle D\rangle^{-\alpha}\|\|[D, b]\langle D\rangle^{-\alpha}\|$$

so $ab \in \mathrm{Lip}_\alpha(D)$ and

$$\begin{aligned}\|ab\|_{D,\alpha} &= \|ab\| + C\|[D, ab]\langle D\rangle^{-\alpha}\| \\ &\leq \|a\|\|b\| + C\left(\|[D, a]\langle D\rangle^{-\alpha}\|\|b\| + \|a\|\|[D, b]\langle D\rangle^{-\alpha}\| + C\|[D, a]\langle D\rangle^{-\alpha}\|\|[D, b]\langle D\rangle^{-\alpha}\right)\| \\ &= \left(\|a\| + C\|[D, a]\|\langle D\rangle^{-\alpha}\right)\left(\|b\| + C\|[D, b]\langle D\rangle^{-\alpha}\|\right) \\ &= \|a\|_{D,\alpha}\|b\|_{D,\alpha}.\end{aligned}$$

Hence, $\mathrm{Lip}_\alpha(D)$ is a normed algebra.

To check completeness, let $(a_n)_{n=1}^\infty$ be a Cauchy sequence in $\mathrm{Lip}_\alpha(D)$ (for the norm $\|\cdot\|_{D,\alpha}$). Since, $\|\cdot\|_{\mathrm{End}^*_B(E)} \leq \|b\|_{D,\alpha}$, $(a_n)_{n=1}^\infty$ is Cauchy in $\mathrm{End}^*_B(E)$, converging to some limit $a \in \mathrm{End}^*_B(E)$. For fixed $\xi \in \operatorname{dom} D$ and $\eta \in \operatorname{dom} D\langle D\rangle^{-\alpha}$,

$$\langle D\xi \mid (a - a_n)\langle D\rangle^{-\alpha}\eta\rangle_B - \langle \xi \mid (a - a_n)D\langle D\rangle^{-\alpha}\eta\rangle_B$$

converges to zero as $n \to \infty$. Further,

$$\begin{aligned}&\left|\langle D\xi \mid (a - a_n)\langle D\rangle^{-\alpha}\eta\rangle_B - \langle \xi \mid (a - a_n)D\langle D\rangle^{-\alpha}\eta\rangle_B\right| \\ &\quad= \lim_{m\to\infty}\left|\langle D\xi \mid (a_m - a_n)\langle D\rangle^{-\alpha}\eta\rangle_B - \langle \xi \mid (a_m - a_n)D\langle D\rangle^{-\alpha}\eta\rangle_B\right| \\ &\quad\leq \|\xi\|\|\eta\| \limsup_{m\to\infty}\left\|[D, a_m - a_n]\langle D\rangle^{-\alpha}\right\|.\end{aligned}$$

Applying Proposition A.3.11, we find that $a - a_n \in \mathrm{Lip}_\alpha(D)$ and so that $\|a - a_n\|_{D,\alpha} \to 0$. Hence a_n converges to $a \in \mathrm{Lip}_\alpha(D)$ in the norm $\|\cdot\|_{D,\alpha}$. □

Proposition A.4.10. *Let D be a self-adjoint, regular operator on a right Hilbert B-module E. Any element a of the Banach algebra* $\mathrm{Lip}_0(D)$ *has the bound on its spectral radius*

$$r_{\mathrm{Lip}_0(D)}(a) = \lim_{n\to\infty}\|a^n\|_{D,0}^{1/n} \leq \|a\|.$$

Hence, $\mathrm{Lip}_0(D)$ *is closed under the holomorphic functional calculus of* $\mathrm{End}^*_B(E)$.

Proof. First, note the algebraic identity

$$[D, a^n] = \sum_{k=1}^{n} a^{k-1}[D, a]a^{n-k}. \tag{A.4.11}$$

From this, we estimate

$$\|[D, a^n]\| \leq \sum_{k=1}^{n} \|a\|^{k-1} \|[D, a]\| \|a\|^{n-k} \leq n\|a\|^{n-1} \|[D, a]\|,$$

giving us

$$\|a^n\|_{D,0} \leq \|a\|^n + n\|a\|^{n-1}\|[D, a]\| = \|a\|^n(1 + n\|a\|^{-1}\|[D, a]\|).$$

Finally,

$$\lim_{n\to\infty} \|a^n\|_{D,0}^{1/n} \leq \lim_{n\to\infty} \|a\|(1 + n\|a\|^{-1}\|[D, a]\|)^{1/n} = \|a\|$$

and the conclusion follows by Lemma A.4.5. □

Proposition A.4.12. *Let D be a self-adjoint, regular operator on a right Hilbert B-module E. An element a of the Banach algebra $\mathrm{Lip}_\alpha(D)$ has the bound on its spectral radius*

$$r_{\mathrm{Lip}_\alpha(D)}(a) = \lim_{n\to\infty} \|a^n\|_{D,\alpha}^{1/n} \leq \|a\|_{\langle D\rangle^\alpha,\beta}$$

for any $0 \leq \beta \leq 1$ such that $2 - a^{-1} < \beta$. Hence, $\mathrm{Lip}_\alpha(D)$ is closed under the holomorphic functional calculus of $\mathrm{Lip}_\beta(\langle D\rangle^\alpha)$.

Similarly, any element a of the Banach algebra $\mathrm{Lip}_\alpha(\langle D\rangle^\delta)$ has the bound on its spectral radius

$$r_{\mathrm{Lip}_\alpha(\langle D\rangle^\delta)}(a) = \lim_{n\to\infty} \|a^n\|_{\langle D\rangle^\delta,\alpha}^{1/n} \leq \|a\|_{\langle D\rangle^{\alpha\delta},\beta}$$

for any $0 \leq \beta \leq 1$ such that $2-\alpha^{-1} < \beta$. Hence, $\mathrm{Lip}_\alpha(\langle D\rangle^\delta)$ is closed under the holomorphic functional calculus of $\mathrm{Lip}_\beta(\langle D\rangle^{\alpha\delta})$.

Proof. First, using (A.4.11), we estimate

$$\begin{aligned}
\left\|[D, a^n]\langle D\rangle^{-\alpha}\right\| &\leq \sum_{k=1}^{n} \|a^{k-1}\| \left\|[D, a]\langle D\rangle^{-\alpha}\right\| \left\|\langle D\rangle^\alpha a^{n-k}\langle D\rangle^{-\alpha}\right\| \\
&\leq \left\|[D, a]\langle D\rangle^{-\alpha}\right\| \sum_{k=1}^{n} \|a\|^{k-1} \left\|\langle D\rangle^\alpha a\langle D\rangle^{-\alpha}\right\|^{n-k} \\
&= \left\|[D, a]\langle D\rangle^{-\alpha}\right\| \frac{\left\|\langle D\rangle^\alpha a\langle D\rangle^{-\alpha}\right\|^n - \|a\|^n}{\left\|\langle D\rangle^\alpha a\langle D\rangle^{-\alpha}\right\| - \|a\|} \\
&= c_1 \left(\left\|\langle D\rangle^\alpha a\langle D\rangle^{-\alpha}\right\|^n - \|a\|^n\right)
\end{aligned}$$

with $c_1 = \left\|[D, a]\langle D\rangle^{-\alpha}\right\| \left(\left\|\langle D\rangle^\alpha a\langle D\rangle^{-\alpha}\right\| - \|a\|\right)^{-1}$, provided that $\|\langle D\rangle^\alpha a\langle D\rangle^{-\alpha}\| \neq \|a\|$. If, in fact, $\|\langle D\rangle^\alpha a\langle D\rangle^{-\alpha}\| = \|a\|$,

$$\left\|[D, a^n]\langle D\rangle^{-\alpha}\right\| \leq \left\|[D, a]\langle D\rangle^{-\alpha}\right\| n\|a\|^{n-1}$$

so that

$$\lim_{n\to\infty} \|a^n\|_{D,0}^{1/n} \leq \lim_{n\to\infty} \|a\|(1 + n\|a\|^{-1}\|[D, a]\langle D\rangle^{-\alpha}\|)^{1/n} = \|a\|.$$

Otherwise,

$$\|a^n\|_{D,\alpha} \leq \|a\|^n + K_\alpha c_1 \left(\left\|\langle D\rangle^\alpha a\langle D\rangle^{-\alpha}\right\|^n - \|a\|^n\right) = K_\alpha c_1 \left\|\langle D\rangle^\alpha a\langle D\rangle^{-\alpha}\right\|^n + (1 - K_\alpha c_1)\|a\|^n.$$

If $\|\langle D\rangle^\alpha a\langle D\rangle^{-\alpha}\| > \|a\|$, $c_1 > 0$ and

$$\lim_{n\to\infty} \|a^n\|_{D,\alpha}^{1/n} \leq \lim_{n\to\infty} \left(K_\alpha c_1 \left\|\langle D\rangle^\alpha a\langle D\rangle^{-\alpha}\right\|^n + (1 - K_\alpha c_1)\|a\|^n\right) = \|\langle D\rangle^\alpha a\langle D\rangle^{-\alpha}\|.$$

If $\|\langle D\rangle^\alpha a\langle D\rangle^{-\alpha}\| < \|a\|$, $c_1 < 0$ and

$$\lim_{n\to\infty} \|a^n\|_{D,\alpha}^{1/n} \leq \lim_{n\to\infty} \left(K_\alpha c_1 \left\|\langle D\rangle^\alpha a\langle D\rangle^{-\alpha}\right\|^n + (1-K_\alpha c_1)\|a\|^n\right) = \|a\|.$$

By definition,

$$\|a\| \leq \|a\|_{\langle D\rangle^\alpha,\beta}$$

for any α,β. For $0\leq\beta\leq 1$, the real function

$$x \mapsto \frac{(1+(1+x^2)^\alpha)^{\beta/2}}{(1+x^2)^{\alpha/2}}$$

is bounded by $2^{\beta/2}$, so that

$$\left\|\langle\langle D\rangle^\alpha\rangle^\beta\langle D\rangle^{-\alpha}\right\| \leq 2^{\beta/2}$$

Then

$$\begin{aligned}
\|\langle D\rangle^\alpha a\langle D\rangle^{-\alpha}\| &\leq \|a\| + \|[\langle D\rangle^\alpha, a]\langle D\rangle^{-\alpha}\| \\
&= \|a\| + \|[\langle D\rangle^\alpha, a]\langle\langle D\rangle^\alpha\rangle^{-\beta}\langle\langle D\rangle^\alpha\rangle^\beta\langle D\rangle^{-\alpha}\| \\
&\leq \|a\| + 2^{\beta/2}\|[\langle D\rangle^\alpha, a]\langle\langle D\rangle^\alpha\rangle^{-\beta}\| \\
&\leq \|a\| + K_\beta\|[\langle D\rangle^\alpha, a]\langle\langle D\rangle^\alpha\rangle^{-\beta}\| \\
&= \|a\|_{\langle D\rangle^\alpha,\beta}
\end{aligned}$$

which is finite, by Corollary A.4.8, for $\alpha-\alpha\beta+\alpha<1$ and $\alpha<1$. The conclusion follows by Lemma A.4.5.

For the second part, one can proceed in the same way, the only difference that being that one should begin by estimating $\|[\langle D\rangle^\delta, a^n]\langle D\rangle^{-\alpha\delta}\|$. □

Theorem A.4.13. *Let D be a self-adjoint, regular operator on a right Hilbert B-module E. For any $0\leq\alpha<1$, $\mathrm{Lip}_\alpha(D)$ is closed under the holomorphic functional calculus of $\mathrm{End}^*_B(E)$.*

Proof. First, we shall construct a sequence $(\beta_n)_{n=1}^N \subset [0,1)$ such that $\beta_1=\alpha$, $\beta_N=0$, and

$$2-\beta_{n-1}^{-1} < \beta_n$$

Pick $N > (1-\alpha)^{-1}$ and, with

$$c = \frac{(1-\alpha)^{-1}-1}{N-1}$$

let

$$\beta_n = 1 - \frac{1}{(1-\alpha)^{-1}-(n-1)c} = \frac{(1-\alpha)^{-1}-(n-1)c-1}{(1-\alpha)^{-1}-(n-1)c}$$

Because $0<c<1$,

$$\begin{aligned}
2-\beta_{n-1} &= 2 - \frac{(1-\alpha)^{-1}-(n-2)c}{(1-\alpha)^{-1}-(n-2)c-1} \\
&= 1 - \frac{1}{(1-\alpha)^{-1}-(n-1)c+(c-1)} \\
&< 1 - \frac{1}{(1-\alpha)^{-1}-(n-1)c} \\
&= \beta_n
\end{aligned}$$

Furthermore,

$$\beta_N = 1 - \frac{1}{(1-\alpha)^{-1}-(N-1)c} = 1 - \frac{1}{(1-\alpha)^{-1}-((1-\alpha)^{-1}-1)} = 0$$

as required. Now we have a chain of inclusions

$$\mathrm{Lip}_{\beta_1=\alpha}(D) \subseteq \mathrm{Lip}_{\beta_2}(\langle D\rangle^{\beta_1}) \subseteq \mathrm{Lip}_{\beta_3}(\langle D\rangle^{\beta_1\beta_2}) \subseteq \cdots \subseteq \mathrm{Lip}_{\beta_N=0}(\langle D\rangle^{\prod_{k=1}^{N-1}\beta_k}) \subseteq \mathrm{End}^*_B(E)$$

where each is closed under the holomorphic functional calculus of the next, by Propositions A.4.10 and A.4.12. By Lemma A.4.4, we are done. □

A.4.2 Closure under the smooth functional calculus

The approach is originally due to [Pow75, Theorem 3], corrected by [BR76, §2], reproduced as [BR87, Theorem 3.2.32]. An approach for dealing with higher derivatives is [BEJ84, Lemma 3.2].

Lemma A.4.14. *Let D be a self-adjoint, regular operator on a right Hilbert B-module E. Let S be a bounded operator such that $S\,\mathrm{dom}\,D \subseteq \mathrm{dom}\,D$. Then $e^S\,\mathrm{dom}\,D = \mathrm{dom}\,D$.*

Proof. For a bounded operator T, the condition that $T\mathrm{dom}\,D \subseteq \mathrm{dom}\,D$ is equivalent to $\langle D\rangle T\langle D\rangle^{-1}$ being everywhere-defined and bounded. Since $e^x = \sum_{k=0}^{\infty} x^k/k!$ converges everywhere,

$$e^{\langle D\rangle S\langle D\rangle^{-1}} = \sum_{k=0}^{\infty}\left(\langle D\rangle S\langle D\rangle^{-1}\right)^k/k! = \langle D\rangle\sum_{k=0}^{\infty} S^k/k!\langle D\rangle^{-1} = \langle D\rangle e^S\langle D\rangle^{-1}$$

is everywhere-defined and bounded, and $e^S\,\mathrm{dom}\,D \subseteq \mathrm{dom}\,D$. Similarly,

$$e^{\langle D\rangle(-S)\langle D\rangle^{-1}} = \langle D\rangle e^{-S}\langle D\rangle^{-1}$$

and $e^{-S}\,\mathrm{dom}\,D \subseteq \mathrm{dom}\,D$. Because $e^Se^{-S} = 1$,

$$\mathrm{dom}\,D = e^Se^{-S}\,\mathrm{dom}\,D \subseteq e^S\,\mathrm{dom}\,D$$

and we obtain the required equality, $e^S\,\mathrm{dom}\,D = \mathrm{dom}\,D$. □

Lemma A.4.15. *Let D be a self-adjoint, regular operator on a right Hilbert B-module E. Let $S = -S^*$ be a bounded operator such that $S\,\mathrm{dom}\,D \subseteq \mathrm{dom}\,D$ and $[D,S]$ extends to a bounded operator. Then $[D,e^S]$ has a norm bound $\left\|[D,e^S]\right\| \leq \left\|[D,S]\right\|$, so that $\|e^S\|_{D,0} \leq 1 + \|S\|_{D,0}$.*

Proof. We have

$$[D,e^S] = \int_0^1 \frac{d}{dx}e^{(1-x)S}De^{xS}dx = \int_0^1 e^{(1-x)S}[D,S]e^{xS}dx,$$

which has norm bound

$$\left\|[D,e^S]\right\| \leq \int_0^1 \left\|e^{(1-x)S}\right\|\left\|[D,S]\right\|\left\|e^{xS}\right\|dx \leq \int_0^1 \left\|[D,S]\right\|dx = \left\|[D,S]\right\|$$

as required. □

Lemma A.4.16. *Let D be a self-adjoint, regular operator on a right Hilbert B-module E. Let $S = -S^*$ be a bounded operator such that $S\,\mathrm{dom}\,D \subseteq \mathrm{dom}\,D$ and $[D,S]\langle D\rangle^{-\alpha}$ extends to a bounded operator for some $0 \leq \alpha < 1$. For $0 \leq \beta < 1$,*

$$\|[D,e^S]\langle D\rangle^{-\alpha}\| \leq \|S\|_{D,\alpha}\sup_{x\in[0,1]}\left\|e^{xS}\right\|_{\langle D\rangle^{\alpha},\beta}$$

and so $\|e^S\|_{D,\alpha} \leq (1+\|S\|_{D,\alpha})\sup_{x\in[0,1]}\left\|e^{xS}\right\|_{\langle D\rangle^{\alpha},\beta}$. Furthermore,

$$\left\|[\langle D\rangle^{\gamma},e^S]\langle\langle D\rangle^{\gamma}\rangle^{-\alpha}\right\| \leq \|S\|_{\langle D\rangle^{\gamma},\alpha}\sup_{x\in[0,1]}\left\|e^{xS}\right\|_{\langle D\rangle^{\alpha\gamma},\beta}$$

and so $\|e^S\|_{\langle D\rangle^{\gamma},\alpha} \leq (1+\|S\|_{\langle D\rangle^{\gamma},\alpha})\sup_{x\in[0,1]}\left\|e^{xS}\right\|_{\langle D\rangle^{\alpha\gamma},\beta}$.

Proof. We have

$$\begin{aligned}
[D,e^S]\langle D\rangle^{-\alpha} &= \int_0^1 \frac{d}{dx} e^{(1-x)S} D e^{xS}\langle D\rangle^{-\alpha}dx \\
&= \int_0^1 e^{(1-x)S}[D,S]e^{xS}\langle D\rangle^{-\alpha}dx \\
&= \int_0^1 e^{(1-x)S}[D,S]\langle D\rangle^{-\alpha}\langle D\rangle^{\alpha}e^{xS}\langle D\rangle^{-\alpha}dx \\
&= \int_0^1 e^{(1-x)S}[D,S]\langle D\rangle^{-\alpha}\left([\langle D\rangle^{\alpha},e^{xS}]\langle D\rangle^{-\alpha}+e^{xS}\right)dx
\end{aligned}$$

This has norm bound

$$\begin{aligned}
\left\|[D,e^S]\langle D\rangle^{-\alpha}\right\| &\le \int_0^1 \left\|e^{(1-x)S}\right\| \left\|[D,S]\langle D\rangle^{-\alpha}\right\| \left(\left\|[\langle D\rangle^{\alpha},e^{xS}]\langle D\rangle^{-\alpha}\right\|+\left\|e^{xS}\right\|\right)dx \\
&\le \left\|[D,S]\langle D\rangle^{-\alpha}\right\| \left(1+\int_0^1 \left\|[\langle D\rangle^{\alpha},e^{xS}]\langle D\rangle^{-\alpha}\right\|dx\right) \\
&\le \left\|[D,S]\langle D\rangle^{-\alpha}\right\| \left(1+\sup_{x\in[0,1]} \left\|[\langle D\rangle^{\alpha},e^{xS}]\langle D\rangle^{-\alpha}\right\|\right) \\
&\le \left\|[D,S]\langle D\rangle^{-\alpha}\right\| \left(1+2^{\beta/2}\sup_{x\in[0,1]} \left\|[\langle D\rangle^{\alpha},e^{xS}]\langle\langle D\rangle^{\alpha}\rangle^{-\beta}\right\|\right).
\end{aligned}$$

So, because of the choice $K_\beta \ge 2^{\beta/2}$,

$$\begin{aligned}
\|e^S\|_{D,\alpha} &= \|e^S\| + K_\alpha \left\|[D,e^S]\langle D\rangle^{-\alpha}\right\| \\
&\le 1+K_\alpha \left\|[D,S]\langle D\rangle^{-\alpha}\right\| \left(1+K_\beta\sup_{x\in[0,1]} \left\|[\langle D\rangle^{\alpha},e^{xS}]\langle\langle D\rangle^{\alpha}\rangle^{-\beta}\right\|\right) \\
&= 1+K_\alpha \left\|[D,S]\langle D\rangle^{-\alpha}\right\| \sup_{x\in[0,1]} \left\|e^{xS}\right\|_{\langle D\rangle^{\alpha},\beta} \\
&\le 1+\|S\|_{D,\alpha} \sup_{x\in[0,1]} \left\|e^{xS}\right\|_{\langle D\rangle^{\alpha},\beta} \\
&\le (1+\|S\|_{D,\alpha}) \sup_{x\in[0,1]} \left\|e^{xS}\right\|_{\langle D\rangle^{\alpha},\beta}
\end{aligned}$$

as required. Similarly,

$$\begin{aligned}
\left\|[\langle D\rangle^{\gamma},e^S]\langle\langle D\rangle^{\gamma}\rangle^{-\alpha}\right\| &\le \left\|[\langle D\rangle^{\gamma},e^S]\langle D\rangle^{-\alpha\gamma}\right\| \\
&\le \left\|[\langle D\rangle^{\gamma},S]\langle D\rangle^{-\alpha\gamma}\right\| \left(1+\sup_{x\in[0,1]} \left\|[\langle D\rangle^{\alpha\gamma},e^{xS}]\langle D\rangle^{-\alpha\gamma}\right\|\right) \\
&\le \left\|[\langle D\rangle^{\gamma},S]\langle D\rangle^{-\alpha\gamma}\right\| \left(1+2^{\beta/2}\sup_{x\in[0,1]} \left\|[\langle D\rangle^{\alpha\gamma},e^{xS}]\langle\langle D\rangle^{\alpha\gamma}\rangle^{-\beta}\right\|\right)
\end{aligned}$$

and

$$\begin{aligned}
\|e^S\|_{\langle D\rangle^{\gamma},\alpha} &= \|e^S\| + K_\alpha \left\|[\langle D\rangle^{\gamma},e^S]\langle\langle D\rangle^{\gamma}\rangle^{-\alpha}\right\| \\
&\le 1+K_\alpha \left\|[\langle D\rangle^{\gamma},S]\langle D\rangle^{-\alpha\gamma}\right\| \left(1+K_\beta\sup_{x\in[0,1]} \left\|[\langle D\rangle^{\alpha\gamma},e^{xS}]\langle\langle D\rangle^{\alpha\gamma}\rangle^{-\beta}\right\|\right) \\
&\le (1+\|S\|_{\langle D\rangle^{\gamma},\alpha}) \sup_{x\in[0,1]} \left\|e^{xS}\right\|_{\langle D\rangle^{\alpha\gamma},\beta}
\end{aligned}$$

as required. $\square$

Lemma A.4.17. *Let D be a self-adjoint, regular operator on a right Hilbert B-module E. Let $S = -S^*$ be a bounded operator such that $S\operatorname{dom} D \subseteq \operatorname{dom} D$ and $[D,S]\langle D\rangle^{-\alpha}$ extends to a bounded operator for some $0 \leq \alpha < 1$. Then $[D,e^S]\langle D\rangle^{-\alpha}$ extends to a bounded operator.*

Proof. Let $N = \lfloor (1-\alpha)^{-1}\rfloor + 1$ and let $(\beta_n)_{n=1}^N \subset [0,1)$ be the sequence constructed in the Proof of Theorem A.4.13, with $\beta_1 = \alpha$, $\beta_N = 0$, and $2 - \beta_{n-1}^{-1} < \beta_n$. As in the Proof of Theorem A.4.13, we have a chain of inclusions

$$\operatorname{Lip}_{\beta_1=\alpha}(D) \subseteq \operatorname{Lip}_{\beta_2}(\langle D\rangle^{\beta_1}) \subseteq \operatorname{Lip}_{\beta_3}(\langle D\rangle^{\beta_1\beta_2}) \subseteq \cdots \subseteq \operatorname{Lip}_{\beta_N=0}(\langle D\rangle^{\prod_{k=1}^{N-1}\beta_k}) \subseteq \operatorname{End}^*_B(E).$$

We now compute that

$$\begin{aligned}
&\|e^S\|_{D,\alpha=\beta_1}\\
&\quad\leq (1+\|S\|_{D,\beta_1}) \sup_{x_1\in[0,1]} \left\|e^{x_1S}\right\|_{\langle D\rangle^{\beta_1},\beta_2}\\
&\quad\leq (1+\|S\|_{D,\beta_1}) \sup_{x_1,x_2\in[0,1]} (1+x_1\|S\|_{\langle D\rangle^{\beta_1},\beta_2}) \left\|e^{x_1x_2S}\right\|_{\langle D\rangle^{\beta_1\beta_2},\beta_3}\\
&\quad\leq (1+\|S\|_{D,\beta_1}) \sup_{x_1,x_2,x_3\in[0,1]} (1+x_1\|S\|_{\langle D\rangle^{\beta_1},\beta_2})(1+x_1x_2\|S\|_{\langle D\rangle^{\beta_1\beta_2},\beta_3}) \left\|e^{x_1x_2x_3S}\right\|_{\langle D\rangle^{\beta_1\beta_2\beta_3},\beta_4}\\
&\quad\leq (1+\|S\|_{D,\beta_1}) \sup_{x_1,x_2,\dots x_{N-1}\in[0,1]} (1+x_1\|S\|_{\langle D\rangle^{\beta_1},\beta_2})(1+x_1x_2\|S\|_{\langle D\rangle^{\beta_1\beta_2},\beta_3}) \times \cdots\\
&\qquad\quad \times (1+x_1x_2\cdots x_{N-2}\|S\|_{\langle D\rangle^{\beta_1\beta_2\cdots\beta_{N-2}},\beta_{N-1}}) \left\|e^{x_1x_2\cdots x_{N-1}S}\right\|_{\langle D\rangle^{\beta_1\beta_2\cdots\beta_{N-1}},\beta_N=0}\\
&\quad\leq (1+\|S\|_{D,\beta_1}) \sup_{x_1,x_2,\dots x_{N-1}\in[0,1]} (1+x_1\|S\|_{\langle D\rangle^{\beta_1},\beta_2})(1+x_1x_2\|S\|_{\langle D\rangle^{\beta_1\beta_2},\beta_3}) \times \cdots\\
&\qquad\quad \times (1+x_1x_2\cdots x_{N-2}\|S\|_{\langle D\rangle^{\beta_1\beta_2\cdots\beta_{N-2}},\beta_{N-1}})(1+x_1x_2\cdots x_{N-1}\|S\|_{\langle D\rangle^{\beta_1\beta_2\cdots\beta_{N-1}},\beta_N=0})\\
&\quad= (1+\|S\|_{D,\beta_1})(1+\|S\|_{\langle D\rangle^{\beta_1},\beta_2})(1+\|S\|_{\langle D\rangle^{\beta_1\beta_2},\beta_3}) \times \cdots\\
&\qquad\quad \times (1+\|S\|_{\langle D\rangle^{\beta_1\beta_2\cdots\beta_{N-2}},\beta_{N-1}})(1+\|S\|_{\langle D\rangle^{\beta_1\beta_2\cdots\beta_{N-1}},\beta_N=0})\\
&\quad= (1+\|S\|_{D,\beta_1})\prod_{n=2}^{N}(1+\|S\|_{\langle D\rangle^{\beta_1\cdots\beta_{n-1}},\beta_n})\\
&\quad< \infty
\end{aligned}$$

as required. $\square$

Theorem A.4.18. *cf. [BEJ84, Lemma 3.2] [BC91, Proposition 6.4] Let D be a self-adjoint, regular operator on a right Hilbert B-module E. Fix $0 \leq \alpha < 1$ and let f be a $\lfloor (1-\alpha)^{-1}\rfloor + 2$-times differentiable function on $\mathbb{R}^d$ for some $d \geq 1$. (For $\alpha = 0$, we may take f to be only twice differentiable.) For any pairwise commuting self-adjoint $a_1,\dots,a_d \in \operatorname{Lip}_\alpha(D)$, we have $f(a_1,\dots,a_d) \in \operatorname{Lip}_\alpha(D)$.*

Proof. Modifying f away from the joint spectrum of $a_1,\dots,a_d$, without loss of generality, we assume that f is compactly supported. With $\hat{f}$ the Fourier transform of f, we may write

$$f(a_1,\dots,a_d) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{it_1a_1+\cdots+it_da_d}\hat{f}(t_1,\dots,t_d)dt^n$$

and

$$[D,f(a_1,\dots,a_d)]\langle D\rangle^{-\alpha} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}[D,e^{it_1a_1+\cdots+it_da_d}]\langle D\rangle^{-\alpha}\hat{f}(t_1,\dots,t_d)dt^n$$

Let $N = \lfloor (1-\alpha)^{-1}\rfloor + 1$ and let $(\beta_n)_{n=1}^N \subset [0,1)$ be the sequence constructed in the Proof of Theorem A.4.13. Because f is $\lfloor (1-\alpha)^{-1}\rfloor + 2$-times differentiable, $|t_1|^{k_1}\cdots|t_d|^{k_d}|\hat{f}(t_1,\dots,t_d)|$ is integrable for

$k_1 + \cdots + k_d < \lfloor (1-\alpha)^{-1} \rfloor + 2$ and so

$$\begin{aligned}
\|f(a_1, \dots, a_d)\|_{D,\alpha} &= \|f(a_1, \dots, a_d)\| + K_\alpha \|[D, f(a_1, \dots, a_d)]\langle D\rangle^{-\alpha}\| \\
&\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \|e^{it_1 a_1 + \cdots + it_d a_d}\| |\hat{f}(t_1, \dots, t_d)| dt \\
&\quad + K_\alpha \frac{1}{2\pi} \int_{-\infty}^{\infty} \|[D, e^{it_1 a_1 + \cdots + it_d a_d}]\langle D\rangle^{-\alpha}\| |\hat{f}(t_1, \dots, t_d)| dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \|e^{it_1 a_1 + \cdots + it_d a_d}\|_{D,\alpha} |\hat{f}(t_1, \dots, t_d)| dt \\
&\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 + \sum_{j=1}^{d} |t_j| \|a_j\|_{D,\beta_1}) \prod_{n=2}^{N} (1 + \sum_{j=1}^{d} |t_j| \|a_j\|_{\langle D\rangle^{\beta_1 \cdots \beta_{n-1}}, \beta_n}) |\hat{f}(t_1, \dots, t_d)| dt \\
&< \infty,
\end{aligned}$$

as required. $\square$

Bibliography

[Aba03] Fernando Abadie, *Enveloping actions and Takai duality for partial actions*, Journal of Functional Analysis **197** (2003), no. 1, 14–67.

[Aba18] ———, *Enveloping actions and Takai duality for partial actions*, `arXiv:math/0007109 [math.OA]`, 2018.

[AFRU21] Artur Avila, Giovanni Forni, Davide Ravotti, and Corinna Ulcigrai, *Mixing for smooth time-changes of general nilflows*, Advances in Mathematics **385** (2021), 107759.

[AGIR22] P. Antonini, D. Guido, T. Isola, and A. Rubin, *A note on twisted crossed products and spectral triples*, Journal of Geometry and Physics **180** (2022), 104640.

[AH59] M.F. Atiyah and F. Hirzebruch, *Riemann-Roch theorems for differentiable manifolds*, Bulletin of the American Mathematical Society **65** (1959), no. 4, 276–281.

[AH61] ———, *Vector bundles and homogeneous spaces*, Differential Geometry, Proceedings of Symposia in Pure Mathematics, no. 3, American Mathematical Society, 1961, pp. 7–38.

[AJV19] Maria Paula Gomez Aparicio, Pierre Julg, and Alain Valette, *The Baum–Connes conjecture: An extended survey*, Advances in Noncommutative Geometry, Springer International Publishing, 2019, pp. 127–244.

[AK23] Yavar Abdolmaleki and Dan Kucerovsky, *A short proof of an index theorem, II*, Journal of Noncommutative Geometry (2023), 123–142.

[AMY22] Iakovos Androulidakis, Omar Mohsen, and Robert Yuncken, *A pseudodifferential calculus for maximally hypoelliptic operators and the Helffer-Nourrigat conjecture*, `arXiv:2201.12060 [math.AP]`, 2022.

[APT73] Charles A. Akemann, Gert K. Pedersen, and Jun Tomiyama, *Multipliers of C^*-algebras*, Journal of Functional Analysis **13** (1973), no. 3, 277–301.

[Ara01] Pere Ara, *Morita Equivalence and Pedersen Ideals*, Proceedings of the American Mathematical Society **129** (2001), no. 4, 1041–1049.

[Arc78] R.J. Archbold, *On the norm of an inner derivation of a C^*-algebra*, Mathematical Proceedings of the Cambridge Philosophical Society **84** (1978), no. 2, 273–291.

[Asa70] Tetsuya Asai, *The reciprocity of Dedekind sums and the factor set for the universal covering group of $SL(2,\mathbb{R})$*, Nagoya Mathematical Journal **37** (1970), 67–80.

[Ati69] M.F. Atiyah, *Global theory of elliptic operators*, Proceedings of the International Symposium on Functional Analysis, Tokyo, University of Tokyo Press, 1969, pp. 21–30.

[Bär07] Christian Bär, *Conformal structures in noncommutative geometry*, Journal of Noncommutative Geometry (2007), 385–395.

[BC91] B. Blackadar and J. Cuntz, *Differential Banach algebra norms and smooth subalgebras of C^*-algebras*, Journal of Operator Theory **26** (1991), no. 2, 255–282.

[BCL06] Paolo Bertozzini, Roberto Conti, and Wicharn Lewkeeratiyutkul, *A category of spectral triples and discrete groups with length function*, Osaka Journal of Mathematics **43** (2006), no. 2, 327 – 350.

[BCR15] Chris Bourne, Alan L. Carey, and Adam Rennie, *The bulk-edge correspondence for the quantum Hall effect in Kasparov theory*, Letters in Mathematical Physics **105** (2015), no. 9, 1253–1273.

[BDF73] L.G. Brown, R.G. Douglas, and P.A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of C^*-algebras*, Proceedings of a Conference on Operator Theory: Dalhousie University, Halifax, Nova Scotia, April 13th and 14th, 1973 (P.A. Fillmore, ed.), Lecture Notes in Mathematics, no. 345, Springer-Verlag, 1973, pp. 58–128.

[BEJ84] O. Bratteli, G.A. Elliott, and P.E.T. Jørgensen, *Decomposition of unbounded derivations into invariant and approximately inner parts*, Journal für die reine und angewandte Mathematik **1984** (1984), no. 346, 166–193.

[BEW15] Alcides Buss, Siegfried Echterhoff, and Rufus Willett, *Exotic crossed products and the Baum–Connes conjecture*, Journal für die reine und angewandte Mathematik (Crelles Journal) **2018** (2015), no. 740, 111–159.

[BFGM07] Uri Bader, Alex Furman, Tsachik Gelander, and Nicolas Monod, *Property (T) and rigidity for actions on Banach spaces*, Acta Mathematica **198** (2007), no. 1, 57–105.

[BGH19] J. Brodzki, E. Guentner, and N. Higson, *A differential complex for CAT(0) cubical spaces*, Advances in Mathematics **347** (2019), 1054–1111.

[BGHN20] Jacek Brodzki, Erik Guentner, Nigel Higson, and Shintaro Nishikawa, *On the Baum–Connes conjecture for groups acting on CAT(0)-cubical spaces*, International Mathematics Research Notices **2021** (2020), no. 5, 3698–3728.

[BH99] Martin R. Bridson and André Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer Berlin, Heidelberg, 1999.

[BJ83] Saad Baaj and Pierre Julg, *Théorie bivariante de Kasparov et opérateurs non bornés dans les C^*-modules hilbertiens*, CR Acad. Sci. Paris Sér. I Math **296** (1983), no. 21, 875–878.

[BL92] J. Brüning and M. Lesch, *Hilbert complexes*, Journal of Functional Analysis **108** (1992), no. 1, 88–132.

[Bla98] Bruce Blackadar, *K-Theory for Operator Algebras*, Mathematical Sciences Research Institute Publications, no. 5, Cambridge University Press, Cambridge, England, September 1998.

[Bla06] Bruce Blackadar, *Operator Algebras*, Springer Berlin Heidelberg, 2006.

[BMR10] Jean V. Bellissard, Matilde Marcolli, and Kamran Reihani, *Dynamical systems on spectral metric spaces*, `arXiv:1008.4617 [math.OA]`, August 2010.

[Bou04] Nicolas Bourbaki, *Integration II: Chapters 7–9*, Elements of Mathematics, Springer-Verlag Berlin Heidelberg, 2004.

[BR76] Ola Bratteli and Derek W. Robinson, *Unbounded derivations of C^*-algebras II*, Communications in Mathematical Physics **46** (1976), no. 1, 11–30.

[BR87] ______, *Operator Algebras and Quantum Statistical Mechanics 1: C*- and W*-Algebras, Symmetry Groups, Decomposition of States*, 2 ed., Springer Berlin Heidelberg, 1987.

[Bro89] Kenneth S. Brown, *Buildings*, 1 ed., Springer, New York, NY, 1989.

[BS89] Saad Baaj and Georges Skandalis, *C*-algèbres de Hopf et théorie de Kasparov équivariante*, K-Theory **2** (1989), no. 6, 683–721.

[BS93] ______, *Unitaires multiplicatifs et dualité pour les produits croisés de C*-algèbres*, Annales scientifiques de l'École normale supérieure **26** (1993), no. 4, 425–488.

[BS10] Jyotishman Bhowmick and Adam Skalski, *Quantum isometry groups of noncommutative manifolds associated to group C*-algebras*, Journal of Geometry and Physics **60** (2010), no. 10, 1474–1489.

[BS22] Bram Bekker and Maarten Solleveld, *The buildings gallery: Visualizing buildings*, Journal of Mathematics and the Arts **16** (2022), no. 1–2, 11–28.

[BSV03] Saad Baaj, Georges Skandalis, and Stefaan Vaes, *Non-semi-regular quantum groups coming from number theory*, Communications in Mathematical Physics **235** (2003), no. 1, 139–167.

[Buc58] R. Creighton Buck, *Bounded continuous functions on a locally compact space*, Michigan Mathematical Journal **5** (1958), no. 2, 95–104.

[BV05] Saad Baaj and Stefaan Vaes, *Double crossed products of locally compact quantum groups*, Journal of the Institute of Mathematics of Jussieu **4** (2005), no. 1, 135–173.

[Cas23] Christopher H. Cashen, *Notes on Coxeter Groups*, `https://mat.univie.ac.at/~cashen/Notes_on_Coxeter_groups.pdf`, 2023.

[CC92] B.P. Cohen and A. Connes, *Conformal geometry of the irrational rotation algebra*, Preprint. MPI / 92-23, 1992.

[CC06] Ali H. Chamseddine and Alain Connes, *Scale invariance in the spectral action*, Journal of Mathematical Physics **47** (2006), no. 6, 063504–1–19.

[CCJJV01] Pierre-Alain Cherix, Michael Cowling, Paul Jolissaint, Pierre Julg, and Alain Valette, *Groups with the Haagerup Property: Gromov's a-T-menability*, Progress in Mathematics, no. 197, Springer Basel AG, 2001.

[CG90] Lawrence J. Corwin and Frederick P. Greenleaf, *Representations of nilpotent Lie groups and their applications, Part I: Basic theory and examples*, Cambridge Studies in Advanced Mathematics, no. 18, Cambridge University Press, 1990.

[CH16] Yves Cornulier and Pierre de la Harpe, *Metric Geometry of Locally Compact Groups*, EMS Tracts in Mathematics, no. 25, EMS Press, September 2016.

[Cha07] Ruth Charney, *An introduction to right-angled Artin groups*, Geometriae Dedicata **125** (2007), no. 1, 141–158.

[Che96] ZhiQiang Chen, *Séries complémentaires des groupes de Lorentz et KK-théorie*, Journal of Functional Analysis **137** (1996), no. 1, 76–96.

[CM95] Alain Connes and Henri Moscovici, *The local index formula in noncommutative geometry*, Geometric and Functional Analysis **5** (1995), no. 2, 174–243.

[CM08] ______, *Type III and spectral triples*, Traces in number theory, geometry and quantum fields (Sergio Albeverio, Matilde Marcolli, Sylvie Paycha, Jorge Plazas, and Max-Planck-Institut für Mathematik, eds.), Vieweg Aspects of Mathematics, no. 38, Vieweg, Wiesbaden, 2008, pp. 57–72.

[CM14] ______, *Modular curvature for noncommutative two-tori*, Journal of the American Mathematical Society **27** (2014), no. 3, 639–684.

[CMRV08] G. Cornelissen, M. Marcolli, K. Reihani, and A. Vdovina, *Noncommutative geometry on trees and buildings*, Traces in number theory, geometry and quantum fields (Sergio Albeverio, Matilde Marcolli, Sylvie Paycha, Jorge Plazas, and Max-Planck-Institut für Mathematik, eds.), Vieweg Aspects of Mathematics, no. 38, Vieweg, Wiesbaden, 2008, pp. 73–98.

[CNNR11] Alan L. Carey, Sergey Neshveyev, Ryszard Nest, and Adam Rennie, *Twisted cyclic theory, equivariant KK-theory and KMS states*, Journal für die reine und angewandte Mathematik (Crelles Journal) **2011** (2011), no. 650, 161–191.

[Con89] Alain Connes, *Compact metric spaces, Fredholm modules, and hyperfiniteness*, Ergodic Theory and Dynamical Systems **9** (1989), no. 2, 207–220.

[Con94] ______, *Noncommutative Geometry*, Academic Press, 1994.

[Con04] ______, *Cyclic cohomology, quantum group symmetries and the local index formula for* $SU_q(2)$, Journal of the Institute of Mathematics of Jussieu **3** (2004), no. 1, 17–68.

[CP98] Alan Carey and John Phillips, *Unbounded Fredholm modules and spectral flow*, Canadian Journal of Mathematics **50** (1998), no. 4, 673–718.

[CR17] Michael Christ and Marc A. Rieffel, *Nilpotent group C*-algebras as compact quantum metric spaces*, Canadian Mathematical Bulletin **60** (2017), no. 1, 77–94.

[CS84] Alain Connes and Georges Skandalis, *The longitudinal index theorem for foliations*, Publications of the Research Institute for Mathematical Sciences **20** (1984), no. 6, 1139–1183.

[CS86] J. Cuntz and G. Skandalis, *Mapping cones and exact sequences in KK-theory*, Journal of Operator Theory **15** (1986), no. 1, 163–180.

[ČS09] Andreas Čap and Jan Slovák, *Parabolic Geometries I: Background and General Theory*, Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, September 2009 (en).

[ČSS01] Andreas Čap, Jan Slovák, and Vladimír Souček, *Bernstein–Gelfand–Gelfand sequences*, The Annals of Mathematics **154** (2001), no. 1, 97.

[CT11] Alain Connes and Paula Tretkoff, *The Gauss–Bonnet theorem for the noncommutative two torus*, Noncommutative Geometry, Arithmetic, and Related Topics: Proceedings of the Twenty-First Meeting of the Japan-U.S. Mathematics Institute (Caterina Consani and Alain Connes, eds.), Johns Hopkins University Press, 2011, pp. 141–158.

[Cun83] Joachim Cuntz, *K-theoretic amenability for discrete groups*, Journal für die reine und angewandte Mathematik (Crelles Journal) **1983** (1983), no. 344, 180–195.

[DDLW07] Francesco D'Andrea, Ludwik Dąbrowski, Giovanni Landi, and Elmar Wagner, *Dirac operators on all Podleś quantum spheres*, Journal of Noncommutative Geometry **1** (2007), no. 2, 213–239.

[DE14] Anton Deitmar and Siegfried Echterhoff, *Principles of Harmonic Analysis*, Universitext, Springer International Publishing, 2014.

[DFLM18] A. Devastato, S. Farnsworth, F. Lizzi, and P. Martinetti, *Lorentz signature and twisted spectral triples*, Journal of High Energy Physics **2018** (2018), no. 3, 89.

[DGM18] Robin J. Deeley, Magnus Goffeng, and Bram Mesland, *The bordism group of unbounded KK-cycles*, Journal of Topology and Analysis **10** (2018), no. 02, 355–400.

[DH17] Shantanu Dave and Stefan Haller, *Graded hypoellipticity of BGG sequences*, `arXiv:1705.01659v2 [math.DG]`, 2017.

[DH19] ———, *The heat asymptotics on filtered manifolds*, The Journal of Geometric Analysis **30** (2019), no. 1, 337–389.

[DH22] ———, *Graded hypoellipticity of BGG sequences*, Annals of Global Analysis and Geometry **62** (2022), no. 4, 721–789.

[Dix77] Jacques Dixmier, *C*-algebras*, North-Holland Mathematical Library, vol. 15, North-Holland Publishing Company, 1977.

[DM20] Koen van den Dungen and Bram Mesland, *Homotopy equivalence in unbounded KK-theory*, Annals of K-Theory **5** (2020), no. 3, 501–537.

[DS03] Ludwik Dąbrowski and Andrzej Sitarz, *Dirac operator on the standard Podleś quantum sphere*, Noncommutative Geometry and Quantum Groups, Institute of Mathematics, Polish Academy of Sciences, 2003, pp. 49–58.

[DS22] ———, *Spectral triples with multitwisted real structure*, Journal of Noncommutative Geometry **16** (2022), no. 2, 625–635.

[Dun18] Koen van den Dungen, *Locally bounded perturbations and (odd) unbounded KK-theory*, Journal of Noncommutative Geometry **12** (2018), no. 4, 1445–1467.

[Dun20] ———, *The Kasparov product on submersions of open manifolds*, Journal of Topology and Analysis **14** (2020), no. 01, 147–181.

[Dun22] ———, *Localisations of half-closed modules and the unbounded Kasparov product*, International Mathematics Research Notices **2023** (2022), no. 9, 7578–7615.

[EE11] Siegfried Echterhoff and Heath Emerson, *Structure and K-theory of crossed products by proper actions*, Expositiones Mathematicae **29** (2011), no. 3, 300–344.

[EKQR06] Siegfried Echterhoff, S. Kaliszewski, John Quigg, and Iain Raeburn, *A Categorical Approach to Imprimitivity Theorems for C*-Dynamical Systems*, vol. 180, Memoirs of the American Mathematical Society, no. 850, American Mathematical Society, March 2006.

[EL97] Ruy Exel and Marcelo Laca, *Continuous Fell bundles associated to measurable twisted actions*, Proceedings of the American Mathematical Society **125** (1997), no. 3, 795–799.

[EN02] Ruy Exel and Chi-Keung Ng, *Approximation property of C*-algebraic bundles*, Mathematical Proceedings of the Cambridge Philosophical Society **132** (2002), no. 3, 509–522.

[Eng89] Ryszard Engelking, *General Topology*, 2 ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, Germany, December 1989.

[ENN93] George A. Elliott, Toshikazu Natsume, and Ryszard Nest, *The Heisenberg group and K-theory*, K-Theory **7** (1993), no. 5, 409–428.

[Exe97] Ruy Exel, *Twisted partial actions: A classification of regular C*-algebraic bundles*, Proceedings of the London Mathematical Society **74** (1997), no. 2, 417–443.

[Exe17] ———, *Partial Dynamical Systems, Fell Bundles and Applications*, American Mathematical Society, September 2017.

[EY17a] Erik van Erp and Robert Yuncken, *A groupoid approach to pseudodifferential calculi*, Journal für die reine und angewandte Mathematik (Crelles Journal) **2019** (2017), no. 756, 151–182.

[EY17b] ———, *On the tangent groupoid of a filtered manifold*, Bulletin of the London Mathematical Society **49** (2017), no. 6, 1000–1012.

[FD88a] J.M.G. Fell and R.S. Doran, *Representations of *-Algebras, Locally Compact Groups, and Banach *-Algebraic Bundles, Volume 1: Basic Representation Theory of Groups and Algebras*, Pure and Applied Mathematics, no. 125, Academic Press, Inc., 1988.

[FD88b] ———, *Representations of *-Algebras, Locally Compact Groups, and Banach *-Algebraic Bundles, Volume 2: Banach *-Algebraic Bundles, Induced Representations, and the Generalized Mackey Analysis*, Pure and Applied Mathematics, no. 126, Academic Press, Inc., 1988.

[Fef79] Charles Fefferman, *Parabolic invariant theory in complex analysis*, Advances in Mathematics **31** (1979), no. 2, 131–262.

[Fel61] J.M.G. Fell, *The structure of algebras of operator fields*, Acta Mathematica **106** (1961), no. 3–4, 233–280.

[Fer96] Jacqueline Ferrand, *The action of conformal transformations on a Riemannian manifold*, Mathematische Annalen **304** (1996), no. 1, 277–291.

[FGM25] Magnus Fries, Magnus Goffeng, and Ada Masters, *Parabolic noncommutative geometry*, `arXiv:2503.12938 [math.OA]`, 2025.

[FGMR19] Iain Forsyth, Magnus Goffeng, Bram Mesland, and Adam Rennie, *Boundaries, spectral triples and K-homology*, Journal of Noncommutative Geometry **13** (2019), no. 2, 407–472.

[FMR14] Iain Forsyth, Bram Mesland, and Adam Rennie, *Dense domains, symmetric operators and spectral triples*, New York Journal of Mathematics **20** (2014), 1001–1020.

[Frą04] Krzysztof Frączek, *On diffeomorphisms with polynomial growth of the derivative on surfaces*, Colloquium Mathematicum **99** (2004), no. 1, 75–90.

[Fra05] Maria Fragoulopoulou, *Topological Algebras with Involution*, North-Holland Mathematics Studies, vol. 200, Elsevier B.V., 2005.

[Fri25] Magnus Fries, *Relative K-homology of higher-order differential operators*, Journal of Functional Analysis **288** (2025), no. 1, 110678.

[Gab21] Jamie Gabe, *Strict topology on the multiplier algebra*, MathOverflow, August 2021, `https://mathoverflow.net/q/400856`.

[GB16] Debashish Goswami and Jyotishman Bhowmick, *Quantum Isometry Groups*, Springer India, 2016.

[GBVF01] José M. Gracia-Bondía, Joseph C. Várilly, and Héctor Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser Boston, 2001.

[GG19] Heiko Gimperlein and Magnus Goffeng, *Commutator estimates on contact manifolds and applications*, Journal of Noncommutative Geometry **13** (2019), no. 1, 363–406.

[GH25] Magnus Goffeng and Bernard Helffer, *The index of sub-Laplacians: Beyond contact manifolds*, `arXiv:2408.00091 [math.AP]`, 2025.

[GK24] Magnus Goffeng and Alexey Kuzmin, *Index theory of hypoelliptic operators on Carnot manifolds*, `arXiv:2203.04717 [math.DG]`, 2024.

[GM15] Magnus Goffeng and Bram Mesland, *Spectral triples and finite summability on Cuntz-Krieger algebras*, Documenta Mathematica **20** (2015), 89–170.

[GMR19] Magnus Goffeng, Bram Mesland, and Adam Rennie, *Untwisting twisted spectral triples*, International Journal of Mathematics **30** (2019), no. 14, 1950076.

[GMT14] M. Greenfield, M. Marcolli, and K. Teh, *Twisted spectral triples and quantum statistical mechanical systems*, p-Adic Numbers, Ultrametric Analysis, and Applications **6** (2014), no. 2, 81–104.

[Gof09] Magnus Goffeng, *Equivariant KK-theory and twists*, Licentiate thesis, University of Gothenburg, 2009.

[Gof24] ———, *Solving the index problem for (curved) Bernstein-Gelfand-Gelfand sequences*, `arXiv:2406.07033 [math.KT]`, 2024.

[Gre12] Martin Grensing, *Universal cycles and homological invariants of locally convex algebras*, Journal of Functional Analysis **263** (2012), no. 8, 2170–2204.

[GRU19] Magnus Goffeng, Adam Rennie, and Alexandr Usachev, *Constructing KMS states from infinite-dimensional spectral triples*, Journal of Geometry and Physics **143** (2019), 107–149.

[Hal22] Stefan Haller, *Analytic torsion of generic rank two distributions in dimension five*, The Journal of Geometric Analysis **32** (2022), no. 10, 248.

[Har23] Sean Harris, *Self-similar states and projections in noncommutative metric spaces*, `arXiv:2304.13340 [math.OA]`, 2023.

[Has14] Stefan Hasselmann, *Spectral triples on Carnot manifolds*, Ph.D. thesis, Fakult¨at fu¨r Mathematik und Physik der Gottfried Wilhelm Leibniz Universita¨t Hannover, 2014.

[Haw13] Andrew Hawkins, *Constructions of spectral triples on C*-algebras*, Ph.D. thesis, University of Nottingham, December 2013.

[Hel01] Sigurdur Helgason, *Differential geometry, Lie groups and symmetric spaces*, Graduate studies in mathematics, American Mathematical Society, Providence, RI, 2001.

[Hij86] Oussama Hijazi, *A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors*, Communications in Mathematical Physics **104** (1986), no. 1, 151–162.

[Hil10] Michel Hilsum, *Bordism invariance in KK-theory*, Mathematica Scandinavica **107** (2010), no. 1, 73.

[Hit74] Nigel Hitchin, *Harmonic spinors*, Advances in Mathematics **14** (1974), no. 1, 1–55.

[HK02] Boris Hasselblatt and Anatole Katok, *Survey I: Principal structures*, Handbook of Dynamical Systems (B. Hasselblatt and A. Katok, eds.), vol. 1A, Elsevier, 2002, pp. 1–203.

[HK07] István Heckenberger and Stefan Kolb, *Differential forms via the Bernstein–Gelfand–Gelfand resolution for quantized irreducible flag manifolds*, Journal of Geometry and Physics **57** (2007), no. 11, 2316–2344.

[HR00] Nigel Higson and John Roe, *Analytic K-Homology*, Oxford Mathematical Monographs, Oxford University Press, London, England, 2000.

[HSWZ13] A. Hawkins, A. Skalski, S. White, and J. Zacharias, *On spectral triples on crossed products arising from equicontinuous actions*, Mathematica Scandinavica **113** (2013), no. 2, 262.

[IM16] Bruno Iochum and Thierry Masson, *Crossed product extensions of spectral triples*, Journal of Noncommutative Geometry **10** (2016), no. 1, 61–129.

[JK95] Pierre Julg and Gennadi Kasparov, *Operator K-theory for the group* $SU(n,1)$, Journal für die reine und angewandte Mathematik **463** (1995), 99–152.

[JL15] Ja A. Jeong and Jae Hyup Lee, *Finite groups acting on higher dimensional noncommutative tori*, Journal of Functional Analysis **268** (2015), no. 2, 473–499.

[Jos21] Michael Joswig, *Essentials of Tropical Combinatorics*, Graduate Studies in Mathematics, vol. 219, American Mathematical Society, Providence, RI, December 2021.

[Jul89] Pierre Julg, *C*-algèbres associées à des complexes simpliciaux*, C. R. Acad. Sci. Paris, Série I **308** (1989), 97–100.

[Jul95] ______, *Complexe de Rumin, suite spectrale de Forman et cohomologie* L^2 *des espaces symétriques de rang 1*, C. R. Acad. Sci. Paris, Série I **320** (1995), 451–456.

[Jul19] ______, *How to prove the Baum–Connes conjecture for the groups* $Sp(n,1)$*?*, Journal of Geometry and Physics **141** (2019), 105–119.

[JV84] Pierre Julg and Alain Valette, *K-theoretic amenability for* $SL_2(\mathbb{Q}_p)$*, and the action on the associated tree*, Journal of Functional Analysis **58** (1984), no. 2, 194–215.

[JV87] ______, *Fredholm modules associated to Bruhat-Tits buildings*, Miniconferences on Harmonic Analysis and Operator Algebras: Canberra, 5–8 August and 2–3 December 1987 (Michael Cowling, Christopher Meaney, and William Moran, eds.), Proceedings of the Centre for Mathematics and its Applications, no. 16, Australian National University, 1987, pp. 143–155.

[Kaa17] Jens Kaad, *Differentiable absorption of Hilbert C*-modules, connections, and lifts of unbounded operators*, Journal of Noncommutative Geometry **11** (2017), no. 3, 1037–1068.

[Kaa20] ______, *On the unbounded picture of KK-theory*, Symmetry, Integrability and Geometry: Methods and Applications (2020), 082.

[Kaa21] ______, *The unbounded Kasparov product by a differentiable module*, Journal of Noncommutative Geometry **15** (2021), no. 2, 423–487.

[Kaa24] ______, *Morita invariance of unbounded bivariant K-theory*, Annals of Functional Analysis **15** (2024), no. 4, 88.

[Kas75] G.G. Kasparov, *Topological invariants of elliptic operators. I: K-homology*, Mathematics of the USSR-Izvestiya **9** (1975), no. 4, 751–792.

[Kas81] ———, *The operator K-functor and extensions of C*-algebras*, Mathematics of the USSR-Izvestiya **16** (1981), no. 3, 513–572.

[Kas84] ———, *Lorentz groups: K-theory of unitary representations and crossed products*, Dokl. Akad. Nauk SSSR **275** (1984), no. 3, 541–545.

[Kas88] ———, *Equivariant KK-theory and the Novikov conjecture*, Inventiones Mathematicae **91** (1988), no. 1, 147–201.

[Kas95] ———, *K-theory, group C*-algebras, and higher signatures (Conspectus)*, Novikov Conjectures, Index Theorems, and Rigidity: Oberwolfach 1993 (Steven C. Ferry, Andrew Ranicki, and Jonathan M.Editors Rosenberg, eds.), London Mathematical Society Lecture Note Series, vol. 1, Cambridge University Press, 1995, pp. 101–146.

[KK20] Jens Kaad and David Kyed, *Dynamics of compact quantum metric spaces*, Ergodic Theory and Dynamical Systems **41** (2020), no. 7, 2069–2109.

[KK25] ———, *The Quantum Metric Structure of Quantum $SU(2)$*, Memoirs of the European Mathematical Society, no. 18, EMS Press, February 2025.

[KL12] Jens Kaad and Matthias Lesch, *A local global principle for regular operators in Hilbert C*-modules*, Journal of Functional Analysis **262** (2012), no. 10, 4540–4569.

[KL13] ———, *Spectral flow and the unbounded Kasparov product*, Advances in Mathematics **248** (2013), 495–530.

[KL17] ———, *Corrigendum to "A local global principle for regular operators in Hilbert C*-modules" [J. Funct. Anal. 262 (10) (2012) 4540–4569]*, Journal of Functional Analysis **272** (2017), no. 10, 4403–4406.

[KMQW10] S. Kaliszewski, Paul S. Muhly, John Quigg, and Dana P. Williams, *Coactions and Fell bundles*, New York Journal of Mathematics **16** (2010), 315–359.

[Kor08] Yuri A. Kordyukov, *Noncommutative geometry of foliations*, Journal of K-Theory **2** (2008), no. 2, 219–327.

[KRS12] Ulrich Krähmer, Adam Rennie, and Roger Senior, *A residue formula for the fundamental Hochschild 3-cocycle for $SU_q(2)$.*, Journal of Lie Theory **22** (2012), 557–585.

[KS91] G.G. Kasparov and G. Skandalis, *Groups acting on buildings, operator K-theory, and Novikov's conjecture*, K-Theory **4** (1991), 303–337.

[KS97] Anatoli Klimyk and Konrad Schmüdgen, *Quantum Groups and Their Representations*, Texts and Monographs in Physics, Springer Berlin Heidelberg, 1997.

[KS12] Jens Kaad and Roger Senior, *A twisted spectral triple for quantum $SU(2)$*, Journal of Geometry and Physics **62** (2012), no. 4, 731–739.

[Kuc94] Dan Kucerovsky, *Kasparov products in KK-theory, and unbounded operators with applications to index theory*, Ph.D. thesis, Magdalen College, University of Oxford, 1994.

[Kuc97] ———, *The KK-product of unbounded modules*, K-Theory **11** (1997), no. 1, 17–34.

[Kuc03] ———, *The assembly map in the unbounded picture*, Proper Group Actions and the Baum-Connes Conjecture, Birkhäuser Basel, 2003, pp. 119–124.

[Kus01] Johan Kustermans, *Locally compact quantum groups in the universal setting*, International Journal of Mathematics **12** (2001), no. 3, 289–338.

[KZPS76] M.A. Krasnoselskii, P.P. Zabreiko, E.I. Pustylnik, and P.E. Sobolevskii, *Integral operators in spaces of summable functions*, Noordhoff International Publishing, Leyden, Netherlands, 1976.

[Lan95] E. Christopher Lance, *Hilbert C*-Modules: A toolkit for operator algebraists*, London Mathematical Society Lecture Note Series, no. 210, Cambridge University Press, Cambridge, England, March 1995.

[LD17] Enrico Le Donne, *A primer on Carnot groups: Homogenous groups, Carnot-Carathéodory spaces, and regularity of their isometries*, Analysis and Geometry in Metric Spaces **5** (2017), no. 1, 116–137.

[Les05] Matthias Lesch, *The uniqueness of the spectral flow on spaces of unbounded self-adjoint Fredholm operators*, Spectral Geometry of Manifolds with Boundary and Decomposition of Manifolds (Bernhelm Booß-Bavnbek, Gerd Grubb, and Krzysztof P. Wojciechowski, eds.), Contemporary Mathematics, no. 366, American Mathematical Society, 2005, pp. 193–224.

[LM89] H. Blaine Lawson and Marie-Louise Michelsohn, *Spin Geometry*, Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989.

[LM19] Matthias Lesch and Bram Mesland, *Sums of regular self-adjoint operators in Hilbert-C*-modules*, Journal of Mathematical Analysis and Applications **472** (2019), no. 1, 947–980.

[LMN05] Robert Lauter, Bertrand Monthubert, and Victor Nistor, *Spectral invariance for certain algebras of pseudodifferential operators*, Journal of the Institute of Mathematics of Jussieu **4** (2005), no. 3, 405–442.

[LPRS87] M.B. Landstad, J. Phillips, I. Raeburn, and C.E. Sutherland, *Representations of crossed products by coactions and principal bundles*, Transactions of the American Mathematical Society **299** (1987), no. 2, 747–784.

[LRV12] Steven Lord, Adam Rennie, and Joseph C. Várilly, *Riemannian manifolds in noncommutative geometry*, Journal of Geometry and Physics **62** (2012), no. 7, 1611–1638.

[LT76] A.J. Lazar and D.C. Taylor, *Multipliers of Pedersen's Ideal*, vol. 5, Memoirs of the American Mathematical Society, no. 169, American Mathematical Society, Providence, RI, March 1976.

[Luk77] Glenys Luke, *Fredholm representations of uniform subgroups*, Arkiv för Matematik **15** (1977), no. 1-2, 101–116.

[Mes12] Bram Mesland, *Unbounded bivariant K-theory and correspondences in noncommutative geometry*, Journal für die reine und angewandte Mathematik (Crelles Journal) **2014** (2012), no. 691, 101–172.

[Mes24] ———, *Lecture notes on unbounded KK-theory*, Hausdorff School "Noncommutative Geometry and Operator Algebras", May 2–5, 2023, Bonn, 2024.

[Mil76] John Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Mathematics **21** (1976), no. 3, 293–329.

[Miš74] A.S. Miščenko, *Infinite-dimensional representations of discrete groups, and higher signatures*, Mathematics of the USSR-Izvestiya **8** (1974), no. 1, 85–111.

[MMW16] Sarah M. Moffat, Walaa M. Moursi, and Xianfu Wang, *Nearly convex sets: Fine properties and domains or ranges of subdifferentials of convex functions*, Mathematical Programming **160** (2016), no. 1–2, 193–223.

[MN06] Ralf Meyer and Ryszard Nest, *The Baum–Connes conjecture via localisation of categories*, Topology **45** (2006), no. 2, 209–259.

[Mos10] Henri Moscovici, *Local index formula and twisted spectral triples*, Quanta of Maths (Providence, RI) (Etienne Blanchard, David Ellwood, Masoud Khalkhali, Matilde Marcolli, Henri Moscovici, and Sorin Popa, eds.), Clay Mathematics Proceedings, vol. 11, American Mathematical Society, 2010.

[MR16] Bram Mesland and Adam Rennie, *Nonunital spectral triples and metric completeness in unbounded KK-theory*, Journal of Functional Analysis **271** (2016), no. 9, 2460–2538.

[MR25] Ada Masters and Adam Rennie, *Conformal transformations and equivariance in unbounded KK-theory*, `arXiv:2412.17220 [math.OA]`, 2025.

[Mun14] James Munkres, *Topology*, 2 ed., Pearson, 2014.

[MW08] Paul S. Muhly and Dana P. Williams, *Equivalence and disintegration theorems for Fell bundles and their C*-algebras*, Dissertationes Mathematicae **456** (2008), 1–57.

[MY19] Marco Matassa and Robert Yuncken, *Regularity of twisted spectral triples and pseudodifferential calculi*, Journal of Noncommutative Geometry **13** (2019), no. 3, 985–1009.

[Nee97] Tristan Needham, *Visual Complex Analysis*, Oxford University Press, London, England, 1997.

[Nis19] Shintaro Nishikawa, *Direct splitting method for the Baum–Connes conjecture*, Journal of Functional Analysis **277** (2019), no. 7, 2259–2287.

[NR98] G.A. Niblo and L.D. Reeves, *The geometry of cube complexes and the complexity of their fundamental groups*, Topology **37** (1998), no. 3, 621–633.

[NT10] Sergey Neshveyev and Lars Tuset, *The Dirac operator on compact quantum groups*, Journal für die reine und angewandte Mathematik (Crelles Journal) **2010** (2010), no. 641, 1–20.

[NV10] Ryszard Nest and Christian Voigt, *Equivariant Poincaré duality for quantum group actions*, Journal of Functional Analysis **258** (2010), no. 5, 1466–1503.

[Oli09] César R. de Oliveira, *Intermediate Spectral Theory and Quantum Dynamics*, Birkhäuser Basel, 2009.

[OR05] Narutaka Ozawa and Marc A. Rieffel, *Hyperbolic group C*-algebras and free-product C*-algebras as compact quantum metric spaces*, Canadian Journal of Mathematics **57** (2005), no. 5, 1056–1079.

[OV94] A.L. Onishchik and E.B. Vinberg (eds.), *Lie Groups and Lie Algebras III: Structure of Lie Groups and Lie Algebras*, Encyclopaedia of Mathematical Sciences, vol. 41, Springer-Verlag, Berlin, Germany, June 1994.

[Pal61] Richard S. Palais, *On the existence of slices for actions of non-compact Lie groups*, Annals of Mathematics **73** (1961), no. 2, 295–323.

[Pal99] Arupkumar Pal, *Regular operators on Hilbert C*-modules*, Journal of Operator Theory **42** (1999), no. 2, 331–350.

[Par95a] Efton Park, *Isometries of noncommutative metric spaces*, Proceedings of the American Mathematical Society **123** (1995), no. 1, 97–105.

[Par95b] ______, *Isometries of unbounded Fredholm modules over reduced group C*-algebras*, Proceedings of the American Mathematical Society **123** (1995), no. 6, 1839–1843.

[Pat14] Alan L.T. Paterson, *Contractive spectral triples for crossed products*, Mathematica Scandinavica **114** (2014), no. 2, 275.

[Phi88] N. Christopher Phillips, *Inverse limits of C*-algebras*, Journal of Operator Theory **19** (1988), no. 1, 159–195.

[Pie06] Francois Pierrot, *Opérateurs réguliers dans les C*-modules et structure des C*-algèbres de groupes de Lie semisimples complexes simplement connexes*, Journal of Lie Theory **16** (2006), no. 4, 651–689.

[Pim86] Mihai V. Pimsner, *KK-groups of crossed products by groups acting on trees*, Inventiones Mathematicae **86** (1986), no. 3, 603–634.

[Ply86] R.J. Plymen, *Strong Morita equivalence, spinors and symplectic spinors*, Journal of Operator Theory **16** (1986), no. 2, 305–324.

[Pod95] Piotr Podleś, *Symmetries of quantum spaces. Subgroups and quotient spaces of quantum* $SU(2)$ *and* $SO(3)$ *groups*, Communications in Mathematical Physics **170** (1995), no. 1, 1–20.

[Pow75] Robert T. Powers, *A remark on the domain of an unbounded derivation of a C*-algebra*, Journal of Functional Analysis **18** (1975), no. 1, 85–95.

[Pus11] Michael Puschnigg, *Finitely summable Fredholm modules over higher rank groups and lattices*, Journal of K-Theory **8** (2011), no. 2, 223–239.

[Qui96] John C. Quigg, *Discrete C*-coactions and C*-algebraic bundles*, Journal of the Australian Mathematical Society. Series A. Pure Mathematics and Statistics **60** (1996), no. 2, 204–221.

[Rae88] Iain Raeburn, *On crossed products and Takai duality*, Proceedings of the Edinburgh Mathematical Society **31** (1988), no. 2, 321–330.

[Rag72] M.S. Raghunathan, *Discrete Subgroups of Lie Groups*, Springer-Verlag, 1972.

[Ree83] Howard D. Rees, *Special manifolds and Novikov's conjecture*, Topology **22** (1983), no. 3, 365–378.

[Rie02] Marc A. Rieffel, *Group C*-algebras as compact quantum metric spaces*, Documenta Mathematica **7** (2002), 605–651.

[RS78] Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators*, Academic Press, San Diego, CA, 1978.

[RS80] ______, *Methods of Modern Mathematical Physics I: Functional Analysis*, 2 ed., Academic Press, San Diego, CA, 1980.

[RS12] Michel Rumin and Neil Seshadri, *Analytic torsions on contact manifolds*, Annales de l'institut Fourier **62** (2012), no. 2, 727–782.

[RTW15] Bertrand Rémy, Amaury Thuillier, and Annette Werner, *Bruhat-Tits buildings and analytic geometry*, Berkovich Spaces and Applications (Antoine Ducros, Charles Favre, and Johannes Nicaise, eds.), Lecture Notes in Mathematics, vol. 2119, Springer International Publishing, 2015, pp. 141–202.

[Rub22] Alessandro Rubin, *Discrete group actions and spectral geometry of crossed products*, Ph.D. thesis, SISSA, 2022.

[Rum94] Michel Rumin, *Formes différentielles sur les variétés de contact*, Journal of Differential Geometry **39** (1994), no. 2, 281–330.

[Rum00] ———, *Sub-Riemannian limit of the differential form spectrum of contact manifolds*, Geometric And Functional Analysis **10** (2000), no. 2, 407–452.

[Rum05] ———, *An introduction to spectral and differential geometry in Carnot-Carathéodory spaces*, Proceedings of the 24th Winter School "Geometry and Physics" (Jan Slovák and Martin Čadek, eds.), Circolo Matematico di Palermo, 2005, pp. 139–196.

[RW98] Iain Raeburn and Dana P Williams, *Morita Equivalence and Continuous-Trace C*-Algebras*, Mathematical Surveys and Monographs, no. 60, American Mathematical Society, Providence, RI, 1998.

[Sch95] R. Schoen, *On the conformal and CR automorphism groups*, Geometric and Functional Analysis **5** (1995), no. 2, 464–481.

[Sen11] Roger John Senior, *Modular spectral triples and KMS states in noncommutative geometry*, Ph.D. thesis, Australian National University, 2011.

[Ser80] Jean-Pierre Serre, *Trees*, Springer Monographs in Mathematics, Springer, Berlin, Germany, 1980.

[Sit15] Andrzej Sitarz, *Conformally rescaled noncommutative geometries*, Geometric Methods in Physics: XXXIII Workshop, Białowieża, Poland, June 29 – July 5, 2014 (Piotr Kielanowski, Pierre Bieliavsky, Anatol Odzijewicz, Martin Schlichenmaier, and Theodore Voronov, eds.), Trends in Mathematics, Springer International Publishing, 2015, pp. 83–100.

[Sta70] Joseph G. Stampfli, *The norm of a derivation*, Pacific Journal of Mathematics **33** (1970), no. 3, 737–747.

[Swa62] Richard G. Swan, *Vector bundles and projective modules*, Transactions of the American Mathematical Society **105** (1962), no. 2, 264–277.

[SWW98] Elmar Schrohe, Markus Walze, and Jan-Martin Warzecha, *Construction de triplets spectraux à partir de modules de Fredholm*, Comptes Rendus de l'Académie des Sciences - Series I - Mathematics **326** (1998), no. 10, 1195–1199.

[Tak03] Masamichi Takesaki, *Theory of Operator Algebras III*, Encyclopaedia of Mathematical Sciences, vol. 127, Springer Berlin Heidelberg, 2003.

[TB16] Gerard Thompson and Giriraj Bhattarai, *Sectional and Ricci curvature for three-dimensional Lie groups*, Journal of Mathematics **2016** (2016), 1–10.

[Tho01] Klaus Thomsen, *Excision in equivariant KK-theory*, Preprint Series No. 4, Department of Mathematics, University of Aarhus, June 2001.

[Tho18] Anne Thomas, *Geometric and Topological Aspects of Coxeter Groups and Buildings*, Zurich Lectures in Mathematics, European Mathematical Society, Zürich, Switzerland, June 2018.

[Tim08] Thomas Timmermann, *An Invitation to Quantum Groups and Duality*, EMS Textbooks in Mathematics, European Mathematical Society, Zürich, Switzerland, February 2008.

[Tu99] Jean-Louis Tu, *La conjecture de Baum–Connes pour les feuilletages moyennables*, K-Theory **17** (1999), no. 3, 215–264.

[Tu00] ———, *The Baum-Connes Conjecture for Groupoids*, C*-Algebras: Proceedings of the SFB-Workshop on C*-Algebras, Münster, Germany, March 8–12, 1999 (Joachim Cuntz and Siegfried Echterhoff, eds.), Springer-Verlag, 2000, pp. 227–242.

[Uuy11] Otgonbayar Uuye, *The Baum-Connes conjecture for KK-theory*, Journal of K-Theory **8** (2011), no. 1, 3–29.

[Vae05] Stefaan Vaes, *A new approach to induction and imprimitivity results*, Journal of Functional Analysis **229** (2005), no. 2, 317–374.

[Val85] Jean-Michel Vallin, *C*-algèbres de Hopf et C*-algèbres de Kac*, Proceedings of the London Mathematical Society **3** (1985), no. 50, 131–174.

[Val02] Alain Valette, *Introduction to the Baum–Connes Conjecture*, Birkhäuser Basel, 2002.

[Ver02] Roland Vergnioux, *KK-théorie équivariante et opérateur de Julg-Valette pour les groupes quantiques*, Ph.D. thesis, Université Paris-Diderot - Paris VII, December 2002.

[Voi11] Christian Voigt, *The Baum–Connes conjecture for free orthogonal quantum groups*, Advances in Mathematics **227** (2011), no. 5, 187–1913.

[VVD25] Brian Villegas-Villalpando and Koen van den Dungen, *Fredholm complexes of Hilbert C*-modules*, `arXiv:2505.07568 [math.OA]`, 2025.

[VY15] Christian Voigt and Robert Yuncken, *Equivariant Fredholm modules for the full quantum flag manifold of* $\mathrm{SU}_q(3)$, Documenta Mathematica **20** (2015), 433–490.

[VY20] ———, *Complex Semisimple Quantum Groups and Representation Theory*, Springer Nature, Cham, Switzerland, September 2020.

[Wah07] Charlotte Wahl, *On the noncommutative spectral flow*, Journal of the Ramanujan Mathematical Society **22** (2007), no. 2, 135–187.

[WDGO22] Elmar Wagner, Fredy Díaz García, and Réamonn O'Buachalla, *A Dolbeault–Dirac spectral triple for the* B_2*-irreducible quantum flag manifold*, Communications in Mathematical Physics **395** (2022), no. 1, 365–403.

[Web04] Corran Webster, *On unbounded operators affiliated with C*-algebras*, Journal of Operator Theory **51** (2004), no. 2, 237–244.

[Wil70] Stephen Willard, *General Topology*, Addison-Wesley Series in Mathematics, Addison-Wesley Publishing Company, 1970.

[Wil07] Dana P. Williams, *Crossed products of C*-algebras*, Mathematical surveys and monographs, no. 134, American Mathematical Society, Providence, RI, 2007, OCLC: ocm76794891.

[WN92] S.L. Woronowicz and K. Napiórkowski, *Operator theory in the C*-algebra framework*, Reports on Mathematical Physics **31** (1992), no. 3, 353–371.

[Wor80] S.L. Woronowicz, *Pseudospaces, pseudogroups and Pontriagin duality*, Mathematical Problems in Theoretical Physics: Proceedings of the International Conference on Mathematical Physics Held in Lausanne, Switzerland, August 20–25, 1979 (Konrad Osterwalder, ed.), Springer Berlin Heidelberg, 1980, pp. 407–412.

[Wor91] ———, *Unbounded elements affiliated with C*-algebras and non-compact quantum groups*, Communications in Mathematical Physics **136** (1991), no. 2, 399–432.

[Yun11] Robert Yuncken, *The Bernstein–Gelfand–Gelfand complex and Kasparov theory for* $SL(3,\mathbb{C})$, Advances in Mathematics **226** (2011), no. 2, 1474–1512.

[Yun18] ———, *On pseudodifferential operators on filtered and multifiltered manifolds*, `arXiv:1810.10272 [math.OA]`, 2018.

[Zet83] Heinrich Zettl, *A characterization of ternary rings of operators*, Advances in Mathematics **48** (1983), no. 2, 117–143.